Name: Solutions

Math 224 Exam 3
October 12, 2012

1. Consider the linear system $\frac{d\mathbf{Y}}{dt} = \mathbf{BY} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \mathbf{Y}$.

(a) Find the eigenvalues of B.

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$$\det(B - \lambda I) = \det \begin{bmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix} = (2-\lambda)(3-\lambda) - 2$$
$$= \lambda^2 - 5\lambda + 4$$
$$= (\lambda - 4)(\lambda - 1)$$

The eigenvalues of B are $\{1, 4\}$.

(b) Find the corresponding eigenvectors.

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$\lambda = 1$:

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

take $x = 2$, $y = -1$:

eigenvector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\lambda = 4$:

$$\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

take $x = 1$, $y = 1$:

eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(c) What form does the general solution to this system have?

6 General solutions have the form

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = k_1 e^{t} \begin{pmatrix} 2 \\ -1 \end{pmatrix} + k_2 e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(d) Solve the initial value problem $\frac{d\mathbf{Y}}{dt} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \mathbf{Y}$ and $\mathbf{Y}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

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$$Y(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = k_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{matrix} 2k_1 + k_2 = 2 \\ -k_1 + k_2 = 0 \end{matrix} \Rightarrow k_1 = k_2,\ 3k_1 = 2$$

$$\Rightarrow k_1 = k_2 = 2/3$$

Solution: $Y(t) = 2/3\, e^{t} \begin{pmatrix} 2 \\ -1 \end{pmatrix} + 2/3\, e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{bmatrix} 4/3\, e^{t} + 2/3\, e^{4t} \\ -2/3\, e^{t} + 2/3\, e^{4t} \end{bmatrix}$

(e) What is the long-term behavior of your solution as $t \to \infty$? What about $t \to -\infty$?

6 As $t \to \infty$, the solution approaches ∞ in both the x and y directions. It runs parallel to the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, shifted by $2/3 \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

As $t \to -\infty$, the solution approaches the origin, tangent to $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

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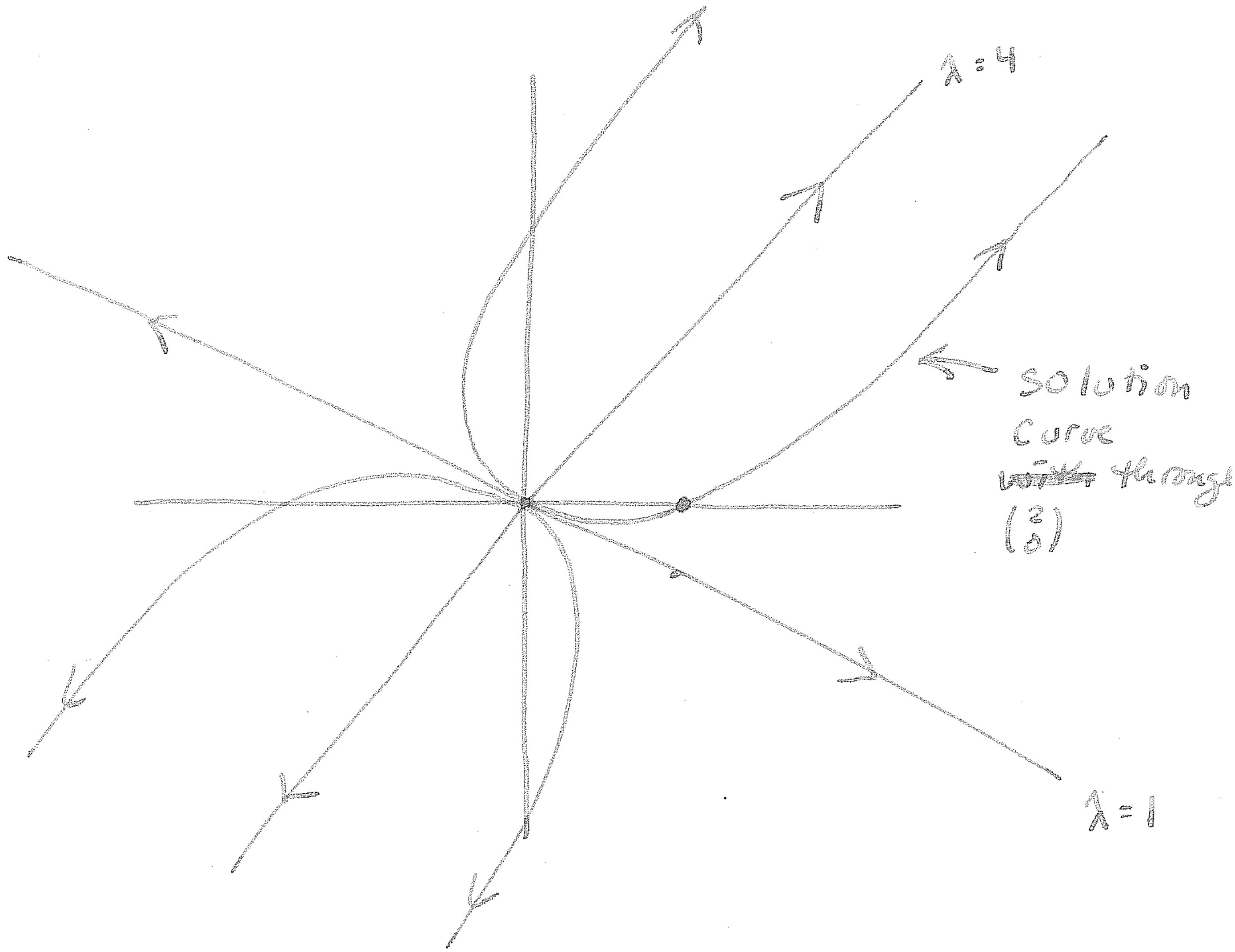
(f) Sketch the phase plane for this system. Make sure to include any straight-line solutions, indicate direction of solution curves in time, and include the solution curve you found above to the initial value problem.



2. (a) What does it mean to say that λ is an eigenvalue of the 2×2 matrix $\mathbf{A}$ with eigenvector $\mathbf{V}$?

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It means that $V \neq 0$ and that

$AV = \lambda V$.

(b) Suppose that λ is an eigenvalue of the 2×2 matrix $\mathbf{A}$ with eigenvector $\mathbf{V}$. Show that $\mathbf{Y}(t) = e^{\lambda t}\mathbf{V}$ is a solution to the linear system $\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$.

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$$\frac{d\vec{Y}}{dt} = \lambda e^{\lambda t} V = e^{\lambda t}[\lambda V] = e^{\lambda t}[AV]$$

$$= A[e^{\lambda t} V]$$

$$= A\vec{Y}.$$

3. Recall the SIR Model of an epidemic:

$$\frac{dS}{dt} = -\alpha IS \qquad \frac{dI}{dt} = \alpha SI - \beta I \qquad \frac{dR}{dt} = \beta I,$$

where S is the portion of the population that is susceptible, I is the portion infected, and R is the portion "recovered"; i.e., not infected or susceptible.

(a) Suppose a fraction v of the population is vaccinated when the first case of a disease is discovered. What initial conditions would reflect this?

Appropriate initial conditions would be

6 $(S_0, I_0, R_0) = (1-v-\varepsilon, \varepsilon, v)$ for small ε.

We know $S_0+I_0+R_0=1$ and $R_0 = v$. We assume the initial infected population is small, so we denote it by ε.

(b) What initial conditions (in terms of α and β) guarantee that at time 0, $\frac{dI}{dt} < 0$?

7 $\frac{dI}{dt}(0) = \alpha S_0 I_0 - \beta I_0 = I_0(\alpha S_0 - \beta)$.

We know $I_0 > 0$, so if we want $\frac{dI}{dt}(0) < 0$, we need $\alpha S_0 - \beta < 0$; i.e., $S_0 < \beta/\alpha$.

(c) Given α and β, what proportion of the population should be vaccinated to prevent an epidemic?

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Since $S_0 = 1-v-\varepsilon$, if we choose $v > 1-\beta/\alpha$, then $S_0 < 1-v < \beta/\alpha$ and there won't be an epidemic.

Note that $\frac{dS}{dt} < 0$ no matter what, so S is decreasing, so if $S_0 < \beta/\alpha$, then $S(t) < \beta/\alpha$ for all t, and so $\frac{dI}{dt} < 0$ for all t.

4. Recall the undamped harmonic oscillator $m\frac{d^2y}{dt^2} = -ky$.

(a) Verify that $y(t) = \sin\left(t\sqrt{\frac{k}{m}}\right)$ and $y(t) = \cos\left(t\sqrt{\frac{k}{m}}\right)$ are solutions to this differential equation.

$y_1(t) = \sin(t\sqrt{\tfrac{k}{m}}) \Rightarrow y_1'(t) = \sqrt{\tfrac{k}{m}}\cos(\sqrt{\tfrac{k}{m}}t),\ y_1''(t) = -\tfrac{k}{m}\sin(\sqrt{\tfrac{k}{m}}$

$\Rightarrow\ m\frac{d^2y_1}{dt^2} = -k\sin(\sqrt{\tfrac{k}{m}}t) = -ky_1(t)$ ✓

$y_2(t) = \cos(t\sqrt{\tfrac{k}{m}}) \Rightarrow y_2'(t) = -\sqrt{\tfrac{k}{m}}\sin(\sqrt{\tfrac{k}{m}}t),\ y_2''(t) = -\tfrac{k}{m}\cos($

$\Rightarrow\ m\frac{d^2y_2}{dt^2} = -k\cos(\sqrt{\tfrac{k}{m}}t) = -ky_2(t)$ ✓

(b) Solve the initial value problem $m\frac{d^2y}{dt^2} = -ky$ with $y(0) = -1$, $y'(0) = \sqrt{\frac{k}{m}}$.

We know, since we have two linearly independent solutions to the corresponding coupled system, that all solutions have the form

$y(t) = c_1\sin(t\sqrt{\tfrac{k}{m}}) + c_2\cos(\sqrt{\tfrac{k}{m}}t)$

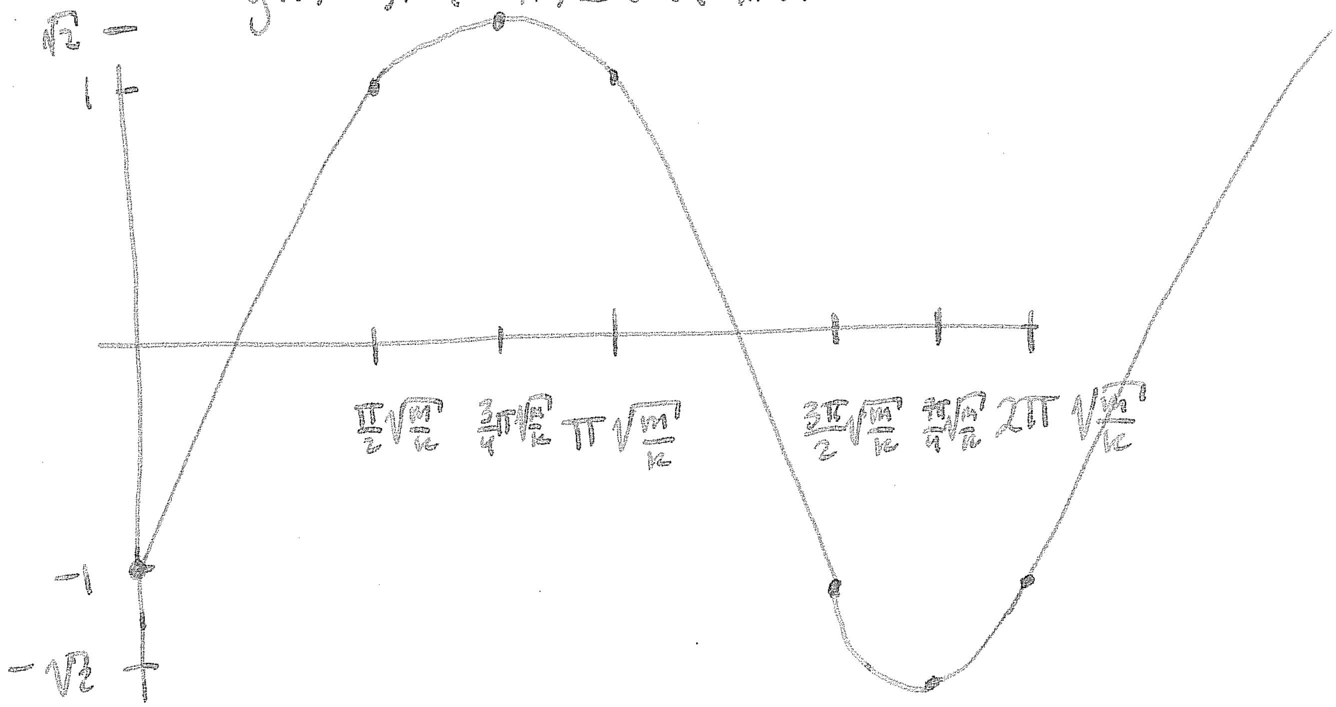
$\left(\text{so } y'(t) = c_1\sqrt{\tfrac{k}{m}}\cos(t\sqrt{\tfrac{k}{m}}) - c_2\sqrt{\tfrac{k}{m}}\sin(\sqrt{\tfrac{k}{m}}t)\right)$

$\Rightarrow \left.\begin{array}{l} y(0) = c_2 = -1 \\ y'(0) = c_1\sqrt{\tfrac{k}{m}} = \sqrt{\tfrac{k}{m}} \end{array}\right\} \Rightarrow c_2 = -1,\ c_1 = 1$

$\Rightarrow y(t) = \sin(t\sqrt{\tfrac{k}{m}}) - \cos(t\sqrt{\tfrac{k}{m}})$

(c) Sketch your solution to the previous part. What is its long-term behavior? (Suggestion: plot the points $y(t)$ for $t = 0, \frac{\pi}{2}\sqrt{\frac{m}{k}}, \frac{3\pi}{4}\sqrt{\frac{m}{k}}, \pi\sqrt{\frac{m}{k}}, \frac{3\pi}{2}\sqrt{\frac{m}{k}}, \frac{7\pi}{4}\sqrt{\frac{m}{k}}, 2\pi\sqrt{\frac{m}{k}}$.)

$y(t) = \sin\left(t\sqrt{\frac{k}{m}}\right) - \cos\left(\sqrt{\frac{k}{m}}\,t\right)$



The solution is $2\pi\sqrt{\frac{m}{k}}$ - periodic, oscillating between $\sqrt{2}$ and $-\sqrt{2}$.