Isoperimetric Inequalities in Semi-Convex Settings

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Outline

- Define isoperimetric inqs and recall usefulness.
- Survey methods of obtaining isoperimetric inqs.
- Sample applications.
- Contraction method for transferring isoperimetric inqs.

Isoperimetric Inequalities

(Ω, d, μ) - measure metric space; d - metric, μ - Borel measure.

Assume: $\Omega \subset (M^n, g)$ Riemannian manifold, d induced geodesic distance on M, $\mu = h\, vol_M|_\Omega$.

Isoperimetric Inqs compare between $\mu(A)$ and $\mu^+(A)$ (Minkowski's exterior boundary measure):

$$\mu^+(A) := \liminf_{\varepsilon \to 0} \frac{\mu(A^d_\varepsilon) - \mu(A)}{\varepsilon},$$

$$A^d_\varepsilon := \{x \in \Omega; d(x, A) < \varepsilon\}.$$

Isoperimetric profile: $\mathcal{I} : [0, \mu(\Omega)] \ni v \mapsto \inf\{\mu^+(A); \mu(A) = v\}$.

Typically $\mu^+(A) = \mu^+(\Omega \setminus A)$; if $\mu(\Omega) = 1$ then $\mathcal{I}(1 - v) = \mathcal{I}(v)$. So we restrict to $\mu(A) \leq 1/2$, and $\mathcal{I} : [0, 1/2] \to \mathbb{R}_+$.

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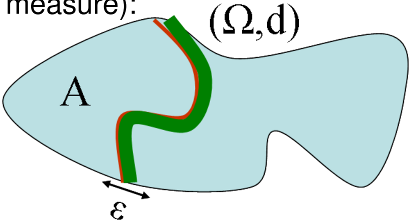

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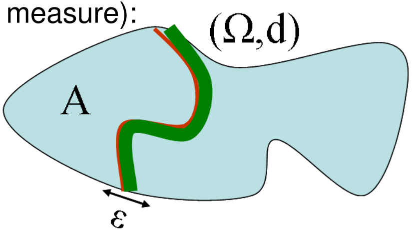

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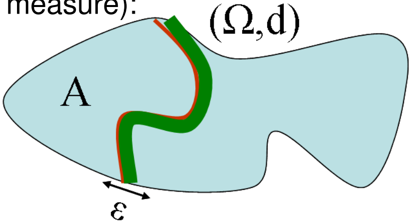

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Examples

Classical isoperimetric inequality in $(\mathbb{R}^n, |\cdot|, Leb)$:
Euclidean balls minimize boundary measure: $\mathcal{I}(v) = c_n v^{\frac{n-1}{n}}$.

Other known solutions to the isoperimetric problem:

- $(S^n, d, Haar)$ (Lévy,Schmidt) - geodesic balls.
- $(\mathbb{R}^n, |\cdot|, Gauss)$ (Sudakov–Tsirelson,Borell) - half spaces.
- Other: H^n, B^n, cones, variations, low-dim spaces.

Open: $([0,1]^3, |\cdot|, Leb)$, Flat Torus, Slabs, Heisenberg, etc.

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Why Important?

Isoperimetric Inqs $\Rightarrow$ Sobolev Inqs $\Rightarrow$ Concentration Inqs.

Concentration (large deviation) Inqs (when $\mu(\Omega) = 1$):

$$\forall r > 0 \quad \forall A \subset \Omega \quad \mu(A) \geq 1/2 \quad \Rightarrow \quad \mu(\Omega \setminus A^d_r) \leq \mathcal{K}(r) \; .$$

Examples: (Federer–Fleming, Maz'ya; Cheeger, Maz'ya; Gromov–V. Milman; Ledoux, Beckner; Herbst)

$\mathcal{I}(v) \geq c_n v^{\frac{n-1}{n}}$ (Euclidean)	$\Rightarrow$	$\| \|\nabla f\| \|_1 \geq c_n \|f\|_{\frac{n}{n-1}}$ (Sobolev)	$\Rightarrow$	...
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  Hierarchy: Isop Inqs $\Rightarrow$ Sobolev Inqs $\Rightarrow$ Conc Inqs.

  - Sobolev inqs (Buser, Ledoux, M.) - Dimension independent!
  - Concentration inqs (M.) - "Dim-indep. Hierarchy Reversal".

Assumption: $Ric_g + Hess_g \psi \geq -\kappa g$, two cases:

- $\kappa = 0$ - "convex case".
- $\kappa > 0$ - "semi-convex case".

Now survey some methods of obtaining isoperimetric inqs in these scenarios.

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  Hierarchy: Isop Inqs $\Rightarrow$ Sobolev Inqs $\Rightarrow$ Conc Inqs.

  - Sobolev inqs (Buser, Ledoux, M.) - Dimension independent!
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Assumption: $Ric_g + Hess_g \psi \geq -\kappa g$, two cases:

- $\kappa = 0$ - “convex case".
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Now survey some methods of obtaining isoperimetric inqs in these scenarios.

How to obtain isoperimetric inqs $\mathcal{I} = \mathcal{I}(\Omega, d, \mu) \geq ...$?

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Semi-group method: $Ric_g + Hess_g\psi \geq -\kappa g$

(p,q) Poincaré inequality: $\forall f \;\; D\|f - \mu(f)\|_{L^p(\mu)} \leq \||\nabla f|\|_{L^q(\mu)}$.

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Idea (Bakry–Émery–Ledoux): $P_t = \exp(t\Delta_\mu)$, curvature lower bound implies contractivity (only state $\kappa = 0$ case):

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$Ric_g + Hess_g \psi \geq 0$ - Hierarchy Reversal

Saw that $(1,\infty)$ Poincaré inq $\Rightarrow \;\; \mathcal{I}(v) \geq c\, Dv$.

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“weakest concentration implies linear isop. (hence exponential conc.)"

Thm (M. 08,09,10): If $\exists \lambda_0 \in (0,1/2)\; \exists r_0 > 0$ so that:

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$$\mathcal{K}(r) \leq \exp(-\alpha(r)) \quad \Rightarrow \quad \mathcal{I}(v) \geq \min(cv \frac{\log 1/v}{\alpha^{-1}(\log 1/v)}, c_\alpha) \quad \forall v \in [0,1/2] \ .$$

Remarks: dim-independent constants.

concentration implies back “right” isoperimetric inq.

$Ric_g + Hess_g\psi \geq 0$ - Hierarchy Reversal

Saw that $(1,\infty)$ Poincaré inq $\Rightarrow \;\; \mathcal{I}(v) \geq c\, Dv$.

$$\forall f \;\; D\|f - \mu(f)\|_{L^1(\mu)} \leq \||\nabla f|\|_{L^\infty(\mu)} \quad \text{“}\Leftrightarrow\text{''}$$

$$\forall \text{ 1-Lip functions } \mu(|f - \mu(f)| \geq r) \leq 1/(Dr) \quad \text{“}\Leftrightarrow\text{''}$$

$$\forall A \;\; \mu(A) \geq 1/2 \Rightarrow \mu(\Omega \setminus A^d_r) \leq 1/(Dr) \quad .$$

“weakest concentration implies linear isop. (hence exponential conc.)"

Thm (M. 08,09,10): If $\exists \lambda_0 \in (0,1/2)$ $\exists r_0 > 0$ so that:

$$\forall A \;\; \mu(A) \geq 1/2 \Rightarrow \mu(\Omega \setminus A^d_{r_0}) \leq \lambda_0 \; .$$

Then:

$$\mathcal{I}(v) \geq \frac{1 - 2\lambda_0}{r_0} v \quad \forall v \in [0,1/2] \; .$$

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"stronger than Gaussian conc. implies stronger than Gaussian isop."

Thm (M. 09): Assume that the following growth condition is satisfied:

$$\exists \delta_0 > 1/2 \quad \exists r_0 \geq 0 \quad \forall r \geq r_0 \quad \alpha(r) \geq \delta_0 \kappa r^2 .$$

Then:

$$\mathcal{K}(r) \leq \exp(-\alpha(r)) \quad \Rightarrow \quad \mathcal{I}(v) \geq \min(c_{\delta_0} v \frac{\log 1/v}{\alpha^{-1}(\log 1/v)}, c_{\alpha,\kappa}) \quad \forall v \in [0, 1/2] .$$

Remark: growth condition is necessary even in 1-D case (Chen–Wang 07 - $\delta_0 = 1/2$ is a sharp threshold).

Intuitively, $\alpha(r) > \frac{1}{2}\kappa r^2$ needed to compensate for $-\kappa$ curvature (second derivative).

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Proof: Geometric Method (Gromov)

Existence & regularity of isop minimizers (Geometric Measure Th.: Almgren, Bombieri, De Giorgi, Federer, Fleming, Giusti, Gonzalez–Massari–Tamanini, Morgan, Simons):

- $\partial A \cap \Omega = \partial_r A \cup \partial_s A$ regular and singular parts.
- 1st variation: $\partial_r A$ has constant (gen.) mean-curvature $H_\mu(A)$.
- Gromov: classical comparison theorems apply to ∂A even though not smooth.
- Generalized version of Heintze–Karcher (Morgan):

$$\mu(A^d_{r_1} \setminus A) \leq \mu^+(A) \int_0^{r_1} \exp((n-1)H_\mu(A)t + \kappa t^2/2)dt \; ,$$

- $\mu^+(A) = \mathcal{I}(v)$, and $(n-1)H_\mu(A)" = "\frac{d}{dv}\mathcal{I}(v)$.

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$$\text{conc.} \leq \quad \mu(A^d_{r_1} \setminus A) \leq \mu^+(A) \int_0^{r_1} \exp((n-1)H_\mu(A)t + \kappa t^2/2)dt \; ,$$

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Sample of Applications

Isoperimetric Inqs $\Rightarrow$ Sobolev Inqs $\Rightarrow$ Concentration Inqs.

$$\begin{cases} \text{Under } Ric_g + Hess_g\psi \geq -\kappa g \;,\;\; \kappa \geq 0 \\ \text{additional growth condition if } \kappa > 0 \end{cases}$$

- Recover all previous dim-dependent results:
  - $\int_\Omega \exp(\beta(d(x,x_0)))d\mu(x) < \infty \Rightarrow \mathcal{K}(r) \leq$ Markov.
  - diameter$(\Omega) < D \Rightarrow \mathcal{K}(r) = 0 \quad \forall r > D$.
  - Generalize everything to Riemannian setting.
  - Some generalizations require $CAT(\lambda)$ cond, $\lambda > 0$.
- Example: known concentration on B_p^n (Schechtman–Zinn 00) implies right isoperimetric inequality (Sodin 06).
- *Stability* of Isoperimetric and Sobolev inqs under measure perturbation w.r.t. $d_{TV}, W_1, H(\mu_2|\mu_1)$ (e.g. Klartag 08).
- Other: tightening weak inqs, equivalence of Transport-Entropy inqs, etc...

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Contraction Methods - I

Fact

If $T : (\Omega_1, d_1, \mu_1) \to (\Omega_2, d_2, \mu_2)$, $T_*(\mu_1) = \mu_2$ and $\|T\|_{Lip} \leq L$.
Then $\mathcal{I}(\Omega_2, d_2, \mu_2) \geq \frac{1}{L}\mathcal{I}(\Omega_1, d_1, \mu_1)$.

"Lipschitz maps transfer isoperimetric inequalities".

Application: f even log-concave on $\mathbb{R}^n$,
$K_f := \left\{x \in \mathbb{R}^n; \int_0^\infty r^{n-1} f(rx) dr \geq 1\right\}$ is convex (K. Ball).

Thm (Sodin, M. 07):
$\exists T : (\mathbb{R}^n, \|\cdot\|_{K_f}, \mu_1 = f\, Leb) \to (\mathbb{R}^n, \|\cdot\|_{K_f}, \mu_2 = Leb|_{K_f})$, s.t.
$\|T\|_{Lip} \leq C f(0)^{1/n}$.

Used to transfer isop inequalities on uniformly log-concave f (obtained by localization), to uniformly convex K_f.

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Contraction Methods - II

Thm (Caffarelli 00): If $\mu = Gauss_n$, $\nu = \mu \exp(-V)$, V convex.
Then $\exists T : (\mathbb{R}^n, |\cdot|, \mu) \to (\mathbb{R}^n, |\cdot|, \nu)$ with $\|T\|_{Lip} \leq 1$.
Method: optimal-transport T, employing regularity theory.

Thm (Kim, M. 10): $\mu = \exp(-U)$, $\nu = \mu \exp(-V)$,
$U = Q(x) + \sum_{i=1}^k \rho_i(|x_i|)$, $\rho_i^{(3)} \leq 0$, V convex + unconditional.
Then: $\exists T : (\mathbb{R}^n, |\cdot|, \mu) \to (\mathbb{R}^n, |\cdot|, \nu)$ with $\|T\|_{Lip} \leq 1$.

Method: construct explicit T, obtained as $T^{-1} = \lim_{t\to\infty} S_t$,
$\frac{d}{dt} S_t = -\nabla \log P_t(\exp(-V))(S_t)$, where $P_t(f_0)$ is solution to:
$\frac{d}{dt} f = \Delta_\mu f := \Delta f - \langle \nabla f, \nabla U \rangle$, $f|_{t=0} = f_0$.

Contraction property is reduced to showing:
Thm (Kim, M. 10): $P_t(\exp(-V))$ is log-concave $\forall t \geq 0$.

Proof: PDE methods and maximum principle (Korevaar, Caffarelli–Spruck, Kawohl, etc...) + Geometric Ideas.

Remark: when $\mu = Gauss_n$, P_t is Ornstein–Uhlenbeck semi-group, which preserves log-concavity by Prekopá–Leindler Thm.
This yields trivial proof of Caffarelli's Thm.

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Remark: when $\mu = Gauss_n$, P_t is Ornstein–Uhlenbeck semi-group, which preserves log-concavity by Prekopá–Leindler Thm.
This yields trivial proof of Caffarelli's Thm.

Contraction Methods - II

Thm (Caffarelli 00): If $\mu = Gauss_n$, $\nu = \mu \exp(-V)$, V convex.
Then $\exists T : (\mathbb{R}^n, |\cdot|, \mu) \to (\mathbb{R}^n, |\cdot|, \nu)$ with $\|T\|_{Lip} \leq 1$.
Method: optimal-transport T, employing regularity theory.

Thm (Kim, M. 10): $\mu = \exp(-U)$, $\nu = \mu \exp(-V)$,
$U = Q(x) + \sum_{i=1}^k \rho_i(|x_i|)$, $\rho_i^{(3)} \leq 0$, V convex + unconditional.
Then: $\exists T : (\mathbb{R}^n, |\cdot|, \mu) \to (\mathbb{R}^n, |\cdot|, \nu)$ with $\|T\|_{Lip} \leq 1$.

Method: construct explicit T, obtained as $T^{-1} = \lim_{t\to\infty} S_t$,
$\frac{d}{dt} S_t = -\nabla \log P_t(\exp(-V))(S_t)$, where $P_t(f_0)$ is solution to:
$\frac{d}{dt} f = \Delta_\mu f := \Delta f - \langle \nabla f, \nabla U \rangle$, $f|_{t=0} = f_0$.

Contraction property is reduced to showing:
Thm (Kim, M. 10): $P_t(\exp(-V))$ is log-concave $\forall t \geq 0$.

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