Dichotomy Conjecture on Symmetric Spaces

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Preliminaries

- Let G denote a compact Lie group (like $SU(n)$) or an abelian group (like $\mathbb{R}^n$), and m denote the Haar measure on G (Haar measure is a nonnegative, not identically zero, translation invariant, regular, Borel measure on the group G). For $G = \mathbb{R}^n$, Haar measure is the Lebesgue measure. For $1 \leq p < \infty$, $L^p(G)$ denote the space of p-integrable functions on G and $M(G)$ denote the space of bounded regular Borel measures on G.

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- Let $G = SO(n)$ be the group of orthogonal n by n matrices with determinant 1. G acts on $\mathbb{R}^n$ by matrix multiplication. For any measure $\mu \in M(\mathbb{R}^n)$ and $g \in G$, we define $g.\mu \in M(\mathbb{R}^n)$ by either of the formulae

$$g.\mu(S) = \mu(g^{-1}S), \ S \subset \mathbb{R}^n$$

$$\int_{\mathbb{R}^n} f(x)\, d(g.\mu)(x) = \int_{\mathbb{R}^n} f(gx)\, d\mu(x)$$

A measure μ is called rotation(G) invariant if $g.\mu = \mu$ for all $g \in G$.

Classical Results

- Let μ_r denote the surface measure of the sphere of radius r in $\mathbb{R}^n$ $(n \geq 2)$, $r > 0$. One way to define these measures is: Let $f \in C_o(\mathbb{R}^n)$, then

$$< \mu_r, f > = \int_{SO(n)} f(gx)\ dg \ , \qquad x \in \mathbb{R}^n, \ \ ||x|| = r$$

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- It is well known that μ_r^2 (product here is the convolution) is absolutely continuous. In fact for $n \geq 2$, $\mu_r^2 \in L^p(\mathbb{R}^n)$ for $p < n$.

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- For example, $G = SU(n)$ and for $h = diag(e^i, e^i, \ldots, e^i, e^{-(n-1)i})$, $\dim C_h = 2(n-1)$ and $\dim G = n^2 - 1$.

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- Define a measure μ_h on G as follows: For $f \in C(G)$,

$$(\mu_h, f) = \int_G f(ghg^{-1}) \; dm(x)$$

Then for h not in the center of G, μ_h is a continuous, central, singular measure on G.

Lie Algebra and Adjoint Orbits

- Let $\mathfrak{g}$ be the Lie alegbra of G and $H \in \mathfrak{g}$. Define its adjoint orbit O_H to be the set $\{gHg^{-1} : g \in G\}$ in $\mathfrak{g}$. For $H \neq 0$, O_H is a manifold of dimension less than dim$G$. Also, if $h = exp(H)$, then $C_h = exp(O_H)$. But dimension of C_h can be smaller than the dimension of O_H. For example : let $H =$diag$(i\pi, -i\pi)$ be in the Lie algebra of $SU(2)$, then $h = diag(-1, -1)$. Hence dimension of C_h is zero but the dimension of O_H is 2.

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- Let $H = diag(i, i, \ldots, i, -(n-1)i)$ be in the Lie algebra of $SU(n)$ and $h = exp(H)$. Then $O_H = C_h = 2(n-1)$.
- By the orbital measure μ_H we mean the G-invariant measure supported on O_H given by :

$$\int_{\mathfrak{g}} f d\mu_H = \int_G f(gHg^{-1}) dm(g) \text{ for all } f \in C_c(\mathfrak{g})$$

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- For $H \neq 0$, μ_H is a G-invariant, continuous, singular measure on $\mathfrak{g}$.

Ragozin's Conjecture

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- In fact, he conjectured, μ^r is absolutely continuous if $r \geq \frac{\dim G}{\min(\dim C_h)}$ (It is easy to see that $r \geq \frac{\dim G}{\min(\dim C_h)}$ is a necessary condition for μ^r to be absolutely continuous).

Dooley-Wildberger Conjecture

- Dooley-Wildberger (1993) further conjectured that $\mu^{\text{rank}(G)}$ is absolutely continuous, where $\text{rank}(G)$ denotes the dimension of maximal torus of G (For $SU(n)$, a maximal torus is given by diagonal matrices in $SU(n)$).

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- This conjecture was made based upon the concrete formulas for the convolution of two orbital measures developed by Dooley and Wildberger.

Kathryn, Wilson, Yee, Yeats work

- In a series of papers (1998 onwards), using combinatorial argument based upon Plancherel Theorem and Weyl Character formula, they proved that $\mu^r \in L^2(G)$ for all continuous orbital measures iff $r \geq k(G)$, and $\mu^r \in L^1(G)$ for central continuous measures where

	A_n	B_n	C_n	D_n
G	$SU(n+1)(n \geq 1)$	$SO(2n+1)(n \geq 2)$	$Sp(n)(n \geq 3)$	$SO(2n)(n \geq 4)$
$\dim G$	$(n+1)^2 - 1$	$n(2n+1)$	$n(2n+1)$	$n(2n-1)$
$k(G)$	$n+1$	$2n$	$n(n > 3)$	n
$k_1(G)$	$\frac{n}{2}+1$	$n+\ \frac{1}{2}$	$\frac{n(2n+1)}{4n-4} \approx o(n/2)$	$\frac{n(2n-1)}{4n-4} \approx o(n/2)$

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- In the above array, $k_1(G)$ denotes the number conjectured by Ragozin.

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- These results already gave positive response to Dooley-Wildberger Conjecture for the cases of C_n $(n > 3)$ and D_n. But it remained open for the cases of A_n, B_n and C_3.
- Ragozin's conjecture remained open as for $r \leq k(G)$, it is still possible that μ^r is absolutely continuous.
- Kathryn also proved that on a compact, connected, simple Lie group and for $k > \frac{\dim G}{2}$, μ^k is absolutely continuous for all continuous, central, singular, measures on G.

Kathryn-Sanjiv's work

- This is the point I show up into picture in this project : Kathryn and I proved that for $r < k(G)$,there are orbital measures μ_h such that μ_h^r is singular for the cases of A_n (IJM-2002), C_n $(n > 3)$ and D_n (GAFA-2003).

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- This settled the Ragozin's conjecture in negative for these cases. But our methods didn't work for C_3 and B_n. That is our methods failed to show the numbers obtained by Kathryn... for C_3 and B_n remain sharp for absolute continuity or not. We waited for few years before resolving it (will talk about it later).

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- In these articles, first we solved the corresponding problem for adjoint orbits at the Lie algebra level and then using the exponential mapping between the Lie algebra and the Group, we transferred (using $C_h^{r-1} \subset exp((r-1)O_H)$) the results from the Lie algebra to the Lie group. This method works well if C_h ($h = exp(H)$) and O_H have the same dimension. In case of C_3 and B_n, there are $h \in G$ for which C_h and O_H have different dimensions.

Surprising Observations

- While proving these results, we observed an interesting 'Dichotomy': μ_h^r is absolutely continuous iff $\mu_h^r \in L^2(G)$, for $h \in G$, h not in the center of G such that C_h has minimal dimension (Note that $L^2(G) \subsetneq L^1(G)$).

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- On the other extreme it is also known that $\mu_h^2 \in L^2(G)$ for measures μ_h supported on conjugacy classes of maximal dimension. Hence measures supported on maximal dimension conjugacy classes also satisfy 'Dichotomy' satisfied by measures supported on minimal dimension. In 2005 (Mona. Math.), Kathryn and I found a large class of conjugacy classes in $SU(n)$ satisfying the above property. This led us (Kathryn-Sanjiv) to conjecture :

Dichotomy Conjecture

- **Dichotomy Conjecture (2003-2005)** Let G be a compact, connected simple Lie group with finite center. For $h \in G$, and r a natural number, μ_h^r is absolutely continuous if and only if $\mu_h^r \in L^2(G)$.

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- **Dichotomy Conjecture on Lie Algebras (2003-2005)** Let G be a compact, connected simple Lie group with finite center and $\mathfrak{g}$ be its Lie algebra. For $H \in \mathfrak{g}$, and r a natural number, μ_H^r is absolutely continuous if and only if $\mu_H^r \in L^2(\mathfrak{g})$.

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- (2006-2007) 'Dichotomy Conjecture' is true for Lie groups and Lie algebras of type A_n, B_n, C_n, D_n i.e. it is true for all classical compact simple Lie groups and their Lie algebras. Moreover, the value of r for which exactly the dichotomy occurs for a given h is determined.

Description of r in $SU(n)$-case

- Let $n \geq 2$ and $G = SU(n)$. Set

  $$h = diag(a_1(J_1 - \text{times}), \ldots, a_l(J_l - \text{times}), b_1, \ldots, b_F)$$

  Suppose that a_i, b_j are complex numbers and are all distinct, $|a_i| = |b_j| = 1 \;\; \forall i, j$ and their product is 1. Then $h \in SU(n)$ and describes the most general conjugacy class in $SU(n)$. Suppose that $J_1 \geq J_2 \geq \ldots \geq J_l \geq 0$, $F \geq 0$ and $l \geq 0$, then $\mu_h^r \in L^2(G)$ and μ_h^{r-1} is singular, where r is given as follows :

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- For $l = 2, F = 0, J_1 = J_2$, then $r = 3$.
- If $J_1 > n - J_1$ then $r = \left[\frac{n-1}{n-J_1}\right] + 1$.

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- Otherwise, $r = 2$.

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- (ii)In contrast to previous arguments, we attack the singularity problem directly on the group in this article. We need to show that C_h^{r-1} has zero Haar measure (and therefore μ_h^{r-1} is a singular measure since it is supported on C_h^{r-1}). Consider $F : C_h^{r-1} \to G$ defined by $F(g_1, ..., g_{r-1}) = g_1...g_{r-1}$.

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- The differential satisfies a 'product rule': Given $Y_1, \ldots, Y_{r-1} \in \mathfrak{g}$, we have

$$\begin{aligned} & (dF)_{(g_1,\ldots,g_{r-1})}([Y_1, g_1], \ldots, [Y_{r-1}, g_{r-1}]) \\ = \; & [Y_1, g_1]g_2 \ldots g_{r-1} + g_1[Y_2, g_2]g_3 \ldots g_{r-1} \\ & + \ldots + g_1 \ldots g_{r-2}[Y_{r-1}, g_{r-1}]. \end{aligned}$$

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- With this notation, the rank of F at $(g_1, .., g_{r-1})$ is $\dim W_{g_1,\ldots,g_{r-1}}$. Calculating the rank of F using the formula given in seems to be very difficult, in part because the expressions, $[Y_1, g_1]g_2 \ldots g_{r-1} + \ldots + g_1 \ldots g_{r-2}[Y_{r-1}, g_{r-1}]$, are not even in the Lie algebra. We proved a sequence of lemmas to reduce the computation of $\dim(W_{g_1,\ldots,g_{r-1}})$ to something more manageable. But I will not get into these here.

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- Maple software played a very important role in the development of this project.
- If G is a non-discrete compact abelian group, then there exists continuous singular measures whose all convolution powers remain singular.

3-Difficult Cases

- $G = SU(2n)$ and $h = diag(ia(n - \text{times}), -ia(n - \text{times}), a \neq 0.$ Then $\dim C_h = 2n^2$ and $\dim G = 4n^2 - 1.$ Also, $\mu_h^3 \in L^2$ and μ_h^2 is singular.

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- $G = SO(2n+1), n \geq 3$, $H = (\pi, \dots, \pi)$ and $h = exp(H) = \text{diag}(-1, \dots, -1, 1)$. Then $\dim G = n(2n+1)$, $\dim C_h = 2n$ and $\dim O_H = n(n+1)$. Also, μ_h^{2n-1} is singular, μ_H^{n-1} is singular, $\mu_h^{2n} \in L^2$ and $\mu_H^n \in L^2$.

Open Problems

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- It will be interesting to study the convolutions of orbital measures supported on different conjugacy classes and see if there is some interesting pattern there. In classical set up, it is known that for $n \geq 3, r \neq t$, $\mu_r * \mu_t$ is a compactly supported function on $\mathbb{R}^n$.

Symmetric Spaces

- Let G be any Lie group with $K \subseteq G$ a compact subgroup. Let $M(K \backslash G / K)$ denote the Banach algebra of K bi-invariant finite Borel measures on G with convolution as multiplication. We will identify $\mu \in M(K \backslash G / K)$ with the measure $\widetilde{\mu}$ on G/K given on a Borel set $S \subseteq G/K$ by $\widetilde{\mu}\ (S) = \mu(\pi^{-1}(S))$, where $\pi : G \longrightarrow G/K$ is the natural projection. Of course, the measure $\widetilde{\mu}$ is invariant under the natural action of K on G/K on the left. Such $K-$invariant measures are called zonal measures on G/K.

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- The algebra $M(K\backslash G/K)$ contains an identity-m_K, normalized Haar measure on K considered as a measure on G. In fact we have $M(K\backslash G/K) = \{\mu \in M(G) : m_K * \mu * m_K = \mu\}$.

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- We say a $\mu \in M(K\backslash G/K)$ is continuous (in sense of zonal measures) if $\mu(gK) = 0$ for all $g \in G$.

Ragozin's Conjecture on Symmetric Spaces

- We are interested in K bi-invariant measures on G which are the analogues of orbital measures μ_h on conjugacy classes of groups. Let $a \in G \backslash N(K)$ ($N(K)$ denotes the normaliser of K in G). Then $\mu_a = \mu_K * \delta_a * \mu_K$ is supported on the double coset KaK, K bi-invariant and continuous. These measures μ_a are the analogues of μ_h on conjugacy classes of groups.

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- Indeed, let G be a Lie group and θ a Cartan involution on G and K a maximal compact subgroup of G fixed by θ.Then G/K is a symmetric space. If G is a compact Lie group then GxG/D (D is the diagonal in GxG) can be identified with G. Under this identification conjugacy classes on G correspond to double cosets on the symmetric space GxG/D.

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- Ragozin proved: if $\mu_i, i = 1, \ldots, n = \dim G/K$ are continuous zonal measures on G, then $\mu_1 * \mu_2 * \cdots * \mu_n$ is absolutely continuous. He further conjectured that $\mu_1 * \mu_2 * \cdots * \mu_k$ is absolutely continuous for $k = [\dim((G/K) - 1)/j] + 1$, where $j =$minimum dimension of any non-finite K orbit in G/K.

Dichotomy Conjecture on Symmetric Spaces

Let $a \in G\backslash N(K)$ ($N(K)$ denotes the normaliser of K in G). Then for every natural number r, either $\mu_a^r = (\mu_K * \delta_a * \mu_K)^r$ is singular or belongs to $L^2(G) \bigcap L^1(G)$.

Gupta-Hare's work on Symmetric Spaces

- Let G be a classical, connected, compact simple Lie group. Consider it as a compact symmetric space GxG/D and let $G^{\mathbf{C}}$ be the complexification of G. Then $G^{\mathbf{C}}/G$ is the non-compact dual of GxG/D.(For $G = SU(n)$, $G^{\mathbf{C}} = SL(n, \mathbf{C})$

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- **(Dichotomy Theorem for Complex Groups)** Let G be a classical, compact, connected, simple Lie group and let $G^{\mathbb{C}}$ be its complexification. Suppose $a \in \mathcal{A}\backslash\{e\}$, say $a = exp(iH)$ with $H \in \mathfrak{t}\backslash\{0\}$. Then $\mu_a^k \in L^2(G^{\mathbb{C}}) \bigcap L^1(G^{\mathbb{C}})$ for all $k \geq k_0(H)$ and μ_a^k is singular to the Haar measure on $G^{\mathbb{C}}$ for all $k < k_0(H)$.

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- Let $G = SU(n)$ and $K = SO(n)$. Then $\mu_a^k \in L^1(G)$ for all $a \in G \diagdown N_G(K)$ if and only if $k \geq n$. Moreover, there is such a measure with μ_a^{n-1} singular to Haar measure on G.

Ragozin proved that the product of any $\dim G/K$, continuous, K-bi-invariant measures belonged to $L^1(G)$. It would be interesting to know if $\dim G/K$ could be replaced by n if $G = SU(n)$ and $K = SO(n)$. In this case, $\dim G/K = (n^2 + n - 2)/2$ and Ragozin's conjectured constant is approximately $n/2$. Using the methods of the proof of previous result, we are not able to determine, for example, if $\mu_{a_1} * \cdots * \mu_{a_n} \in L^1(G)$ when $a_j \notin N_G(K)$, but are different.