Math 423 Homework 1

1. A family of sets $\mathcal{R} \subseteq \mathcal{P}(X)$ is called a **ring** if it is closed under finite unitions and differences (i.e., if $E_1, \ldots, E_n \in \mathcal{R}$, then $\cup_{j=1}^n E_j \in \mathcal{R}$, and if $E, F \in \mathcal{R}$, then $F \setminus E \in \mathcal{R}$). A ring which is closed under countable unions is called a **σ-ring**. Prove the following statements:

   (a) Rings (resp. σ-rings) are closed under finite (resp. countable) intersections.

   (b) If $\mathcal{R}$ is a σ-ring, then $\{E \subseteq X : E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}$ is a σ-algebra.

   (c) If $\mathcal{R}$ is a σ-ring, then $\{E \subseteq X : E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$ is a σ-algebra.

2. Show that an algebra $\mathcal{A}$ is a σ-algebra if and only if whenever $\{E_j\}_{j=1}^\infty \subseteq \mathcal{A}$ with $E_1 \subseteq E_2 \subseteq \cdots$, then $\cup_{j=1}^\infty E_j \in \mathcal{A}$.

3. Show that if $\mathcal{M}$ is the σ-algebra generated by $\mathcal{E}$, then $\mathcal{M} = \cup \mathcal{M}(\mathcal{F})$, where the union is over $\mathcal{F} \subseteq \mathcal{E}$ countable. *Hint:* Show that this last object is a σ-algebra.

4. If $(X, \mathcal{M}, \mu)$ is a measure space and $\{E_j\}_{j=1}^\infty \subseteq \mathcal{M}$, then $\mu(\liminf E_j) \leq \liminf \mu(E_j)$. If $\mu\left(\cup_{j=1}^\infty E_j\right) < \infty$, then $\mu(\limsup E_j) \geq \limsup \mu(E_j)$.