Random subspaces of tensor products and QI

Marius Junge

University of Illinois

Joint work with Carlos Palazuelos

Cleveland, August 2010

Dvoretzky's theorem

Theorem (Dvoretzky) Let X be a finite dimensional Banch space and $\varepsilon > 0$. Then there exists a subspace $F \subset X$ and $u : \ell_2^d \to F$ such that

$$\|u\|\|u^{-1}\| \ \le \ (1+\varepsilon) \, .$$

Later: (Lindenstrauss, Tzafriri/Milmann/Gordon). Assume that

$$\|x\| \ \le \ \|x\|_2 \, .$$

Then d can be chosen $C(\varepsilon)$ proportional to

$$\mathbb{E}\|\sum_{j=1}^{n} g_j e_j\|_X^2 \, .$$

Applications

- ★ Let $X = S_p^{nm}$, Schatten p-class with n columns and m rows, $n \leq m$. Then Chevet's inequality implies
$$\mathbb{E}\| \sum_{ij} g_{ij} e_{ij} \|_p \sim n^{1/2-1/p} m^{1/2} \ .$$
- ★ A random subspace $H_n \subset \ell_\infty^n(\ell_1^n)$ of dimension $d = n$ is Hilbertian and $\sqrt{logn}$ complemented.
- ★ Why? Apply above to $X = \ell_\infty^n(\ell_1^n)$ and the dual space $X^* = \ell_1^n(\ell_\infty^n)$.
- ★ According to a result by Bourgain, Casaza, Lindenstrauss, and Tzafriri ℓ_2^n can not be complemented in $\ell_1(\ell_\infty)$, at most up to $(\log n)^\beta$, $\beta > 0$.
- ★ Why interesting? Applications to Quantum information theory.

Tensor norms

✷ A tensor norm α on two Banach spaces X and Y is a norm on the algebraic tensor product $X \otimes Y$ such that

$$\|x \otimes y\| \ = \ \|x\|\|y\| \quad \forall x \in X, y \in Y \ .$$

The completion is denoted by $X \otimes_\alpha Y$.

✷ A tensor norm is a family of such assignments such that $\|1 \otimes 1\|_{\mathbb{K}\otimes\mathbb{K}} = 1$ and

$$\|T \otimes S : X_1 \otimes_\alpha Y_1 \to X_2 \otimes_\alpha Y_2\| \ \le \ \|T\|\|S\|$$

✷ For S_p^{nm} we can use $\ell_2^n \otimes_{\pi_{p,2}} \ell_2^m$ coming from $\Pi_{p,2}(\ell_2^m, \ell_2^n)$.

Largest/Smallest tensor norms

- $\ell_1^n(X) = \ell_1^n \otimes_\pi X$ is given by the largest tensor norm

$$\|z\|_\pi \;=\; \inf_{z=\sum_k x_k \otimes y_k} \sum_k \|x_k\| \|y_k\| \, .$$

- $\ell_\infty^n(Y)$ is given by the smallest tensor norm: Let $z = \sum_k x_k \otimes y_k$. Then

$$\|z\|_\varepsilon \;=\; \sup_{\|x^*\| \|y^*\| \le 1} |\sum_k x^*(x_k) y^*(y_k)| \, .$$

Operator spaces

- An operator space is a subspace X of $B(H)$, bounded operators on a Hilbert space.
- In addition to the norm we add all the matrix norms

$$\|(x_{ij})\|_{M_n(X)} \;=\; \|(x_{ij}) : \ell_2^n(H) \to \ell_2^n(H)\| \,.$$

- Bad news: We have to replace the norm of a linear maps by the cb-norm (completely bounded norm)

$$\|u\|_{cb} \;=\; \sup_n \|id_{M_n} \otimes u : M_n(X) \to M_n(Y)\| \,.$$

$CB(X, Y)$ is the space of *cb*-norms. We don't know how to prove an analogue of Chevet's inequality!

Examples

- ✶ Good news: Operator spaces are closed under duality (Ruan): There exists an isometric embedding such that

$$CB(X, M_m) \; = \; M_m(X^*) \; .$$

- ✶ $X = \ell_\infty =$ diagonal matrices in M_n.
- ✶ $X = \ell_1^n = (\ell_\infty^n)^*$. By a result of Paulsen, this space can be realized using the generators of the full C^*-algebra of arbitrary unitaries.
- ✶ The column space $X = C_n = B(\mathbb{C}, \ell_2^n)$. The dual space is given by the space of row space $R_n = B(\ell_2^n, \mathbb{C})$.

Tensor norms

- The smallest tensor norm for two operator spaces $X \subset B(H)$ and $Y \subset B(K)$ is given by

$$X \otimes_{min} Y \subset B(H \otimes_2 K) .$$

If X or Y is finite dimensional, we have

$$CB(X^*, Y) = X \otimes_{\min} Y .$$

- The biggest tensor norm $X \otimes_\wedge Y$ is defined so that

$$(X \otimes_\wedge Y)^* = CB(X, Y^*) \cong CB(Y, X^*) .$$

Special operator spaces

The space $\ell_1^n(\ell_\infty^m) = \ell_1^n \otimes_\wedge \ell_\infty^m$ is an operator space with a good state space

$$CB(\ell_1^n(\ell_\infty^m), B(H)) \;=\; \ell_\infty^n(CB(\ell_\infty^m, B(H))\,.$$

A map $u : \ell_\infty^m \to B(H)$ has *cb*-norm less than one iff

$$u(e_a) \;=\; v_a w_a \quad \sum_a v_a v_a^* \le 1\,,\; \sum w_a^* w_a \le 1\,.$$

Proposition: Let $m_{a,b,x,y}$ be a multiindexed matrix. Then

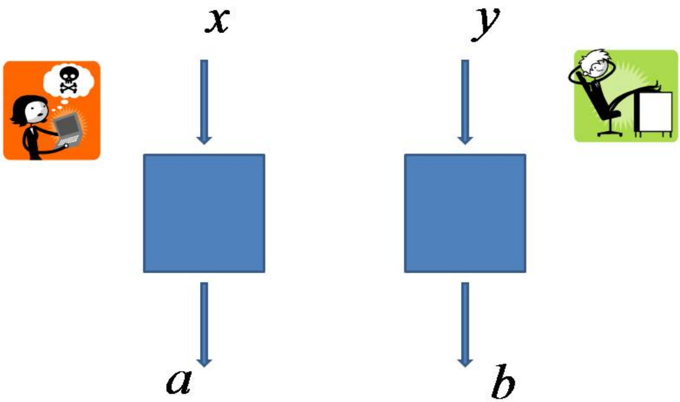
$$\|\sum_{a,b,x,y} m_{a,b,x,y} e_{x,a} \otimes e_{y,b}\|_{\ell_1^n(\ell_\infty^m)\otimes_{\min}\ell_1^n(\ell_\infty^m)} \sim$$

$$\sup_{\sum_a E_a^x \le 1, \sum_y F_b^y \le 1} \|\sum_{a,b,x,y} m_{a,b,x,y} E_a^x \otimes F_b^y\|_{B(H\otimes K)}\,.$$

min norm is about commuting POVM's (i.e. $\sum_a E_a = 1$).

Quantum Information and Violation

- In a so called "Gedankenexperiment" Bell showed that certain quantum mechanical effects can not be "explained" using an average of classical (=local) experiments.
- This experiment leads to an inequality satisfied for all averages, but not satisfies by a quantum mechanical experiments.
- Since then Bell inequalities are used as witness for entanglement, and they can be measured in a lab.



$$p(ab|xy)$$

More precisely: Family of probabilities

- We consider families $p(a,b|x,y)$ of positive real numbers such that
- $\sum_a p(a,b|x,y) = P(b|y)$ does not depend on x,
- $\sum_b p(a,b|x,y) = Q(a|x)$ does not depend on y,
- $\sum_b P(b|y) = 1 = \sum_a Q(a|x)$ for all x, y.
- The classical (=local) model says that Alice has x inputs, B has y inputs and the outputs are given by

$$p_{loc}(a,b|x,y) \ = \ \int_\Omega p_a^x(\lambda) q_b^y(\lambda) d\mu(\lambda) \, .$$

Here $\sum_a p_a^x(\lambda) = 1 = \sum_a q_b^y(\lambda)$ for all x, y, λ. (π-tensor norm)

Quantum version

The quantum version of this experiment replaces the commuting variables $p_a^x(\lambda)$ and $q_b^y(\lambda)$ by commuting operators

$$p_{qua}(a,b|x,y) \;=\; (h|(E_a^x \otimes F_b^y)h)\;,\; h \in H \otimes H$$

such that for all experiments x, y

$$E_a^x \;\geq\; 0, F_y^b \;\geq\; 0 \quad , \quad \sum_x E_a^x \;=\; 1 \;=\; \sum_y F_b^y \;.$$

For tripartite systems one may consider $(h|(E_a^x \otimes F_b^y \otimes G_c^z)h)$.
Theorem: (Bell) There are quantum probabilities which are not local.

Correlations versus probabilities

Suppose that $p_\lambda(a|x)$ is the probability for tossing a sign $\varepsilon_a \in \{\pm 1\}$. Then

$$U_{x,\lambda} = \sum_a \varepsilon_a p_\lambda(a|x) \quad , \quad V_y = \sum_b \varepsilon_b p_\lambda(b|x)$$

gives the correlation matrix

$$C_{x,y} = \int_\Omega U_{x,\lambda} V_{y,\lambda} d\mu(\lambda) \, .$$

In the tripartite situation one has similarly

$$C_{x,y,z} = \int_\Omega U_{x,\lambda} V_{y,\lambda} W_{z,\lambda} d\mu(\lambda) \, .$$

Testing with linear constraints

- Let $m_{x,y}$ be any matrix. By convexity

$$\sup_{C \text{ local}} |\sum_{x,y} C_{x,y} m_{x,y}| \leq \sup_{\varepsilon_x=\pm 1, \delta_y=\pm 1} |\sum_{x,y} \varepsilon_x \delta_y m_{x,y}| \, .$$

- The quantum analogue is

$$\sup_{C \text{ quantum}} |\sum_{x,y} C_{x,y} m_{x,y}|$$
$$= \sup_{\|T_x\|, \|S_y\| \leq 1, \|h\| \leq 1} |\sum_{x,y} (h|(T_x \otimes S_y)h) m_{x,y}| \, .$$

Testing with linear constraints

- Let $m_{x,a,y,b}$ be a matrix. Then we can compare

$$\|m\|_{\varepsilon} = \sup_{p \text{ local}} |\sum_{x,a,y,b} p(a,b|x,y) m_{x,y}|$$

with

$$\|m\|_{\min} = \sup_{p \text{ quantum}} |\sum_{x,a,y,b} p(a,b|x,y) m_{x,y}|$$

$$= \sup_{\sum_x E_a^x = 1 = \sum_y F_b^y, h} |\sum_{x,a,y,b} (h|E_a^x \otimes F_b^y) h) m_{x,y}| \, .$$

Violation

A Bell inequality with violation for a two-partite correlation is a matrix $m_{x,y}$ such that

$$\|\sum_{x,y} m_{x,y} e_x \otimes e_y\|_{\ell_1 \otimes_\varepsilon \ell_1} < \|\sum_{x,y} m_{a,b} e_x \otimes e_y\|_{\ell_1 \otimes_{\min} \ell_1}$$

A Bell inequality with violations for a three-partite correlation is a matrix $m_{x,y,z}$ such that

$$\|\sum_{x,y} m_{a,b,c} e_x \otimes e_y \otimes e_y\|_{\ell_1 \otimes_\varepsilon \ell_1} < \|\sum_{x,y} m_{a,b,c} e_x \otimes e_y \otimes e_c\|_{\ell_1 \otimes_{\min} \ell_1}$$

Violation = min is bigger than ε

A Bell inequality with violations for two partite probabilities is a matrix $m_{x,a,y,b}$ such that

$$\|\sum_{x,y,a,b} m_{x,a,y,b}(e_a \otimes e_x) \otimes (e_b \otimes e_y)\|_{\ell_1(\ell_\infty)\otimes_\varepsilon \ell_1(\ell_\infty)}$$

$$< \|\sum_{x,y} m_{x,a,y,b}(e_x \otimes e_a) \otimes (e_y \otimes e_b)\|_{\ell_1(\ell_\infty)\otimes_{\min} \ell_1(\ell_\infty)} .$$

Results

Theorem

(J-GP-P-V-W-08) There exists a rank n matrix in $\ell_1 \otimes \ell_1 \otimes \ell_1$ *such that*

$$\frac{\|m\|_{\ell_1 \otimes_{\min} \ell_1 \otimes_{\min} \ell_1}}{\|m\|_{\ell_1 \otimes_{\varepsilon} \ell_1 \otimes_{\varepsilon} \ell_1}} \sim \sqrt{n}$$

The rate is optimal and can be achieved using one Hilbert space n-dimensional.

Comment: Coefficients, POV's and testing vector chosen from large random unitaries + large limit perturbation coming from limit model.

Results-II

Theorem

(J-GP-P-V-W-09) There exists a rank n matrix in $\ell_1(\ell_\infty^n) \otimes \ell_1(\ell_\infty^n)$ *such that*

$$\frac{\|m\|_{\min}}{\|m\|_\varepsilon} \ \geq \ c\frac{\sqrt{n}}{\log n} \ .$$

Comment: We use operator space embedding results.

Recent results

Theorem

(J-P-10) There exists a rank n matrix in $\ell_1^n(\ell_\infty^n) \otimes \ell_1^n(\ell_\infty^n)$ *such that*

$$c\frac{\sqrt{n}}{\log n} \le \frac{\|m\|_{\min}}{\|m\|_{\varepsilon}} \le C\sqrt{n} \,.$$

Remark: A similar result with exponential input was also recently obtained for games by Buhrmann and de Wolf (2010).

Comments: Previous estimates of polynomial order $n^{-10^{-5}}$, but no control of input size.

Violation and Entropy

- Let $\xi \in \ell_2^n \otimes \ell_2^m$ with singular value decomposition

$$\xi \;=\; \sum_{k=1}^{\min(n,m)} s_k v_k \otimes u_k \;.$$

- The von Neumann entropy is given by

$$\mathrm{Ent}(\xi) \;=\; \mathrm{Ent}(\rho) \;=\; -\sum_k s_k^2 \log_2(s_k^2)\;.$$

This is the entropy of the density

$$\rho \;=\; \sum_k s_k^2 |v_k\rangle\langle v_k|$$

or $\tilde{\rho} = \sum_k s_k^2 |u_k\rangle\langle u_k|$.

Violation and Entropy

- We can show that if $\|s\|_1\|s\|_\infty \geq 2$, then there exists a Bell inequality and POVM's such that

$$\sum_{a,b,x,y} (\xi | m_{a,b,x,y} E_a^x \otimes F_b^y \xi) \;\geq\; c\frac{\|s\|_1\|s\|_\infty}{\log n}\|m\|_\varepsilon \,.$$

  This gives violation with entropy between $\frac{\delta}{\log n}$ and $\log n - \delta$ of order $c(\delta)\sqrt{n}/\log^2 n$.

- Violation and Entropy are almost independent.

Minimal Entropy

★ A channel $\Phi : M_m \to M_m$ is a trace preserving completely positive map, i.e.

$$\Phi(x) \;=\; \sum_{k=1}^{N} a_k x a_k^* \quad \sum_k a_k^* a_k = 1\;.$$

The linear map $V : \ell_2^m \to \ell_2^m \otimes \ell_2^N$ given by the row matrix $(a_1, ..., a_N)$ is a partial isometry.

★ The minimal entropy is given by

$$Ent(\Phi) \;=\; \min_{tr(\rho)=1} Ent(\Phi(\rho))\;.$$

This is the minimal entropy in the subspace $V(\ell_2^n)$.

★ $Ent(\rho) \;=\; -\frac{d}{dp}\|\rho\|_p\Big|_{p=1}$.

CB-entropy

- ✯ (DJRK) $\|\Phi\|_{p,cb} = \|\Phi : S_1^m \to S_p^m\|_{cb} = \|\Phi\|_{M_m(S_{p'}^m)}$ is multiplicative, i.e.

$$\|\Phi \otimes \Psi\|_{p,cb} \; = \; \|\Phi\|_{p,cb}\|\Psi\|_{p,cb} \; .$$

- ✯ Corollary (06): The *cb*-entropy

$$\mathrm{Ent}_{cb} \; = \; -\frac{d}{dp}\|\Phi\|_{p,cb}$$

is additive (in contrast to the classical entropy!).

CB-entropy determined by subspace

- Let V be as above then $\|\Phi\|_{p,cb} \leq C$ iff

$$\|\sum_j \xi_j\xi_j^*\|_p \ \leq \ C \sup_{\|a\|_{2p'}\leq 1} \|\sum_j a\xi_j\xi_j^* a^*\|_2$$

holds for all $1 \leq k \leq m$ and $\xi_j \in \ell_2^k \otimes \ell_2^m$.

- For $k = 1$ this gives the mininal entropy.
- Again this expression only depends on the subspace determined by V!
- Problem: Calculate *cb*-entropy for random subspaces.

Thanks for listening