Probability Problems for Homework 13

1. Prove that if $\mathbb{P}$ is a probability on S and $E \subseteq F \subseteq S$, then $\mathbb{P}(E) \leq \mathbb{P}(F)$.
2. Prove by induction that for $E_1, \ldots, E_n$ subsets of S and $\mathbb{P}$ a probability on S,
$$\mathbb{P}\left(\bigcup_{j=1}^{n} E_j\right) = \sum_{r=1}^{n} (-1)^{r+1} \left[\sum_{i_1 < i_2 < \cdots < i_r} \mathbb{P}\left(E_{i_1} \cap \cdots \cap E_{i_r}\right)\right].$$
3. (a) Prove that if S is finite and $\mathbb{P}$ is a probability on S such that $\mathbb{P}(\{s\})$ is the same for all $s \in S$, then $\mathbb{P}(\{s\}) = \frac{1}{|P|}$ for all $s \in S$.
   (b) For a fair die, what is the probability that an even number is rolled?
   (c) Prove that if S is infinite, there is no probability $\mathbb{P}$ on S such that $\mathbb{P}(\{s\})$ is the same and non-zero for all $s \in S$.
4. The royal family has two children, one of whom is the king. What is the probability that the other is the king's brother?
5. A lab test for the presence of a certain disease is 95% effective in both directions; that is, the probability of a false positive is .05 and the probability of missing the disease when it's present is also .05. Suppose that 1% of the population has the disease. If a random person tests positive, what is the probability that he or she actually has the disease?