Name: Solutions

# Math 224 Exam 2
September 28, 2012

1. Consider the predator-prey system

$$\frac{dR}{dt} = 2R - 1.2RF,$$
$$\frac{dF}{dt} = -F + 0.9RF.$$

(a) Suppose that the predators find a second food source in limited supply. How would you modify the system to take this into a account?

New:

$$\frac{dR}{dt} = 2R - k_1 RF$$

$$\frac{dF}{dt} = k_2 F\left(1 - \frac{F}{N}\right) + k_3 RF,$$

$\left(\begin{array}{l} 0 < k_1 < 1.2 \\ 0 < k_3 < .9 \\ k_2, N > 0 \end{array}\right)$

Foxes no longer die without rabbits, so the "$-F$" term in $\frac{dF}{dt}$ should be replaced with a growth term. Since the new food is limited, a logistic model is reasonable. It's also reasonable think that the rate of rabbit consumption will go down, so 1.2 and .9 are likely replaced with smaller constants.

(b) Suppose that predators migrate into the area at a constant rate if there are at least ten times as many prey as predators in the area (that is, if $R > 10F$), and they move away at a (possibly different) constant rate if there are fewer than ten times as many predators. How would you modify the system to take this into account? Possibly useful notation:

$$\mathbb{1}(x > 0) = \begin{cases} 1 & x > 0; \\ 0 & x \le 0; \end{cases} \quad \textit{and} \quad \mathbb{1}(x < 0) = \begin{cases} 1 & x < 0; \\ 0 & x \ge 0. \end{cases}$$

If the rate of migration in is $\alpha$ when $R > 10F$ and the rate out is $\beta$ when $R < 10F$, we replace $\frac{dF}{dt}$ (in the original model) by

$$\frac{dF}{dt} = -F + .9RF + \alpha \mathbb{1}(R - 10F > 0) - \beta \mathbb{1}(R - 10F < 0).$$

$\frac{dR}{dt}$ is unchanged!

2. Solve the system

$$\frac{dx}{dt} = 3x + y,$$
$$\frac{dy}{dt} = -y$$

with initial conditions $x(0) = 1$, $y(0) = 2$.

First solve for $y$: $\int \frac{dy}{y} = \int (-1)\, dt \Rightarrow \log|y| = -t + C$

$\Rightarrow y = ke^{-t}$ for some $k$

$y(0) = 2 \Rightarrow y(t) = 2e^{-t}$.

Now the equation for $x$ becomes

$\frac{dx}{dt} = 3x + 2e^{-t}$. The solution to the homogeneous part ($\frac{dx}{dt} = 3x$) is $x_h(t) = ke^{3t}$ for some

We guess a particular solution $x_p(t) = \alpha e^{-t}$.

$\frac{dx_p}{dt} = -\alpha e^{-\alpha t}$. $3x_p(t) + 2e^{-t} = 3\alpha e^{-t} + 2e^{-t}$.

So for $x_p$ to be a solution, we need $-\alpha e^{-t} = (3\alpha+2)e^{-t}$

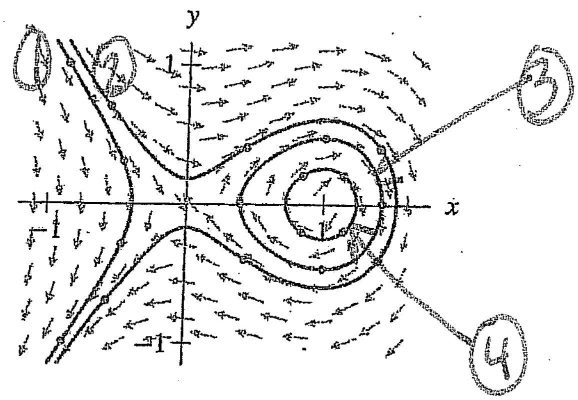
that is, $\alpha = -\frac{1}{2}$. So $x_p(t) = -\frac{1}{2}e^{-t}$ is a solution

So the general solution for $x$ is $x(t) = ke^{3t} - \frac{1}{2}e^{-t}$.

To get $x(0) = 1$, we need $k = 3/2$. Final solution:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \frac{3}{2}e^{3t} - \frac{1}{2}e^{-t} \\ 2e^{-t} \end{pmatrix}.$$

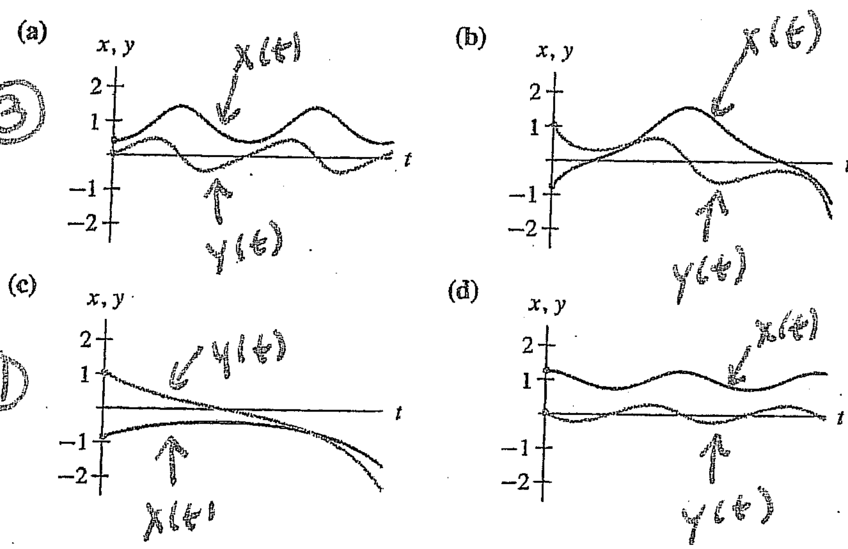
3. The following is a graph of four solution curves $(x(t), y(t))$ to an autonomous system of differential equations, together with the direction field of the system.




Below are four pairs of graphs of $x(t)$ and $y(t)$ versus $t$.

curve 3 curve 2

curve 1 curve 4




Match each of the pairs of graphs to a solution curve in the phase plane. Label which graph is $x$ and which is $y$. Finally, describe the long-term behavior of solutions in all cases.

- For (a) (curve 3), x oscillates periodically about 1, and y oscillates periodically about 0.
- For (b) (curve 2), x initially increases, then decreases to $-\infty$, y fluctuates a little, then decreases to $-\infty$ as well.
- For (c) (curve 1), x increases a little, then decreases to $-\infty$, y decreases monotonically to $-\infty$



- For (d) (curve 4) the behavior is the same as (a) but with small fluctuations

4. Find two nonzero solutions of the differential equation

$$\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 6y = 0$$

which are not constant multiples of each other.

Given the form of the differential equation, we guess that it has solutions of the form $y(t) = e^{st}$ for some values of $s$.

Plugging in:

$$\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 6y = s^2 e^{st} + 7se^{st} + 6e^{st}$$

$$= (s^2+7s+6)\, e^{st}$$

For this to vanish, we need $s^2+7s+6=0$ (since $e^{st}>0 \ \forall s,t$).

By the quadratic formula, this yields

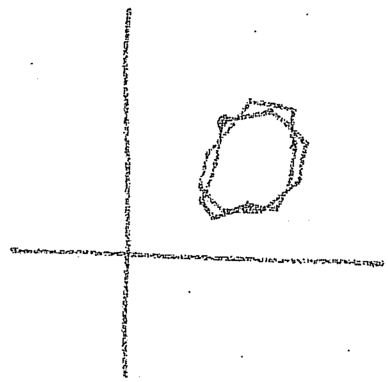
$$s = \frac{-7 \pm \sqrt{49-24}}{2} = \frac{-7 \pm \sqrt{25}}{2} = \{-6, -1\}.$$

We therefore have that $y_1(t) = e^{-6t}$ and $y_2(t) = e^{-t}$ are both solutions; they are clearly not ~~multi~~ constant multiples of each other.

5. Suppose you used Euler's method to approximate the solution to the **autonomous** system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(\mathbf{Y})$$

with initial condition $\mathbf{Y}(0) = \mathbf{Y}_0$, and the resulting solution curve plotted on the phase plane looked like this:



(a) Explain how you can tell that the Euler's method approximation must not be a very good approximation of the true solution.

Solution curves of an autonomous system can't cross themselves - this one appears to, so the approximation isn't very good.

(b) What would you do to try to get a better approximation?

Use a smaller value of $\Delta t$.

(c) What do you guess the true solution looks like, based on the approximation above?

It's probably periodic, and that's why the method is having trouble.