Name: _____

Math 224 Exam 4
October 31, 2012
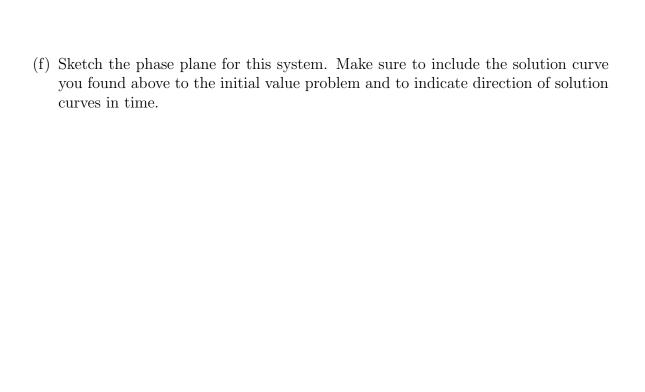
1. Consider the linear system $\dfrac{d\mathbf{Y}}{dt} = \mathbf{BY} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \mathbf{Y}$.

   (a) Find the eigenvalues of B.

   (b) Find the corresponding eigenvectors.

(c) What form does the general solution to this system have?

(d) Solve the initial value problem $\dfrac{d\mathbf{Y}}{dt} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \mathbf{Y}$ and $\mathbf{Y}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(e) What is the long-term behavior of your solution as $t \to \infty$? What about $t \to -\infty$?

(f) Sketch the phase plane for this system. Make sure to include the solution curve you found above to the initial value problem and to indicate direction of solution curves in time.

2. Consider the one-parameter family of systems

$$\frac{d\mathbf{Y}}{dt} = \begin{bmatrix} -2 & a \\ -2 & 0 \end{bmatrix} \mathbf{Y}.$$

Sketch the relevant part of the trace-determinant plane and the curve in it that corresponds to the above systems. Describe the qualitative behavior of the system for all values of a.

3. Consider the following model for the profits of Paul's and Bob's cafés and Glen's ice-cream store. Let $x(t)$ denote Paul's profits, $y(t)$ denote Bobs profits, and $z(t)$ denote Glen's profits, and suppose that they satisfy the system of differential equations:

$$\frac{dx}{dt} = -y + z$$
$$\frac{dy}{dt} = -x + z$$
$$\frac{dz}{dt} = z.$$

(a)   i. If Glen makes a profit, does this help or hurt Paul's and Bob's profits?

   ii. If Paul and Bob are both making profits, does this help or hurt Glen's profit?

(b) Write the system in matrix form, find the eigenvalues, and identify the type of system.

(c) Suppose that at time $t = 0$, $x(0)$ is small but positive, and $y(0) = z(0) = 0$. What happens to all the stores' profits? (*Hint*: think first about z, then think about x and y).