Math 423 Homework 2

1. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $A_1, \ldots, A_n$ be measurable. Prove the inclusion/exclusion principle:

$$\mu(\cup_{j=1}^n A_j) = \sum_{r=1}^n (-1)^{r+1} \sum_{1 \le k_1 < \cdots < k_r \le n} \mu\left(A_{k_1} \cap \cdots \cap A_{k_r}\right).$$

2. Let $\mathcal{A}$ consist of finite unions of sets of the form $(a, b] \cap \mathbb{Q}$, for $-\infty \le a \le b \le \infty$.

   (a) Show that $\mathcal{A}$ is an algebra on $\mathbb{Q}$. (You may use Prop. 1.7 from Folland.)

   (b) Show that the σ-algebra generated by $\mathcal{A}$ is all of $\mathcal{P}(\mathbb{Q})$.

   (c) Define μ_0 on $\mathcal{A}$ by $\mu_0(\emptyset) = 0$ and $\mu_0(A) = \infty$ for all $A \neq \emptyset$. Show that μ_0 is a premeasure on $\mathcal{A}$.

   (d) Show that the extension of μ_0 to $\mathcal{P}(\mathbb{Q})$ is not unique; that is, that there is a measure ν on $\mathcal{P}(\mathbb{Q})$ which agrees with μ_0 on $\mathcal{A}$ but not on all of $\mathcal{P}(\mathbb{Q})$.

3. Let μ be a finite measure on $(X, \mathcal{M})$ and let μ^* be the outer measure induced by μ. Suppose that $E \subseteq X$ (but E may not be in $\mathcal{M}$!) and $\mu^*(E) = \mu^*(X)$.

   (a) Show that if $A, B \in \mathcal{M}$ have $A \cap E = B \cap E$, then $\mu(A) = \mu(B)$.

   (b) Let $\mathcal{M}_E := \{A \cap E : A \in \mathcal{M}\}$. Show that $\mathcal{M}_E$ is a σ-algebra on E.

   (c) Define $\nu : \mathcal{M}_E \to [0, \infty)$ by $\nu(A \cap E) = \mu(A)$. Show that ν is a well-defined measure on $\mathcal{M}_E$.