Mixed volumes of random convex sets

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Outline

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- Polytopes
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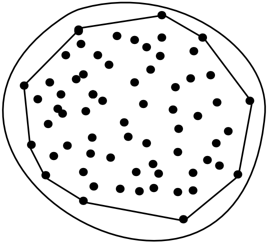
Random polytopes

Assume

- $K \subset \mathbb{R}^n$ - convex body, $\mathrm{vol}(K) = 1$.
- X - random vector distributed uniformly in K, i.e.,

$$\mathbb{P}(X \in A) = \mathrm{vol}(A \cap K)$$

  for measurable $A \subset \mathbb{R}^n$.
- $X_1, \dots, X_N$ independent copies of X,
- convex hull $= \mathrm{conv}\{X_1, \dots, X_N\}$.



Theorem (Groemer, 1974)

Assume

- $K \subset \mathbb{R}^n$ - *convex body,* $\mathrm{vol}\,(K) = 1$.
- $\overline{B_2^n}$ - *Euclidean ball in* $\mathbb{R}^n$ *with* $\mathrm{vol}\left(\overline{B_2^n}\right) = 1$.

Set

$$\mathbb{E}(K, N) := \int_K \dots \int_K \mathrm{vol}\,(\mathrm{conv}\,\{x_1, \dots, x_N\})\, dx_1 \dots dx_N.$$

Then

$$\mathbb{E}(K, N) \geq \mathbb{E}(\overline{B_2^n}, N).$$

Equality holds only for ellipsoids.

- Analogous result holds for $\mathrm{conv}\,\{\pm x_1, \dots, \pm x_N\}$.

Generalizations: [Pfiefer '90], [Giannopoulos-Tsolomitis '03], [Hartzoulaki-Paouris, '03], ...

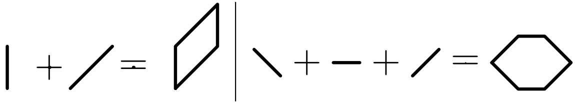
Zonotopes

A *zonotope* is a Minkowski sum of line segments:

$$\sum_{i=1}^{N} [-x_i, x_i],$$

where the $x_i \in \mathbb{R}^n$ and $[-x_i, x_i] := \{\lambda x_i : -1 \leq \lambda \leq 1\}$.

E.g.:



Theorem (Bourgain-Meyer-Milman-Pajor, 1988)

Let $K \subset \mathbb{R}^n$ *be a convex body* $\mathrm{vol}\,(K) = 1$, $p > 0$. *Set*

$$\mathcal{I}_p(K, N) := \int_K \cdots \int_K \mathrm{vol}\left(\sum_{i=1}^{N} [-x_i, x_i]\right)^p dx_1 \cdots dx_N.$$

Then

$$\mathcal{I}_p(K, N) \geq \mathcal{I}_p(\overline{B_2^n}, N).$$

Linear operator viewpoint

Assume

- $e_1, \ldots, e_N$ - standard unit vector basis in $\mathbb{R}^N$.
- $x_1, \ldots, x_N \in \mathbb{R}^n$.
- $T_N : \mathbb{R}^N \to \mathbb{R}^n$,

$$e_i \mapsto x_i, \quad i = 1, \ldots, N.$$

As a matrix,

$$T_N = [x_1 \cdots x_N].$$

Write

$$B_1^N = \mathrm{conv}\{\pm e_1, \ldots, \pm e_N\}, \quad B_\infty^N = [-1, 1]^N.$$

Then

1. $T_N B_1^N = \mathrm{conv}\{\pm x_1, \ldots, \pm x_N\}$.
2. $T_N B_\infty^N = \left\{ \sum_i b_i x_i : (b_i) \in B_\infty^N \right\} = \sum_{i=1}^{N} [-x_i, x_i]$.

Main result

Theorem (P., Paouris, 2010)

Assume

- $K \subset \mathbb{R}^n$ - *bounded, Borel measurable,* $\mathrm{vol}\,(K) = 1$.
- $T_N(x_1, \dots, x_N) : \mathbb{R}^N \to \mathbb{R}^n$, *defined by*

$$T_N e_i = x_i, \quad i = 1, \dots, N.$$

- $C \subset \mathbb{R}^N$ *compact, convex.*

Set

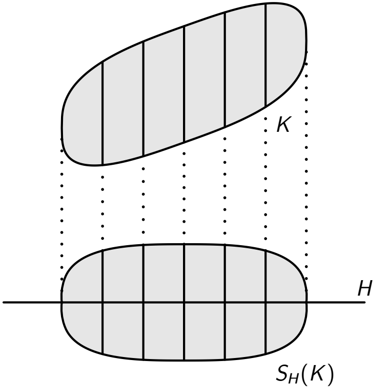
$$\mathbb{E}(K, N, C) := \int_K \dots \int_K \mathrm{vol}\,(T_N(x_1, \dots, x_N)C)\, dx_N \dots dx_1.$$

Then

$$\mathbb{E}(K, N, C) \geq \mathbb{E}(\overline{B_2^n}, N, C).$$

Elements of the proof

Common to all proofs: Steiner symmetrization



Sufficient:

$$\mathbb{E}(K, N, C) \geq \mathbb{E}(S_H(K), N, C).$$

Elements of the proof cont...

- Let H be a hyperplane.
- Assume $y_1, \dots, y_N \in H$ are fixed; $Y = (y_1, \dots, y_N)$.
- Fix $\theta \in S^{n-1} \cap H^\perp$

Let $T_N^Y : \mathbb{R}^N \to \mathbb{R}^n$ be the operator defined by

$$T_N^Y(t)e_i = y_i + t_i\theta \quad \text{for each } i = 1, \dots, N,$$

Lemma

Let $C \subset \mathbb{R}^N$ be a compact convex set. Set $d = \min(n, N, \dim C)$.
Let $F_Y : \mathbb{R}^N \to \mathbb{R}^+$ be defined by

$$F_Y(t) = \mathrm{vol}_d\left(T_N^Y(t)C\right).$$

Then F_Y is a convex function.

Generalizations ...

Recall

$$\mathbb{E}(K, N, C) := \int_K \dots \int_K \mathrm{vol}\,(T_N(x_1, \dots, x_N)C)\, dx_N \dots dx_1.$$

Similar results hold

1. with $\mathrm{vol}\,(\cdot)$ replaced by $f \circ V_k(\cdot)$ for
   - $V_k =$ intrinsic volumes
   - $f : \mathbb{R} \to \mathbb{R}^+$ any increasing function
2. limits of $(T_N C_N)_N$ in the Hausdorff metric.

Consider independent random vectors

$$X_1, X_2, \dots \in K \text{ and } Y_1, Y_2, \dots \in \overline{B_2^n}.$$

For $N = 1, 2, \dots$, set

$$T_N := [X_1 \cdots X_N] \text{ and } S_N = [Y_1 \cdots Y_N].$$

Corollary

Let $C_N \subset \mathbb{R}^N$, be compact, convex sets for $N = 1, 2, \dots$ Let $R > 0$. Assume that

$$T_N C_N \subset RB_2^n \;\; \textit{and } S_N C_N \subset RB_2^n$$

and (in the Hausdorff metric)

$$\mathcal{C}(K) := \lim_{N\to\infty} T_N C_N \; (a.s.) \textit{ and } \mathcal{C}(\overline{B_2^n}) := \lim_{N\to\infty} S_N C_N \; (a.s.).$$

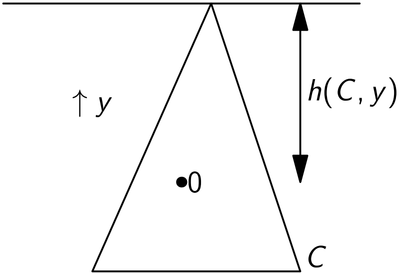
Then

$$\mathbb{E}\,\mathrm{vol}\,(\mathcal{C}(K)) \geq \mathbb{E}\,\mathrm{vol}\left(\mathcal{C}(\overline{B_2^n})\right).$$

Notation:

For a convex body $C \subset \mathbb{R}^n$, let $h(C, \cdot)$ be the support function, i.e.,

$$h(C, y) = \sup_{x \in C} \langle x, y \rangle \quad (y \in S^{n-1}).$$



Hausdorff metric: for convex bodies $C, D \subset \mathbb{R}^n$,

$$\delta^H(C, D) := \sup_{y \in S^{n-1}} |h(C, y) - h(D, y)|.$$

L_p-centroid bodies

Definition

Assume

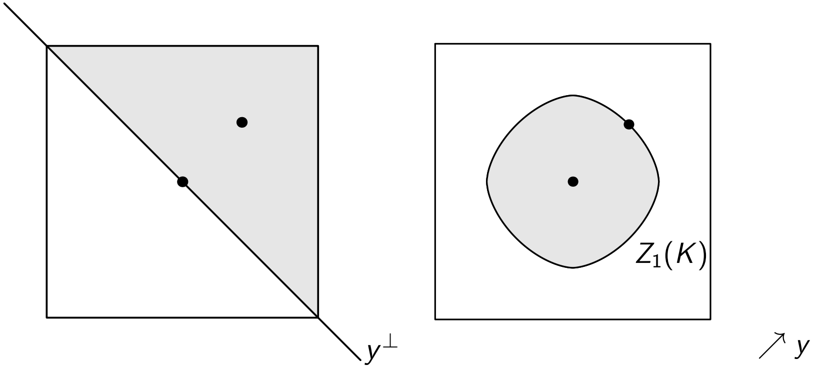
- $K \subset \mathbb{R}^n$ is bounded, Borel measurable.
- $\mathrm{vol}\,(K) = 1$.
- $p \geq 1$.

The L_p-centroid body $Z_p(K)$ is defined by its support function

$$h(Z_p(K), y) := \left(\int_K |\langle x, y \rangle|^p \, dx \right)^{1/p} \quad (y \in S^{n-1}).$$

Example: $K = [-\frac{1}{2}, \frac{1}{2}]^2$, $p = 1$.

$$h(Z_1(K), y) := \int_K |\langle x, y\rangle|\, dx \quad (y \in S^{n-1}).$$



Support function of $Z_p(K)$:

$$h(Z_p(K), y) := \left(\int_K |\langle x, y \rangle|^p \, dx \right)^{1/p} \quad (y \in S^{n-1}).$$

Theorem (Lutwak-Yang-Zhang, 2000)

Let $K \subset \mathbb{R}^n$ be compact. Assume

- $\mathrm{vol}\,(K) = 1$.
- *K is star-shaped (i.e. $\alpha K \subset K$ for each $\alpha \in [0, 1]$).*

Then

$$\mathrm{vol}\,(Z_p(K)) \geq \mathrm{vol}\,\left(Z_p(\overline{B_2^n})\right).$$

Equality holds only for ellipsoids.

A different approach

Theorem (P., Paouris, 2010)

Assume

- $K \subset \mathbb{R}^n$ - *bounded, Borel measurable.*
- $\mathrm{vol}\,(K) = 1$.
- $p \geq 1$.

Then

$$\mathrm{vol}\,(Z_p(K)) \geq \mathrm{vol}\left(Z_p(\overline{B_2^n})\right).$$

Proof: Take q with $1/p + 1/q = 1$ and set

$$B_q^N := \left\{ t \in \mathbb{R}^N : \sum_{i=1}^{N} |t_i|^q \leq 1 \right\}.$$

Apply $T_N = [X_1 \cdots X_N]$ to B_q^N:

In our set-up:

- $T_N = [X_1 \cdots X_N]$, $N = 1, 2, \ldots$

Support function of $T_N B_q^N$:

$$h(T_N B_q^N, y)^p = \sum_{i=1}^{N} |\langle X_i, y\rangle|^p \,.$$

Support function of $Z_p(K)$:

$$h(Z_p(K), y)^p := \int_K |\langle x, y\rangle|^p \, dx$$

So, in the Hausdorff metric,

$$Z_p(K) = \lim_{N\to\infty} N^{-1/p} T_N B_q^N \text{ (a.s.)}$$

Orlicz centroid bodies

Theorem (Lutwak-Yang-Zhang, 2010)

Assume

- $K \subset \mathbb{R}^n$ - *convex body with* $\mathrm{vol}\,(K) = 1$.
- $\psi : [0, \infty) \to [0, \infty)$ - *convex,* $\nearrow$, $\psi(0) = 0$.

Define $Z_\psi(K)$ *by its support function*

$$h(Z_\psi(K), y) := \inf \left\{ \lambda > 0 : \int_K \psi \left(\frac{|\langle x, y \rangle|}{\lambda} \right) dx \leq 1 \right\}.$$

Then

$$\mathrm{vol}\,(Z_\psi(K)) \geq \mathrm{vol}\,\left(Z_\psi(\overline{B_2^n})\right). \tag{1}$$

Equality holds only for ellipsoids.

Remark: For $\psi(s) := s^p$, $Z_\psi(K) = Z_p(K)$.

Orlicz-centroid body inequality in our framework

As before:

- $\psi : [0, \infty) \to [0, \infty)$ - convex, $\nearrow$, $\psi(0) = 0$.

Orlicz norm on $\mathbb{R}^N$

$$\|t\|_{\psi/N} := \inf \left\{ \lambda > 0 : \frac{1}{N} \sum_{i=1}^{N} \psi\left(\frac{|t_i|}{\lambda}\right) \leq 1 \right\}.$$

Consider

$$B_{\psi/N} := \{t \in \mathbb{R}^N : \|t\|_{\psi/N} \leq 1\}.$$

As before,

- $X_1, \dots, X_N$ independent random vectors in K.
- $T_N = [X_1 \cdots X_N]$.

Then (in the Hausdorff metric)

$$Z_\psi(K) = \lim_{N\to\infty} T_N B^\circ_{\psi/N} \quad \text{(a.s.)}$$

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