Math 307 Homework
October 28, 2015

1. Show that the operator norm $\|\boldsymbol{T}\|$ of a linear map $\boldsymbol{T}$ is the smallest constant C such that
$$\|\boldsymbol{T}v\| \leq C\|v\|$$
for all $v \in V$.

2. Prove that none of the following norms is associated to any inner product:
   (a) The ℓ^1 norm on $\mathbb{R}^2$.
   (b) The supremum norm on $C([0,1])$.
   (c) The operator norm on $\mathrm{M}_2(\mathbb{R})$.

3. Prove that if $\mathbf{A} \in \mathrm{M}_{m,n}(\mathbb{R})$ has rank 1, then $\|\mathbf{A}\| = \|\mathbf{A}\|_F$.

   Hint: As you saw on a previous homework, if $\mathbf{A}$ has rank 1, then $\mathbf{A} = \mathbf{v}\mathbf{w}^{\mathrm{T}}$ for some nonzero vectors $\mathbf{v} \in \mathbb{R}^m$ and $\mathbf{w} \in \mathbb{R}^n$. You can compute $\mathbf{A}\mathbf{x}$ and $\|\mathbf{A}\|_F$ explicitly in terms of $\mathbf{v}$ and $\mathbf{w}$.

   Then show that $\|\mathbf{A}\| \leq \|\mathbf{A}\|_F$ by using the Cauchy–Schwarz inequality to show that $\|\mathbf{A}\mathbf{x}\| \leq \|\mathbf{A}\|_F \|\mathbf{x}\|$ for every $\mathbf{x} \in \mathbb{R}^n$.

   Finally, show that $\|\mathbf{A}\| \geq \|\mathbf{A}\|_F$ by finding a specific unit vector $\mathbf{x} \in \mathbb{R}^n$ with $\|\mathbf{A}\mathbf{x}\| \geq \|\mathbf{A}\|_F$.