Math 423 Homework 9

1. Let $f \in L^1(\mathbb{R}^n)$, with $m(\{x : f(x) \neq 0\}) > 0$. Show that there exist $C, R > 0$ such that $Hf(x) \geq \frac{C}{|x|^n}$ whenever $|x| > R$. Show that this implies the existence of another constant $C' > 0$ such that, for α small enough, $m(\{x : Hf(x) > \alpha\}) \geq \frac{C'}{\alpha}$. That is, the estimate in the Hardy-Littlewood maximal theorem is pretty much as good as you can get.

2. Consider the function
$$H^*f(x) = \sup\left\{\frac{1}{m(B)}\int_B |f(y)|dy : B \text{ a ball}, x \in B\right\}.$$
Show that $Hf \leq H^*f \leq 2^n Hf$.

3. Show that if $f \in L^1_{loc}$ and f is continuous at x, then $x \in L_f$.

4. Let $f \in L^+(\mathbb{R}^n)$. Show that the measure fdm is regular if and only if $f \in L^1_{loc}$.