Name: ____________________________________Group:____________

Math 224 Quiz 4 – E. Meckes

1. Consider the system of differential equations

$$\frac{dx}{dt} = -\frac{1}{y}$$
$$\frac{dy}{dt} = -\frac{2}{x}.$$

(a) Show that the system is Hamiltonian and find the Hamiltonian function.

(b) Sketch the phase portrait of the system. Make sure to include directions on solution curves and include curves starting in each of the four quadrants. You will probably want to use the fact that $a\log(s) - \log(t) = \log\left(\frac{s^a}{t}\right)$.

2. Consider the undamped harmonic oscillator $y'' + 4y = 0$.

   (a) What is the natural period of this oscillator?

   (b) Find the solution of the initial value problem

$$y'' + 4y = \sum_{n=1}^{\infty} \delta_{\pi n} \qquad y(0) = 0, y'(0) = 1.$$

(c) Describe the long-term behavior of your solution in part (b). What is the connection between the timing of the delta functions (i.e., the choice of πn) and the natural period of the oscillator?

3. Explain in words the idea behind the derivation of the improved Euler's method. You are welcome to include a picture as well. Your answer should be in complete, clear English sentences.

   Hint: Recall that we started from

$$y(t_{k+1}) = y(t_k) + \int_{t_k}^{t_{k+1}} f(s, y(s))ds.$$

Hamiltonian systems: A two-dimensional system of differential equations is *Hamiltonian* if there is a function $H(x, y)$ such that

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}$$
$$\frac{dy}{dt} = -\frac{\partial H}{\partial x}.$$

The function $H(x, y)$ is called the *Hamiltonian (function)* of the system.

Laplace transforms:

$$\mathcal{L}[y] = \int_0^\infty y(t)e^{-st}\, dt$$

$$\mathcal{L}[y'] = s\mathcal{L}[y] - y(0)$$

$$\mathcal{L}[y''] = s^2\mathcal{L}[y] - sy(0) - y'(0)$$

$y(t)$	$Y(s) = \mathcal{L}[y]$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$u_a(t)$	$\frac{e^{-sa}}{s}$
δ_a	e^{-as}
$u_a(t)f(t-a)$	$e^{-as}\mathcal{L}[f](s)$
$e^{at}f(t)$	$\mathcal{L}[f](s-a)$
$tf(t)$	$-\frac{d}{ds}\left(\mathcal{L}[f]\right)$

Numerical Methods

Euler: $y_{k+1} = y_k + \Delta t f(t_k, y_k)$

Improved Euler:

- $m_k = f(t_k, y_k)$
- $y^*_{k+1} = y_k + (\Delta t)m_k$
- $n_{k+1} = f(t_{k+1}, y^*_{k+1})$
- $y_{k+1} = y_k + (\Delta t)\left(\frac{m_k + n_{k+1}}{2}\right)$