A different proof of Junge's estimate of the isotropy constant of polytopes

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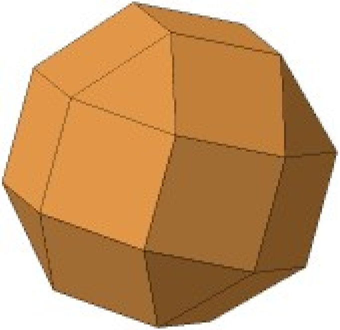
Convex bodies

- A subset $K \subset \mathbb{R}^n$ is said to be a convex body if it is convex, compact and has non-empty interior.



The slicing problem

A convex body K is called isotropic if its volume equals 1, $|K| = 1$, and

- $\int_K x dx = 0$ (center of mass at 0)
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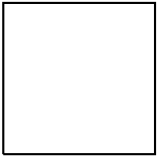

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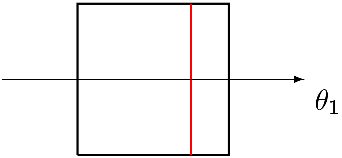

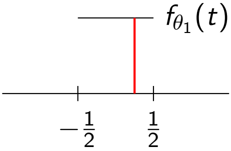

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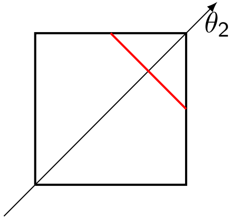

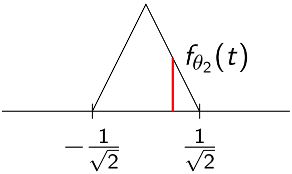

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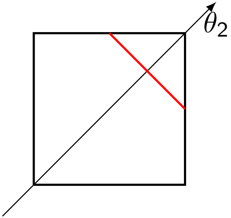

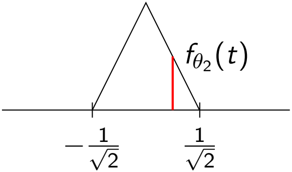

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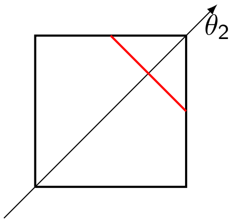

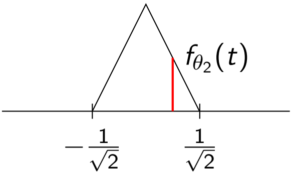

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It is the solution of the following minimization problem

$$\int_{K_1} |x|^2 dx = \min_{\substack{T \in GL(n) \\ a \in \mathbb{R}^n}} \left\{ \frac{1}{|TK|^{1+\frac{2}{n}}} \int_{a+TK} |x|^2 dx \right\}$$

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Hence, we can define the isotropy constant of any convex body by this expression:

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$$L_K \geq L_{B_2^n} \geq c$$

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$$\Updownarrow$$

The hyperplane conjecture

There exists an absolute constant c such that for any convex body of volume 1, there is a hyperplane H such that

$$|K \cap H| > c$$

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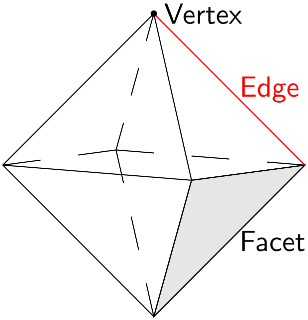
Since any convex body can be approximated by polytopes the hyperplane conjecture is true if it is so for polytopes.

The slicing problem for polytopes

A polytope K is the convex hull of a finite number of points. $F \subset K$ is called a face if given $x, y \in K$, we have $\frac{x+y}{2} \in F \Rightarrow x, y \in F$.



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They need the symmetry of the polytope

Let K be a convex body of volume 1 and $\alpha \geq 1$. The norm ψ_α of a bounded measurable function $f : K \to \mathbb{R}$ is defined by

$$\|f\|_{\psi_\alpha} = \inf\{t > 0 \, : \, \int_K e^{\left(\frac{|f(x)|}{t}\right)^\alpha} dx \leq 2\}$$

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If K is an isotropic convex body, using Borell's lemma, it can be proved that for any $\theta \in S^{n-1}$

$$\|\langle \cdot, \theta \rangle\|_{\psi_1} \leq CL_K$$

The slicing problem for polytopes

Let $\theta_1, \ldots, \theta_N \in S^{n-1}$. If K is an isotropic convex body which satisfies the estimate

$$\|\langle \cdot, \theta \rangle\|_{\psi_\alpha} \leq BL_K$$

for all $\theta \in S^{n-1}$, then

$$\int_K \max_{i=1,\ldots,N} |\langle x, \theta_i \rangle| dx \leq CBL_K (\log N)^{\frac{1}{\alpha}}$$

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Hence, if K is isotropic,

$$\int_K \max_{i=1,\ldots,N} |\langle x, \theta_i \rangle| dx \leq CL_K \log N$$

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Proof of Junge's estimate:

Let K be an isotropic polytope with vertices $P_1, \ldots, P_N$. For any symmetric positive definite T

$$L_K^2 = \frac{1}{\operatorname{tr} T} \int_K \langle x, Tx \rangle dx$$

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The slicing problem for polytopes

$$L_K \leq \frac{C \log N \sup_{i=1,\dots,N} |TP_i|}{n|TK|^{\frac{1}{n}}}$$

There exists an affine map $a + T$ such that $a + TK$ is in Lowner's position (The minimum volume ellipsoid containing K is the Euclidean ball).

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and consequently

$$L_K \leq C \log N$$

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Theorem (D.A, J. Bastero, J. Bernués, P. Wolff)

Let $K \subseteq \mathbb{R}^n$ be a polytope with N vertices. Then

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This estimate is better when the number of vertices is small and worse if the number of vertices is big.

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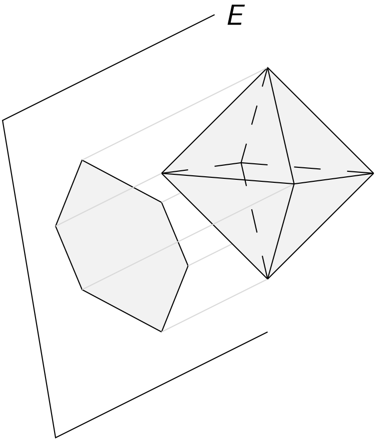
Sketch of the proof (K symmetric with $2N$ vertices)



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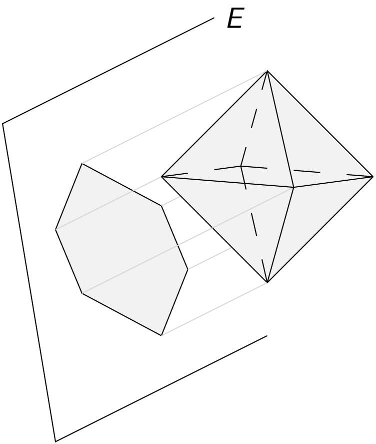

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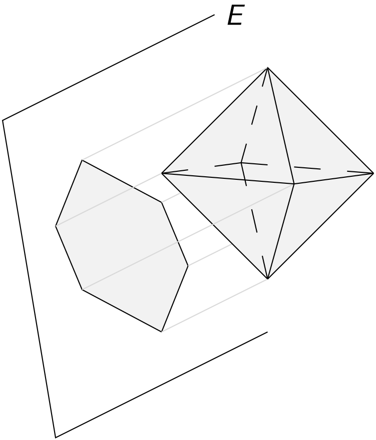

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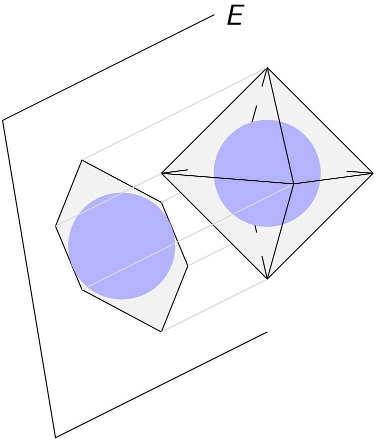

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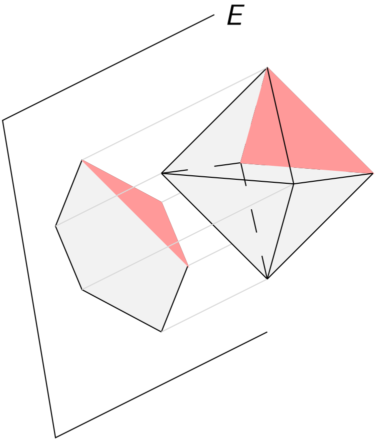

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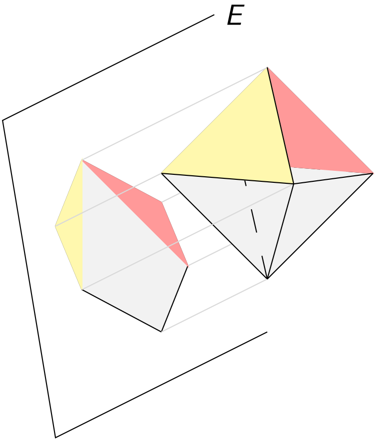

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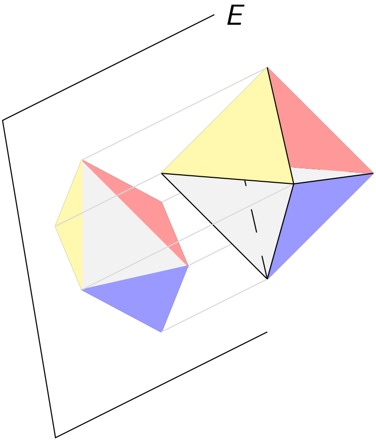

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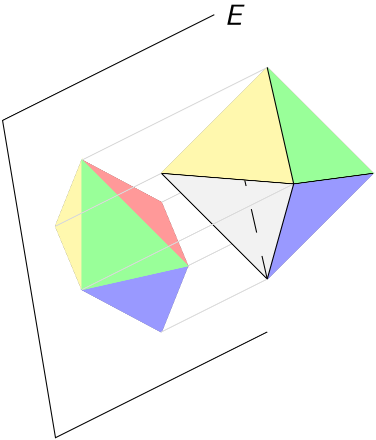

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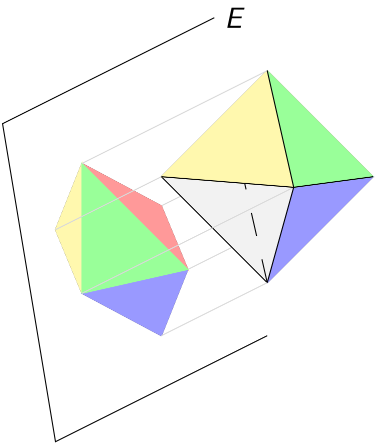

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$$\max_{F \in \mathcal{F}_n(B_1^N)} \frac{1}{|P_E F|} \int_{P_E F} |x|^2 dx$$

The slicing problem for polytopes

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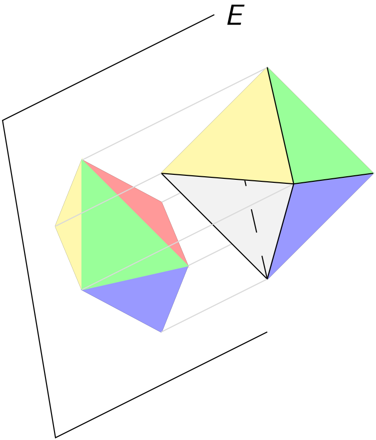

$$nL_K^2 \leq \frac{1}{|P_E B_1^N|^{\frac{2}{n}}} \frac{1}{|P_E B_1^N|} \int_{P_E B_1^N} |x|^2 dx$$

$$|P_E B_1^N|^{\frac{2}{n}} \geq \frac{c}{Nn}$$

$$\frac{1}{|P_E B_1^N|} \int_{P_E B_1^N} |x|^2 dx \leq$$

$$\max_{F \in \mathcal{F}_n(B_1^N)} \frac{1}{|F|} \int_F |P_E x|^2 dx$$

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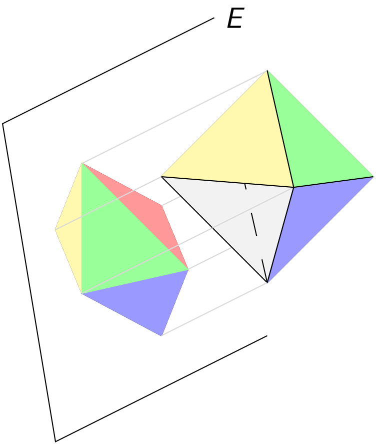

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Theorem

Let $K \subset \mathbb{R}^n$ be a polytope with N facets. Then

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Consequently $nL_K^2 \leq \frac{1}{|K|^{\frac{2}{n}}}\frac{1}{|K|}\int_K |x|^2 dx \leq Cn\log \frac{N}{n}$

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If K is not symmetric, instead of assuming it is in John's position, we assume that

- Its centroid is at 0
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In this case we obtain the same estimates and $L_K \leq C\sqrt{\log \frac{N}{n}}$