Math 307 Homework
October 16, 2015

1. (a) Prove that if V is a real inner product space, then

$$\langle v, w \rangle = \frac{1}{4}\left(\|v+w\|^2 - \|v-w\|^2\right)$$

for each $v, w \in V$.

(b) Prove that if V is a complex inner product space, then

$$\langle v, w \rangle = \frac{1}{4}\left(\|v+w\|^2 - \|v-w\|^2 + i\,\|v+iw\|^2 - i\,\|v-iw\|^2\right)$$

for each $v, w \in V$.

2. Equip $C([0,2\pi])$ with the inner product

$$\langle f, g \rangle = \int_0^{2\pi} f(x)g(x)\ dx.$$

Let $f(x) = \sin x$ and $g(x) = \cos x$. Compute each of the following:

(a) $\|f\|$

(b) $\|g\|$

(c) $\langle f, g \rangle$

(d) $\|af + bg\|$, where $a, b \in \mathbb{R}$ are constants.
Hint: After doing the first three parts, you shouldn't need to do any integrals in the last part.

3. Prove that

$$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \leq \left(a_1^2 + 2a_2^2 + \cdots + na_n^2\right)\left(b_1^2 + \frac{b_2^2}{2} + \cdots + \frac{b_n^2}{n}\right)$$

for all $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$.

4. Suppose that V is a vector space, W is an inner product space, and $\boldsymbol{T} : V \to W$ is an isomorphism. For $v_1, v_2 \in V$, define

$$\langle v_1, v_2 \rangle := \langle \boldsymbol{T}v_1, \boldsymbol{T}v_2 \rangle\,,$$

where the right hand side involves the given inner product on W. Prove that this defines an inner product on V.