1. Two populations, one of predators and one of prey, are modeled by the system of differential equations

$$\frac{dx}{dt} = x(1-x) + xy,$$
$$\frac{dy}{dt} = 4y\left(1 - \frac{y}{2}\right) - 2xy.$$

(a) Which of $x(t)$ and $y(t)$ represents the predator population and which represents the prey population? Explain your answer.

$x(t)$ represents the preddtors; the xy term means that x-y interactions benefit the x's.

$y(t)$ represents the prey; the $-2xy$ term means that x-y interactions hurt the y's.

(b) What would happen to each population in the absence of the other?

- If $x(t) \equiv 0$, then $\frac{dy}{dt} = 4y(1-\frac{y}{2})$, so the y population grows according to a logistic model - if it is non-zero and below 2, it increases to 2, and if it starts above 2, it decreases towards 2.
- If $y(t) \equiv 0$, then $\frac{dx}{dt} = x(1-x)$; the x population is also modeled by a logistic model; if it begins below 1, it increases towards 1 and above 1, it decreases to 1.

(c) Linearize the system about each nontrival equilibrium (that is, equilibria with $x > 0$ and $y > 0$) and classify the equilibria.

$$\frac{dx}{dt} = x\left[1 - x + y\right]$$
$$\frac{dy}{dt} = y\left[4 - 2y - 2x\right]$$

Equilibria: $(0,0)$; $(0,2)$; $(1,0)$; $(\frac{3}{2}, \frac{1}{2})$

(Non-trivial one is if $y = x-1$ and $y = 2-x$
$\Rightarrow x - 1 = 2 - x \Rightarrow x = 3/2 \Rightarrow y = \frac{1}{2}$

If we linearize at $(\frac{3}{2}, \frac{1}{2})$, we need $J(\frac{3}{2}, \frac{1}{2})$.

$$J(x,y) = \begin{bmatrix} 1 - 2x + y & x \\ -2y & 4 - 4y - 2x \end{bmatrix}$$

$$J(\tfrac{3}{2}, \tfrac{1}{2}) = \begin{bmatrix} -3/2 & 3/2 \\ -1 & -1 \end{bmatrix}$$

Char. poly: $\lambda^2 + \frac{5}{2}\lambda + 3$.

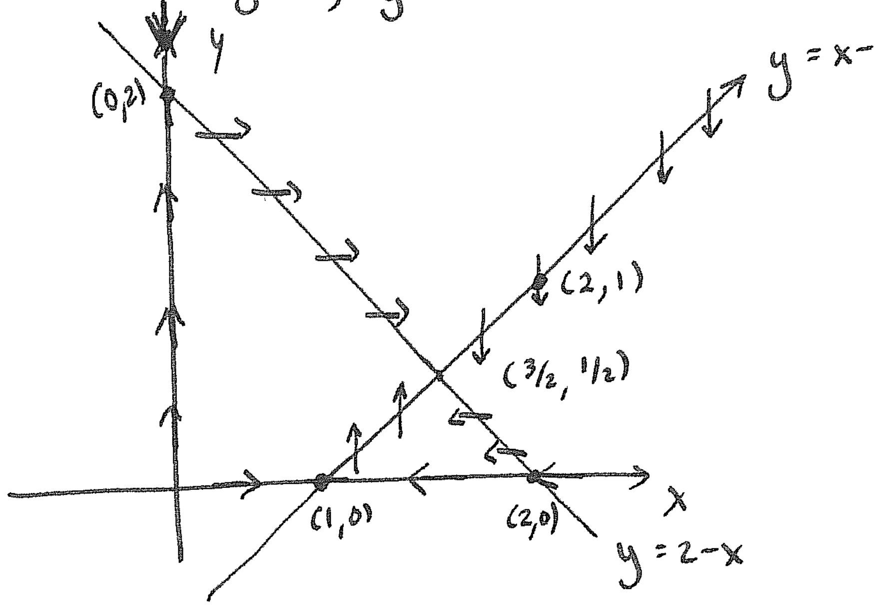
$$\lambda = \frac{-5/2 \pm \sqrt{25/4 - 12}}{2} = \frac{-5/2 \pm \frac{\sqrt{23}}{2}}{2} = -5/4 \pm i\frac{\sqrt{23}}{4}$$

$\Rightarrow$ The equilibrium at $(3/2, 1/2)$ is a spiral sink.

(d) Sketch the x- and y-nullclines of the system for $x \geq 0$ and $y \geq 0$, including arrows indicating the directions of solution curves which cross the nullclines.

x- null clines: $x = 0, \quad y = x - 1$

y- null clines: $y = 0, \quad y = 2 - x$



(e) Suppose that $x(0) = 2$ and $y(0) = 1$. Predict the long-term fates of the populations.

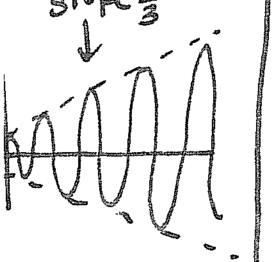
With these initial conditions, the solution curve begins by going straight down. It then moves into a down-left region. It is reasonable to guess then that it moves far enough to the left to cross $y = 2 - x$, where it begins to move up-left, and begins to spiral around the spiral-sink at $(3/2, 1/2)$. We guess the population stabilizes around the non-trivial equilibrium.

2. Consider the forced undamped harmonic oscillator

$$\frac{d^2y}{dt^2} + 9y = 2\sin(3t)$$

with initial conditions $y(0) = y'(0) = 0$. Find the solution $y(t)$ and describe its long-term behavior.

Long-term behavior: the $-\frac{t}{3}\cos(3t)$ term quickly becomes dominant: the period is $\frac{2\pi}{3}$, and the amplitude increases linearly: slope $\frac{1}{3}$



Complexify: $y'' + 9y = 2e^{3it}$

The solution to the homogeneous part (in the real case) is just $k_1\cos(3t) + k_2\sin(3t)$.

For a particular solution, we try $\alpha t e^{3it}$ for some α (we already know that αe^{3it} solves the homogeneous part, so that can't work)

If $y_{p,c}(t) = \alpha t e^{3it}$, $y_{p,c}'(t) = \alpha e^{3it} + 3i\alpha t e^{3it}$

and $y_{p,c}''(t) = 6i\alpha e^{3it} - 9\alpha t e^{3it}$

$\Rightarrow y_{p,c}''(t) + 9y_{p,c}(t) = 6i\alpha e^{3it} - 9\alpha t e^{3it} + 9\alpha t e^{3it}$

$= 6i\alpha e^{3it} \stackrel{?}{=} 2e^{3it}$

$\Rightarrow$ take $\alpha = \frac{1}{3i}$ $\Rightarrow$ $y_{p,c}(t) = -\frac{i}{3}t\,[\cos(3t) + i\sin(3t)]$

$\Rightarrow$ our $y_p(t) = \mathrm{Im}(y_{p,c}(t)) = -\frac{t}{3}\cos(3t)$

$\Rightarrow y_{gen'l}(t) = k_1\cos(3t) + k_2\sin(3t) - \frac{t}{3}\cos(3t)$. $\quad y(0) = k_1 \Rightarrow k_1 = 0$

$\Rightarrow y'(t) = 3k_2\cos(3t) - \frac{1}{3}\cos(3t) + t\sin(3t)$

$y'(0) = 3k_2 - \frac{1}{3} \Rightarrow k_2 = \frac{1}{9} \Rightarrow \boxed{y(t) = \frac{1}{9}\sin(3t) - \frac{t}{3}\cos(3t)}$

3. Find the general solution of

$$\frac{dy}{dt} = \frac{\cos t}{y^2}.$$

Separate: $\int y^2 \, dy = \int \cos t \, dt$

$\Rightarrow \quad \frac{y^3}{3} = \sin t + C$

$\Rightarrow \quad y = (3\sin t + C)^{\frac{1}{3}}$, where C is any constant.

4. Consider the linear system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}, \qquad \text{where } \mathbf{A} = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}.$$

(a) Find the trace, determinant, and eigenvalues of $\mathbf{A}$.

$tr(A) = 1 + 4 = 5$

$det(A) = 1 \cdot 4 - 1 \cdot (-2) = 6$

Char. poly.: $\lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)$

eigenvalues: $2, 3$.

(b) Find eigenvectors corresponding to each eigenvalue.

$\lambda = 2$: $\begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$ take $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$\lambda = 3$: $\begin{bmatrix} -2 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$ take $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(c) Find the general solution of the system.

$$Y_{gen'l}(t) = k_1 e^{2t}\begin{pmatrix}2\\-1\end{pmatrix} + k_2 e^{3t}\begin{pmatrix}1\\-1\end{pmatrix}$$

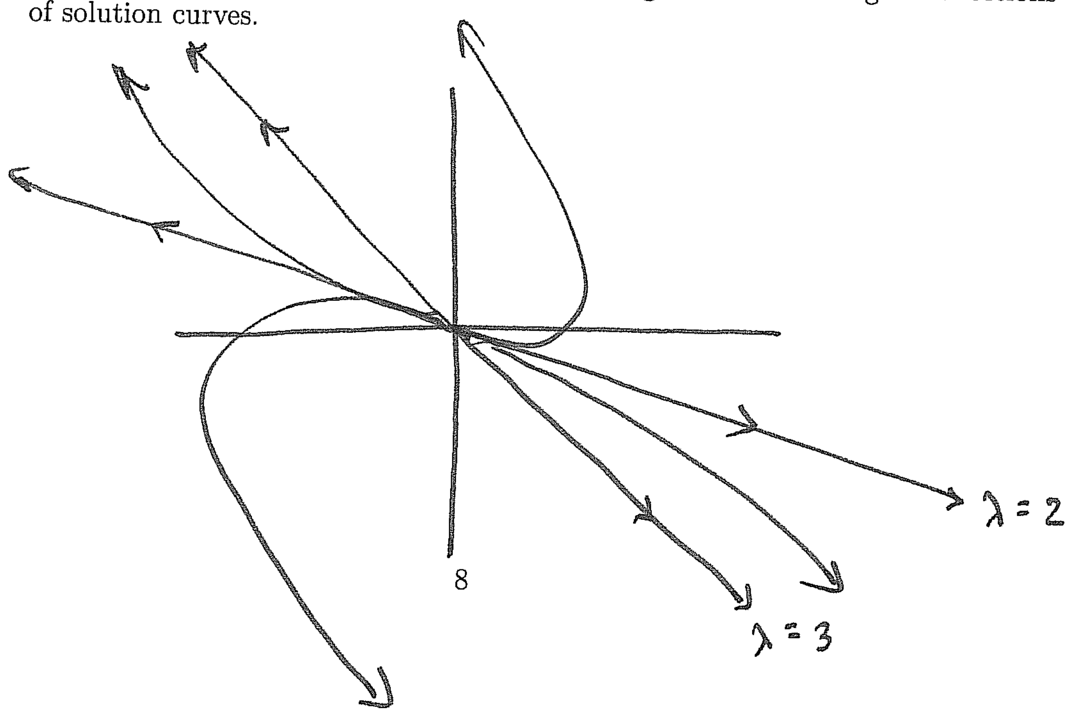
(d) Find the particular solution with $\mathbf{Y}(0) = (1,0)$.

$$\begin{pmatrix}1\\0\end{pmatrix} = k_1\begin{pmatrix}2\\-1\end{pmatrix} + k_2\begin{pmatrix}1\\-1\end{pmatrix} \Rightarrow \begin{aligned} 2k_1 + k_2 &= 1 \\ k_1 + k_2 &= 0 \end{aligned}$$

$$\Rightarrow \quad k_1 = 1 \Rightarrow k_2 = -1$$

$$Y(t) = e^{2t}\begin{pmatrix}2\\-1\end{pmatrix} + e^{3t}\begin{pmatrix}-1\\1\end{pmatrix}$$

(e) Sketch the phase portrait for the system, including arrows indicating the directions of solution curves.



5. Solve

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = \delta_2(t), \qquad y(0) = 1, \qquad y'(0) = 0.$$

Taking Laplace transforms of both sides:

$$s^2\mathcal{L}[y] - s\cdot 1 - 0 + 4s\mathcal{L}[y] - 4\cdot 1 + 3\mathcal{L}[y] = e^{-2s}$$

$$\Rightarrow \quad \mathcal{L}[y] = \frac{s+4}{s^2+4s+3} + \frac{e^{-2s}}{s^2+4s+3}$$

$$= \frac{s+4}{(s+1)(s+3)} + \frac{e^{-2s}}{(s+1)(s+3)} = \frac{(s+3)+1}{(s+1)(s+3)} + \frac{e^{-2s}}{(s+1)(s+3)}$$

Now: $\displaystyle \frac{1}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3} \Rightarrow \frac{1}{s+3} = A + \frac{B(s+1)}{s+3} \overset{s=-1}{\Rightarrow} A = \frac{1}{2}$

$$\Downarrow$$

$$\frac{1}{s+1} = \frac{A(s+3)}{s+1} + B \overset{s=-3}{\Rightarrow} B = -\frac{1}{2}$$

$$\Rightarrow \quad \mathcal{L}[y] = \frac{1}{s+1} + \frac{\frac{1}{2}}{s+1} - \frac{\frac{1}{2}}{s+3} + \frac{\frac{1}{2}e^{-2s}}{s+1} - \frac{\frac{1}{2}e^{-2s}}{s+3}$$

$$\Rightarrow y(t) = e^{-t} + \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} + \frac{1}{2}u_2(t)e^{-(t-2)} - \frac{1}{2}u_2(t)e^{-3(t-2)}$$

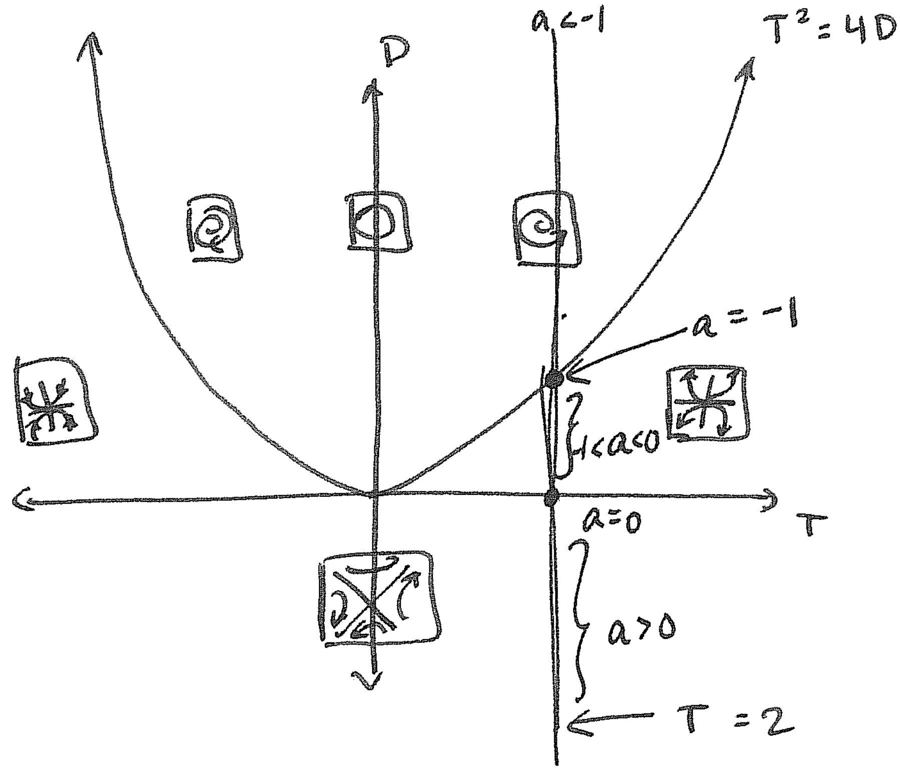
$$= \boxed{\frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t} + \frac{1}{2}u_2(t)\left[e^{-(t-2)} - e^{-3(t-2)}\right]}$$

6. Consider the linear system

$$\frac{d\mathbf{Y}}{dt} = \begin{bmatrix} 2 & 1 \\ a & 0 \end{bmatrix} \mathbf{Y},$$

where a is a parameter. Graph the corresponding curve in the trace-determinant plane and for each value of a classify the type of equilibrium at the origin.

$\mathrm{Tr}\left(\begin{bmatrix} 2 & 1 \\ a & 0 \end{bmatrix}\right) = 2 \qquad \mathrm{Det}\left(\begin{bmatrix} 2 & 1 \\ a & 0 \end{bmatrix}\right) = -a$



• For $a<-1$, we get a spiral source

• For $a=-1$, there is a repeated positive eigenvalue.

Moreover, the matrix is $\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$ with characteristic polynomial $(\lambda-1)^2$, and only one eigenvector (a multiple of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$) $\Rightarrow$ we get .

• For $a \in (-1,0)$, we get a ~~spiral~~ source

• For $a=0$,  • For $a>0$, we get <u>saddles</u>.

7. A certain population grows according to the logistic model

$$\frac{dP}{dt} = 4P\left(1 - \frac{P}{4}\right),$$

where P is measured in thousands of individuals.

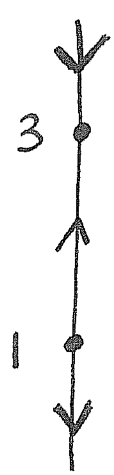
(a) Suppose that the population begins to be hunted at a rate of 3 thousand per unit time. Modify the above logistic equation to reflect this.

$$\frac{dP}{dt} = 4P\left(1-\frac{P}{4}\right) - 3$$

(b) Sketch the phase line for your modified equation and classify all the equilibria.

$\frac{dP}{dt} = 0$ if $P^2 - 4P + 3 = (P-3)(P-1) = 0$

$\Rightarrow P = 3$ or 1.



If $1 \le P \le 3$, $\frac{dP}{dt} > 0$.

If $P > 3$, $\frac{dP}{dt} < 0$.

If $P < 1$, $\frac{dP}{dt} < 0$.

* 3 is a sink.

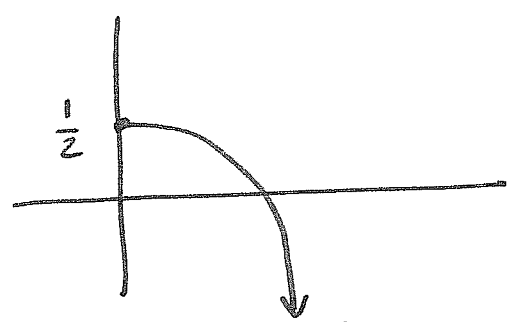
* 1 is a source.

(c) What happens to the hunted population if it starts at size 2?

$2 \in [1,3]$, so the population is increasing and asymptotic to 3.

(d) What happens to the hunted population if it starts at size 1/2?

$\frac{1}{2} < 1$, so the population dies out. (The actual curve looks like



but the model breaks down when P hits zero.)

8. Find the general solution of

$$\frac{dy}{dt} = -2y + \sin t.$$

The solution to the homogeneous part is $y_h(t) = ke^{-2t}$.

To find a particular solution, we guess $y_p(t) = a\sin t + b\cos t$. Then

$y_p'(t) = a\cos t - b\sin t$. ~~and $y_p''(t) = -a\sin t - b\cos t$~~

$\Rightarrow$ We need

$$a\cos t - b\sin t = -2a\sin t - 2b\cos t + \sin t$$

$$\Leftrightarrow (2a - b - 1)\sin t + (a + 2b)\cos t = 0$$

$$\Leftrightarrow \begin{array}{l} 2a - b = 1 \\ a + 2b = 0 \end{array} \Rightarrow 5a = 2 \Rightarrow a = \frac{2}{5}$$

$$\Downarrow$$

$$b = -\frac{1}{5}$$

$$\Rightarrow y_p(t) = \frac{2}{5}\sin t - \frac{1}{5}\cos t$$

$$\Rightarrow y_{gen'l}(t) = ke^{-2t} + \frac{2}{5}\sin t - \frac{1}{5}\cos t.$$

9. Consider the initial value problem

$$\frac{dy}{dt} = t - y^2, \qquad y(0) = 0.$$

(a) Estimate $y(2)$ using Euler's method with $n = 2$.

$\Delta t = \frac{2-0}{2} = 1.$

k	t_k	y_k	$f(t_k, y_k)$	$y_{k+1} = y_k + \Delta t f(t_k, y_k)$
0	0	0	0	0
1	1	0	1	1

$\Rightarrow y(2) \approx y_2 = 1$

(b) Suppose you used a computer to implement improved Euler's method for the initial value problem above to estimate $y(2)$ using $n = 100$, and you believe your error to be approximately 0.005. How large should n be so that you believe your error to be approximately 0.0002?

Improved Euler's method is ~~first~~ second order, so we think $e_n \approx \frac{K}{n^2}$. If $e_{100} = .005$, then $K = 50$. We want n s.t. $e_n = .0002 = \frac{K}{n^2} = \frac{50}{n^2}$

$\Rightarrow n = \sqrt{\frac{25}{.0001}} = \cancel{2500}0.$

$= \sqrt{250{,}000} = 500$

10. Find the general solution of

$$\frac{d\mathbf{Y}}{dt} = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} \mathbf{Y}.$$

Char. poly: $\lambda^2 + 4 = (\lambda + 2i)(\lambda - 2i)$

$\Rightarrow$ eigenvalues: $\pm 2i$

$\lambda = 2i$: $\begin{bmatrix} 1-2i & -5 \\ 1 & -1-2i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow$ take $\begin{bmatrix} 5 \\ 1-2i \end{bmatrix}$

$\Rightarrow Y_{\mathbb{C}}(t) = e^{2it} \begin{pmatrix} 5 \\ 1-2i \end{pmatrix} = [\cos(2t) + i\sin(2t)]\left[\begin{pmatrix} 5 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ -2 \end{pmatrix}\right]$

$= \begin{pmatrix} 5\cos(2t) \\ \cos(2t) + 2\sin(2t) \end{pmatrix} + i \begin{pmatrix} 5\sin(2t) \\ \sin(2t) - 2\cos(2t) \end{pmatrix}$

is a solution. So the real & imaginary parts of $Y_{\mathbb{C}}$ are solutions, and we have

$$Y_{gen'l}(t) = k_1 \begin{pmatrix} 5\cos(2t) \\ \cos(2t) + 2\sin(2t) \end{pmatrix} + k_2 \begin{pmatrix} 5\sin(2t) \\ \sin(2t) - 2\cos(2t) \end{pmatrix}.$$