Name: Solutions

Math 224 Quiz 3 – E. Meckes

1. Consider the non-linear system

$$\frac{dx}{dt} = y - ax^3$$
$$\frac{dy}{dt} = y - x,$$

where a is a parameter.

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(a) Identify the nullclines of the system. Then find all the equilibria of the system, and identify the value(s) of a at which the number and/or location of the equilibria changes.

x-nullcline is where $\frac{dx}{dt} = 0$: $y = ax^3$
y-nullcline is where $\frac{dy}{dt} = 0$: $y = x$.

Equilibria are pts. of intersection of the nullclines: $ax^3 = x$. This is true if $x = 0$ (and then $y = 0$ as well) or $x = \pm\frac{1}{\sqrt{a}}$ (if $a > 0$). So if $a \leq 0$, there is only the one equilibrium at $(0,0)$, and if $a > 0$, there are 3: $(0,0)$, $\left(\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{a}}\right)$, and $\left(-\frac{1}{\sqrt{a}}, -\frac{1}{\sqrt{a}}\right)$.

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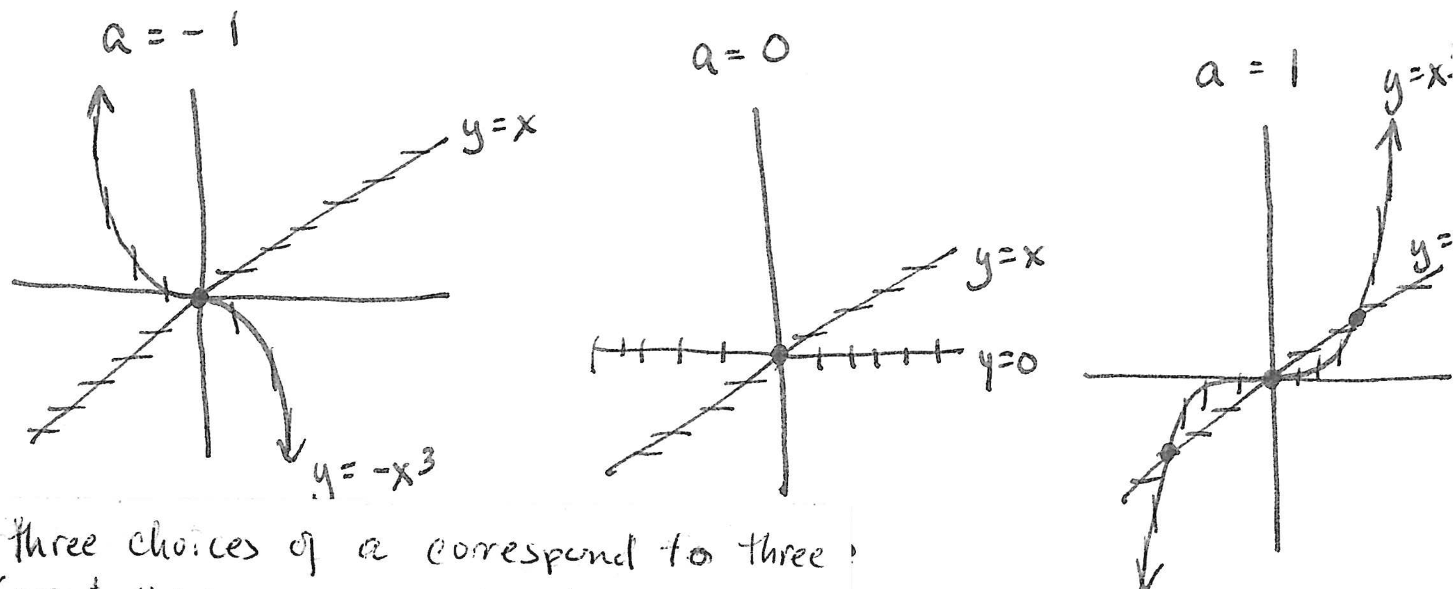
(b) Sketch the nullclines for $a = -1$, $a = 0$, and $a = 1$. You do not need to draw arrows, just the nullclines themselves. Please label everything in your pictures!



The three choices of a correspond to three different systems - each should be drawn separately.

(c) For $a = 1$, compute the Jacobian of the system.

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$$J(x,y) = \begin{bmatrix} -3x^2 & 1 \\ -1 & 1 \end{bmatrix}$$

(d) Linearize the system at each of the equilibria and identify the type of equilibrium. You do not need to calculate eigenvectors.

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$(0,0)$: $J(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$

Char. poly:

$\lambda^2 - \lambda + 1$

Eigenvalues:

$\lambda = \frac{1 \pm \sqrt{1-4}}{2}$: $\boxed{\text{spiral source}}$

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$(1,1)$: $J(1,1) = \begin{bmatrix} -3 & 1 \\ -1 & 1 \end{bmatrix}$

Char. poly:

$\lambda^2 + 2\lambda - 2$

Eigenvalues:

$\lambda = \frac{-2 \pm \sqrt{4+8}}{2}$ $\boxed{\text{Saddle}}$

6 $(-1,-1)$: $J(-1,-1) = \begin{bmatrix} -3 & 1 \\ -1 & 1 \end{bmatrix}$

(same as at $(1,1)$)

$\boxed{\text{Saddle}}$

(e) Sketch the phase portrait of the system with $a = 1$, making use of your answers to the previous parts. Again, you do not need to calculate eigenvectors.

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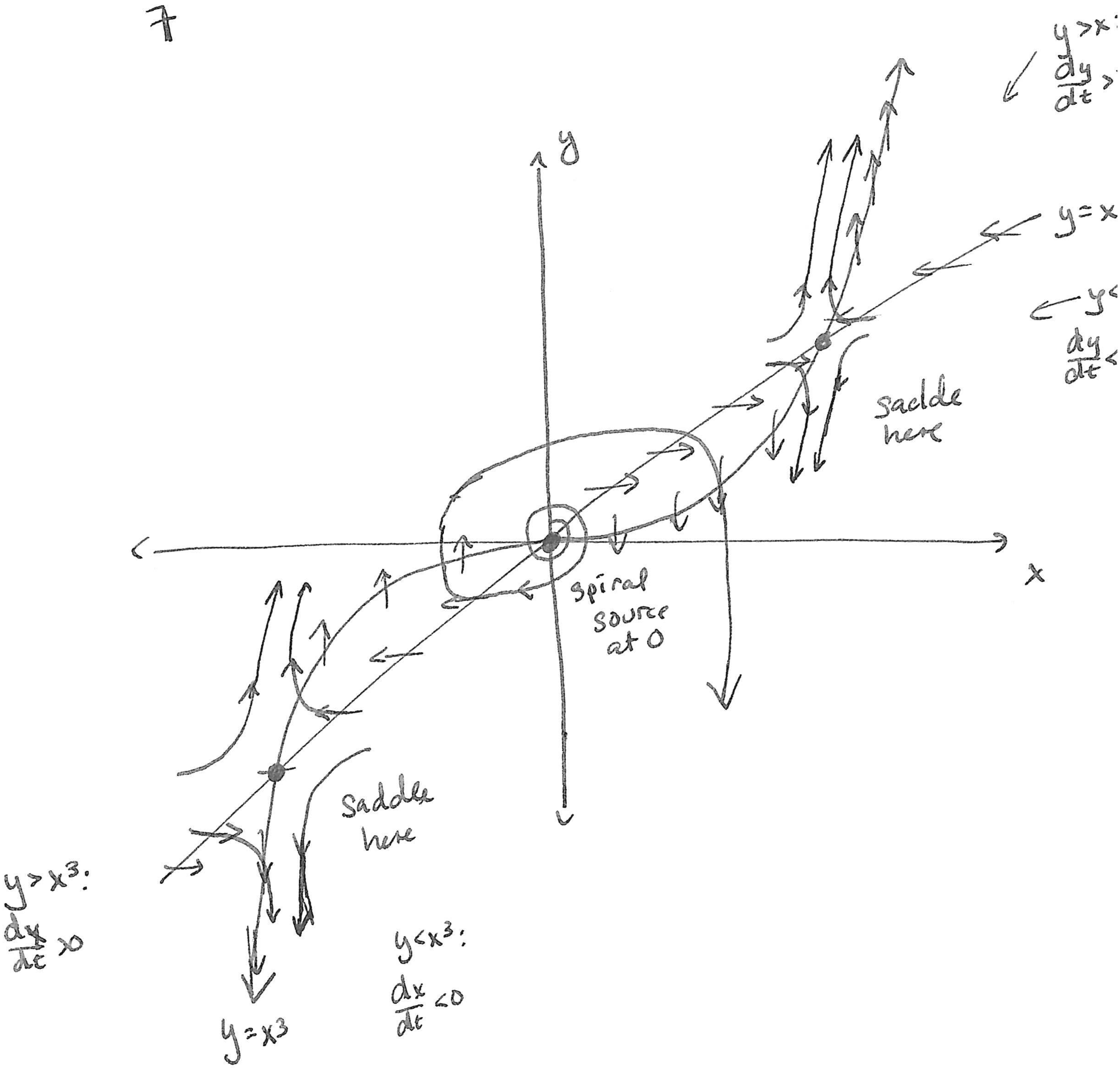

2. Consider the undamped, forced harmonic oscillator modeled by

$$\frac{d^2y}{dt^2} + 4y = 2\cos(\omega t).$$

(a) If $\omega \neq 2$, find the general solution to the equation above.

15 Char. poly: λ^2+4. Ews: $\lambda = \pm 2i$ 3

$\Rightarrow y_h(t) = k_1 \cos(2t) + k_2 \sin(2t)$ 3

Complexify: $y'' + 4y = 2e^{i\omega t}$. Try $y_c(t) = \alpha e^{i\omega t}$ 1

$\Rightarrow y_c'' + 4y_c' = (-\omega^2+4)\alpha e^{i\omega t}$ $\Rightarrow$ take $\alpha = \frac{2}{4-\omega^2}$ 4

$\Rightarrow y_p(t) = \mathrm{Re}(y_c(t)) = \frac{2}{4-\omega^2}\cos(\omega t)$ 1

So the general solution is

$$\boxed{y(t) = k_1\cos(2t) + k_2 \sin(2t) + \frac{2}{4-\omega^2}\cos(\omega t)}$$

3

(b) If $\omega = 3$, give the solution to the initial value problem

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$$\frac{d^2y}{dt^2} + 4y = 2\cos(\omega t), \qquad y(0) = y'(0) = 0.$$

From above, $y(0) = k_1 + \frac{2}{4-9} = k_1 - \frac{2}{5}$: take $k_1 = \frac{2}{5}$. 1

$y'(0) = 2k_2$, so take $k_2 = 0$ 2

$$\Rightarrow \boxed{y(t) = \frac{2}{5}\cos(2t) - \frac{2}{5}\cos(3t)}$$ 2

(c) Now suppose $\omega = 2$. Find the solution to the initial value problem

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$$\frac{d^2y}{dt^2} + 4y = 2\cos(\omega t), \qquad y(0) = y'(0) = 0.$$

Again, $y_h(t) = k_1 \cos(2t) + k_2 \sin(2t)$, and again we complexify the equation to $y'' + 4y = 2e^{i\omega t}$ to find a particular solution. Since $\omega = 2$, we try

$y_c(t) = \alpha t e^{i\omega t} = \alpha t e^{2it}$. 2 Then

$y_c'(t) = \alpha e^{2it} + 2i\alpha t e^{2it}$ 2 and $y_c''(t) = 4i\alpha e^{2it} - 4\alpha t e^{2it}$ 3

$\Rightarrow y_c'' + 4y_c = \alpha e^{2it}[4i - 4t + 4t]$

$= 4i\alpha e^{2it}$ 2 $\Rightarrow$ take $\alpha = \frac{2}{4i} = -\frac{i}{2}$. 2 (So $y_p = \mathrm{Re}($
$= \frac{1}{2}t\sin(2t)$

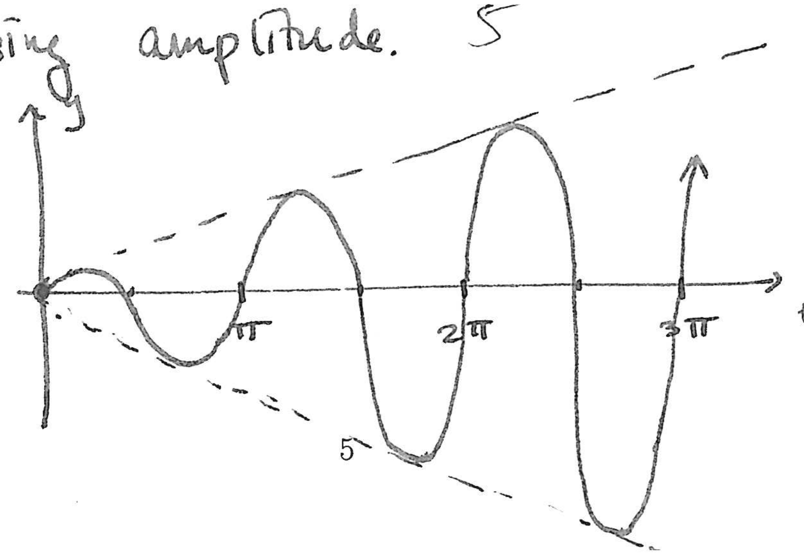
$\Rightarrow y(t) = k_1\cos(2t) + k_2 \sin(2t) + \frac{1}{2}t\sin(2t)$. 2

$y(0) = k_1 \Rightarrow$ take $k_1 = 0$. 2 $y'(0) = 2k_2 \Rightarrow k_2 = 0$ 3

$\Rightarrow \boxed{y(t) = \frac{t}{2}\sin(2t)}$ solves the IVP 2

(d) Describe the long-term behavior of your solution above, and give a rough sketch.

10 This is resonant forcing: we have oscillations about 0 with period π, and linearly increasing amplitude. 5



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