Symmetry of mapping cones with applications in entanglement theory

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"Quantum states"

State spaces

$\mathcal{K}$, $\mathcal{H}$ - two finite-dimensional Hilbert spaces, thus equivalent to $\mathbb{C}^m$, $\mathbb{C}^n$ for some $m, n \in \mathbb{N}$. We fix orthonormal bases $\{f_k\}_{k=1}^m$, $\{e_i\}_{i=1}^n$ of $\mathcal{K}$, $\mathcal{H}$, resp.

Operators

$\mathcal{B}(\mathcal{K})$, $\mathcal{B}(\mathcal{H})$ - linear (bounded) operators on $\mathcal{K}$, $\mathcal{H}$, resp. $\mathcal{B}(\mathcal{K})^+$, $\mathcal{B}(\mathcal{H})^+$ - positive elements of $\mathcal{B}(\mathcal{K})$, $\mathcal{B}(\mathcal{H})$.

$$\langle A, B\rangle' := \mathrm{Tr}\,(A^* B)\,. \tag{1}$$

Hilbert-Schmidt product in $\mathcal{B}(\mathcal{K})$ or $\mathcal{B}(\mathcal{H})$. Canonical orthonormal bases of $\mathcal{B}(\mathcal{K})$, $\mathcal{B}(\mathcal{H})$ - $\{f_{kl}\}_{k,l=1}^m$, $\{e_{ij}\}_{i,j=1}^n$, resp.

“Quantum operations”

Maps

$\mathcal{B}(\mathcal{B}(\mathcal{K}), \mathcal{B}(\mathcal{H}))$ - linear (bounded) operators from $\mathcal{B}(\mathcal{K})$ into $\mathcal{B}(\mathcal{H})$.

$$\langle \Phi, \Psi \rangle'' := \sum_{k,l=1} \langle \Phi(f_{kl}), \Psi(f_{kl}) \rangle' . \qquad (2)$$

Hilbert-Schmidt type product in $\mathcal{B}(\mathcal{K})$ or $\mathcal{B}(\mathcal{H})$.

Property

$$\langle \Phi, \Psi \rangle'' = \langle \mathrm{id}, \Phi^* \circ \Psi \rangle'' = \langle \mathrm{id}, \Psi \circ \Phi^* \rangle'' \qquad (3)$$

Property

$$\langle \alpha \circ \Phi \circ \beta, \Psi \rangle'' = \langle \Phi, \alpha^* \circ \Psi \circ \beta^* \rangle'' \qquad (4)$$

for all $\alpha \in \mathcal{B}(\mathcal{B}(\mathcal{H}))$, $\beta \in \mathcal{B}(\mathcal{B}(\mathcal{K}))$.

Jamiołkowski-Choi isomorphism

The isomorphism

$$J : \Phi \to C_\Phi := \sum_{k,l=1}^{m} f_{kl} \otimes \Phi\left(f_{kl}\right). \tag{5}$$

J goes from $\mathcal{B}\left(\mathcal{B}\left(\mathcal{K}\right), \mathcal{B}\left(\mathcal{H}\right)\right)$ into $\mathcal{B}\left(\mathcal{K} \otimes \mathcal{H}\right)$. The operator C_Φ is usually referred as the *Choi matrix* of Φ.

Property

J is an isometry. One has

$$\langle \Phi, \Psi \rangle'' = \langle C_\Phi, C_\Psi \rangle' \tag{6}$$

for all $\Phi, \Psi \in \mathcal{B}\left(\mathcal{B}\left(\mathcal{K}\right), \mathcal{B}\left(\mathcal{H}\right)\right)$.

Classes of maps. Mapping cones

Positivity, complete positivity, etc.

- $\mathcal{P}$ - positive maps, $\Phi\left(\mathcal{B}\left(\mathcal{K}\right)^{+}\right) \subset \mathcal{B}\left(\mathcal{H}\right)^{+}$ for $\Phi \in \mathcal{P}$
- k-$\mathcal{P}$ - k-positive maps, $\Phi \otimes \mathrm{id}_{M_k(\mathbb{C})} \in \mathcal{P}$
- $\mathcal{CP}$ - completely positive maps, $\mathcal{CP} \subset k\text{-}\mathcal{P}\,\forall_k$, $\Phi = \sum_i \mathrm{Ad}_{V_i}$ for all $\Phi \in \mathcal{CP}$
- k-$\mathcal{SP}$ - k-superpositive maps, $\Phi = \sum_i \mathrm{Ad}_{V_i}$, $\mathrm{rk}\, V_i \leqslant k$ for all $\Phi \in k\text{-}\mathcal{SP}$
- $\mathcal{SP}$ - superpositive maps, $\Phi = \sum_i \mathrm{Ad}_{V_i}$, $\mathrm{rk}\, V_i = 1$ for all $\Phi \in \mathcal{SP}$

Mapping cones

All the cones $\mathcal{P}$, k-$\mathcal{P}$, $\mathcal{CP}$, k-$\mathcal{SP}$, $\mathcal{SP}$ are *mapping cones*, i.e. they are closed subcones of $\mathcal{P}$ and $\Phi \in \mathcal{C} \Rightarrow \Upsilon \circ \Phi \circ \Omega \in \mathcal{C}$ for $\Upsilon, \Omega \in \mathcal{CP}$ and $\mathcal{C} = \mathcal{P}, k\text{-}\mathcal{P}, \mathcal{CP}, k\text{-}\mathcal{SP}, \mathcal{SP}$

Duality between cones

Hermiticity-preserving maps

Consider the $\mathbb{R}$-linear subspace $\mathcal{HP} \subset \mathcal{B}(\mathcal{B}(\mathcal{K}), \mathcal{B}(\mathcal{H}))$, consisting of Hermiticity-preserving maps, i.e. maps Φ s.t. $\Phi(A^*) = \Phi(A)^*$. Since $\mathcal{P} \subset \mathcal{HP}$, $\mathcal{C} \subset \mathcal{HP}$ for all mapping cones $\mathcal{C}$.

Duality between cones

For a cone $\mathcal{C} \subset \mathcal{HP}(\mathcal{B}(\mathcal{K}), \mathcal{B}(\mathcal{H}))$, its *dual* is defined as

$$\mathcal{C}^\circ := \left\{ \Psi \in \mathcal{HP}(\mathcal{B}(\mathcal{K}), \mathcal{B}(\mathcal{H})) \middle| \langle \Psi, \Phi \rangle'' \geqslant 0 \, \forall_{\Phi \in \mathcal{C}} \right\}. \quad (7)$$

Proposition

Let $\mathcal{C} \subset \mathcal{P}(\mathcal{B}(\mathcal{K}), \mathcal{B}(\mathcal{H}))$ be an arbitrary mapping cone. Then $\mathcal{C}^\circ$, defined as in (7), is a mapping cone as well.

Proof of the Proposition (sketchy)

$\mathcal{C}^\circ$ has the mapping cone symmetry

Take $\Phi \in \mathcal{C}$, $\Upsilon, \Omega \in \mathcal{CP}$. Since $\mathcal{C}$ is a mapping cone, $\Upsilon^* \circ \Phi \circ \Omega^* \in \mathcal{C}$, so $\langle \Upsilon^* \circ \Phi \circ \Omega^*, \Psi \rangle'' \geqslant 0$. But $\langle \Upsilon^* \circ \Phi \circ \Omega^*, \Psi \rangle'' = \langle \Phi, \Upsilon \circ \Psi \circ \Omega \rangle''$ by the second property of $\langle ., . \rangle''$. Hence

$$\langle \Phi, \Upsilon \circ \Psi \circ \Omega \rangle'' \geqslant 0 \, \forall_{\Phi \in \mathcal{C}} \forall_{\Upsilon, \Psi \in \mathcal{CP}}, \tag{8}$$

which means that $\Upsilon \circ \Psi \circ \Omega \in \mathcal{C}^*$.

$\mathcal{C}^\circ \subset \mathcal{P}(\mathcal{B}(\mathcal{K}), \mathcal{B}(\mathcal{H}))$

Can be proved using the fact that $\mathrm{Ad}_V \in \mathcal{C}$ for all $V \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, no matter what the mapping cone $\mathcal{C}$ is.

Main characterization result

Theorem

Let $\mathcal{C} \subset \mathcal{P}(\mathcal{B}(\mathcal{K}), \mathcal{B}(\mathcal{H}))$ *be a convex mapping cone. The following conditions are equivalent,*

1. $\Phi \in \mathcal{C}$,
2. $\Psi^* \circ \Phi \in \mathcal{CP}(\mathcal{B}(\mathcal{K}))$ *for all* $\Psi \in \mathcal{C}^{\circ}$,
3. $\Phi \circ \Psi^* \in \mathcal{CP}(\mathcal{B}(\mathcal{H}))$ *for all* $\Psi \in \mathcal{C}^{\circ}$.

Remark

A similar theorem does not hold for a general convex cone $\mathcal{C} \subset \mathcal{HP}(\mathcal{B}(\mathcal{K}), \mathcal{B}(\mathcal{H}))$.

Proof of the main result (sketchy)

Proof of 1 $\Rightarrow$ 2

By the Proposition, $\mathcal{C}^\circ$ is a mapping cone. Thus $\Psi \circ \mathrm{Ad}_V \in \mathcal{C}^\circ$ and $\langle \Psi \circ \mathrm{Ad}_V, \Phi \rangle'' \geqslant 0 \, \forall_{\Phi \in \mathcal{C}}$. But $\langle \Psi \circ \mathrm{Ad}_V, \Phi \rangle'' = \langle \mathrm{id}, \mathrm{Ad}_{V^*} \circ \Psi^* \circ \Phi \rangle'' = \langle \mathrm{Ad}_V, \Psi^* \circ \Phi \rangle''$, which equals $\langle C_{\mathrm{Ad}_V}, C_{\Psi^* \circ \Phi} \rangle' = \langle |v\rangle \langle v| \, , C_{\Psi^* \circ \Phi} \rangle' = \langle v, C_{\Psi^* \circ \Phi} (v) \rangle$, where $v \in \mathcal{K} \otimes \mathcal{H}$ can be arbitrary if there no constraints on V. Thus

$$\langle v, C_{\Psi^* \circ \Phi} (v) \rangle \geqslant 0 \, \forall_{v \in \mathcal{K} \otimes \mathcal{H}} \forall_{\Phi \in \mathcal{C}}. \tag{9}$$

By the Choi theorem, $\Psi^* \circ \Phi \in \mathcal{CP}$ for all $\Phi \in \mathcal{C}$.

Proof of 2 $\Rightarrow$ 1

$\Psi^* \circ \Phi \in \mathcal{CP}$ implies $\langle \Psi^* \circ \Phi, \mathrm{id} \rangle'' \geqslant 0$. But $\langle \Psi^* \circ \Phi, \mathrm{id} \rangle'' = \langle \Phi, \Psi \rangle''$. Thus we get $\langle \Phi, \Psi \rangle'' \geqslant 0 \, \forall_{\Psi \in \mathcal{C}^\circ}$, $\Phi \in \mathcal{C}^{\circ\circ} = \mathcal{C}$.

A version for k-positive maps

Theorem

Denote with $\Pi_k(\mathcal{K})$ and $\Pi_k(\mathcal{H})$ the sets of k-dimensional projections in $\mathcal{K}$ and $\mathcal{H}$, resp. The following conditons are equivalent

1. $\Phi \in k\text{-}\mathcal{P}(\mathcal{B}(\mathcal{K}), \mathcal{B}(\mathcal{H}))$,
2. $\mathrm{Ad}_E \circ \Phi \in \mathcal{CP}(\mathcal{B}(\mathcal{K}), \mathcal{B}(\mathcal{H}))$ *for all* $E \in \Pi_k(\mathcal{H})$,
3. $\Phi \circ \mathrm{Ad}_F \in \mathcal{CP}(\mathcal{B}(\mathcal{K}), \mathcal{B}(\mathcal{H}))$ *for all* $F \in \Pi_k(\mathcal{K})$,
4. $\mathrm{Ad}_E \circ \Phi \circ \mathrm{Ad}_F \in \mathcal{CP}(\mathcal{B}(\mathcal{K}), \mathcal{B}(\mathcal{H}))$ *for all* $E \in \Pi_k(\mathcal{H})$, $F \in \Pi_k(\mathcal{K})$.

An application

Consider $\Phi_\lambda := \mathrm{Tr} - \lambda \mathrm{Ad}_V$, $\mathrm{Ad}_E \circ \Phi_\lambda = E(\mathrm{Tr} - \lambda \mathrm{Ad}_{EV})E$. The same form, $V \to EV \Rightarrow$ maximize $\mathcal{CP}$ condition over E

Norms with respect to a mapping cone

Definition

Given a mapping cone $\mathcal{C} \in \mathcal{B}(\mathcal{B}(\mathcal{K}), \mathcal{B}(\mathcal{H}))$, let us define $\mathcal{S}_{\mathcal{C}} := \{C_\Phi | \Phi \in \mathcal{C}^\circ, \operatorname{Tr} C_\Phi = 1\}$. For example, $\mathcal{S}_{\mathcal{P}}$ equals the set of separable states. It is now possible to define a norm in $\mathcal{B}(\mathcal{K} \otimes \mathcal{H})$ with respect to $\mathcal{S}_{\mathcal{C}}$,

$$\|A\|_{\mathcal{S}_{\mathcal{C}}} := \sup_{B \in \mathcal{S}_{\mathcal{C}}} \left| \langle A, B \rangle' \right| = \sup_{B \in \mathcal{S}_{\mathcal{C}}} \left| \operatorname{Tr}(AB^*) \right| . \tag{10}$$

A corresponding norm in $\mathcal{B}(\mathcal{B}(\mathcal{K}), \mathcal{B}(\mathcal{H}))$,

$$\|\Psi\|_{\mathcal{C}} := \sup_{\Phi \in J^{-1}(\mathcal{S}_{\mathcal{C}})} \left| \langle \Psi, \Phi \rangle'' \right| = \sup_{\Phi \in J^{-1}(\mathcal{S}_{\mathcal{C}})} \left| \langle C_\Psi, C_\Phi \rangle' \right| , \tag{11}$$

where the latter equalities follow because $J : \Phi \mapsto C_\Phi$ is an isometry.

A 2-positivity criterion for $\Phi \otimes \Phi$

A norm condition for maps in a mapping cone

Let $C_{\Psi_\lambda} = \mathbb{1} - \lambda A$, $A \geqslant 0$. Then $\Phi_\lambda \in \mathcal{C} \Leftrightarrow \lambda \|A\|_{\mathcal{S}_\mathcal{C}} \leqslant 1$

An application

For $\mathcal{C} = k\text{-}\mathcal{P}, \|.\|_{\mathcal{S}_{k\text{-}\mathcal{P}}} = \|.\|_{S(k)}$, k-th Schmidt norm defined by Johnston&Kribs. Using their upper bound for $\|A\|_{S(k)}$ in terms of eigendecomposition of A and other facts, one gets

$$\|\mathbb{1} \otimes |v\rangle\langle v| + |v\rangle\langle v| \otimes \mathbb{1}\|_{S(k)} \leqslant 4\|v\|_{s(1)}^2, \tag{12}$$

where $\|v\|_{s(1)}^2$ is the square of the greatest singular value of the coefficient matrix for v in the basis $\{f_k \otimes e_i\}_{k,i}$ of $\mathcal{K} \otimes \mathcal{H}$.

Consequently, $(\mathbb{1} - \lambda |v\rangle\langle v|)^{\otimes 2}$, corresponds to a 2-positive map if

$$4\lambda \|v\|_{s(1)}^2 \leqslant 1$$

Conclusion

Summary

- The class of mapping cones is closed under the operation $\mathcal{C} \to \mathcal{C}^\circ$
- A mapping cone can be characterized by saying that the products of its elements with the conjugates of the elements in the convex dual cone are $\mathcal{CP}$ maps
- Several mapping cones have applications in the theory of quantum information, where the above characterization can be used

Question

How large is the class of mapping cones?

Related papers

- Ł. Skowronek, *Theory of Mapping Cones in the Finite-Dimensional Case*, Preprint 2010
- Ł. Skowronek, E. Størmer, *Choi matrices, norms and entanglement associated with positive maps of matrix algebras*, Preprint 2010

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