INTRINSIC VOLUMES

Rick Vitale
University of Connecticut

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Case Western Reserve University
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Steiner formula for $K \subset \mathbb{R}^d$

$$\mathrm{vol}_d(K + \lambda B) = \sum_{j=0}^{d} \mathrm{vol}_j(B_j)\lambda^j \, V_{d-j}(K)$$

$V_j(K)$: intrinsic volumes

Proof Take *iid* isotropic line segments $L_1, L_2, \ldots, L_n$, such that $\mathrm{E}L_1 = B_d$. By the LLN for random convex bodies: for large n,

$$B_d \approx (1/n)\,(L_1 + L_2 + \cdots + L_n)$$

so

$$\mathrm{vol}_d\,(K + \lambda B_d) \approx \mathrm{vol}_d\,[K + (\lambda/n)\,(L_1 + L_2 + \cdots + L_n)]\,.$$

For one line segment (i.e., $n = 1$):

$$\mathrm{vol}_d(K + (\lambda/n)L_1) = \mathrm{vol}_d(K) + (\lambda/n)|L_1| \cdot \mathrm{vol}_{d-1}(\Pi_{L_1^\perp} K).$$

By induction,

$$\mathrm{vol}_d\left[K + (\lambda/n)\left(L_1 + L_2 + \cdots + L_n\right)\right] =$$

$$\sum_{\substack{S \subseteq \{1,2,\ldots,n\} \\ 0 \le |S| \le d}} (\lambda/n)^{|S|}\, \mathrm{vol}_{|S|}\left(L_S\right) \mathrm{vol}_{d-|S|}\left(\Pi_{L_S^\perp} K\right),$$

where $L_S = \sum_{i \in S} L_i$. Equivalently,

$$\mathrm{vol}_d\left[K + (\lambda/n)\left(L_1 + L_2 + \cdots + L_n\right)\right] = \sum_{j=0}^{d} \frac{\binom{n}{j}}{n^j} \lambda^j U_{jn}.$$

$$U_{jn} = \frac{1}{\binom{n}{j}} \sum_{|S|=j} \mathrm{vol}_j\left(L_S\right) \mathrm{vol}_{d-j}\left(\Pi_{L_S^\perp} K\right) \qquad \text{(U-statistic)}\ .$$

$$
\begin{aligned}
\mathrm{vol}_d\left(K+\lambda B_d\right) &= \lim_n \mathrm{vol}_d\left[K+(\lambda/n)\left(L_1+L_2+\cdots+L_n\right)\right] \\
&= \sum_{j=0}^{d} \mathrm{vol}_j(B_j)\lambda^j\ V_{d-j}(K),
\end{aligned}
$$

where

$$
V_j(K) = \binom{d}{j}\frac{\mathrm{vol}_d(B_d)}{\mathrm{vol}_j(B_j)\,\mathrm{vol}_{d-j}(B_{d-j})}\mathrm{E}\,\mathrm{vol}_j\left(\Pi_j K\right).
$$

Key property: For an independent, random orthogonal O,

$$\Pi_j O \stackrel{\mathrm{d}}{=} \Pi_j.$$

But also

$$Z_{[j,d]} O \stackrel{\mathrm{d}}{=} Z_{[j,d]},$$

where $Z_{[j,d]}$,is a $j \times d$ matrix of independent $N(0,1)$ variables.

Alternate representation (V, 2008):

$$V_j(K) = \frac{(2\pi)^{j/2}\,\mathsf{E}\,\mathrm{vol}_j\left(Z_{[j,d]}K\right)}{j!\ \mathrm{vol}_j(B_j)} \quad .$$

Commonly Encountered Intrinsic Volumes

$$
\begin{aligned}
V_0(K) &= 1 \\
V_1(K) &= \text{intrinsic width} \\
&= \sqrt{2\pi}\,\mathsf{E}h_K(Z) = \sqrt{2\pi}\,\mathsf{E}\sup_{t\in K} X_t \\
&\vdots \\
V_{d-1}(K) &= 1/2 \cdot \text{surface area of } K \\
V_d(K) &= d\text{-dimensional volume of } K \\
V_j(K) &= 0 \quad \text{for} \quad j > d
\end{aligned}
$$

$$
V_j([a_1, b_1] \times \cdots \times [a_n, b_n]) = \sum_{i_1 < i_2 < \cdots < i_j} (b_{i_1} - a_{i_1}) \cdots (b_{i_j} - a_{i_j})
$$

$$
V_1(B_d) \sim \sqrt{2\pi d}
$$

Alexandrov–Fenchel Inequalities for Intrinsic Volumes:

For a fixed K, $\{j! V_j(K)\}_{j=0}^{\infty}$ is a log-concave sequence.

- Beckenbach and Bellman, *An Introduction to Inequalities*, vol. 2 (!) .

A Curious Sharpening

Recall the planar isoperimetric inequality: for any convex body K with area $A(K)$ and perimeter $L(K)$

$$4\pi \cdot A(K) \leq L^2(K) \, . \tag{1}$$

Consider now a triangle K, a 2×2 matrix M of independent $N(0,1)$ variables, and the image body MK. Insert this into (1) and take expectations:

$$4\pi \cdot \mathsf{E}\left[A(MK)\right] \leq \mathsf{E}\left[L^2(MK)\right] \, . \tag{2}$$

However, it is the case that a stronger inequality holds:

$$4\pi \cdot \mathsf{E}\left[A(MK)\right] \leq \left[\mathsf{E} L(MK)\right]^2 \quad . \tag{3}$$

That is, the *realization-wise* bound (2) yields to the sharpened form (3).

Conjecture: The Alexandrov-Fenchel inequalities can be regarded *in general* as sharpenings of realization-wise geometric bounds for random convex bodies.

Extension of intrinsic volumes to convex bodies in ℓ_2

$$\begin{aligned} \mathcal{K}_{\mathsf{d}} &= \text{convex bodies in } I\!R^d \\ \mathcal{K} &= \text{convex bodies in } \ell_2 \\ \mathcal{K}_{\mathsf{FD}} &= \text{finite–dimensional convex bodies in } \ell_2. \end{aligned}$$

For arbitrary $K \in \mathcal{K}$, define

$$V_j(K) \;=\; \sup\left\{ V_j(\widehat{K}) \,:\, \widehat{K} \subseteq K,\;\; \widehat{K} \in \mathcal{K}_{\mathsf{FD}} \right\}$$

$$\mathcal{K}_{\mathsf{GB}} \;=\; \{K \in \mathcal{K} : V_1(K) < \infty\}\,.$$

- $\mathcal{K}_{\mathsf{FD}} \subset \mathcal{K}_{\mathsf{GB}} \subset \mathcal{K}$
- $K \in \mathcal{K}_{\mathsf{GB}} \implies V_j(K) < \infty, \quad j = 2, 3, \ldots$

- $K \in \mathcal{K}_{\text{GB}} \implies V_j(K) = \frac{(2\pi)^{j/2}\,\mathrm{E}\,\mathrm{vol}_j\left(Z_{[j,\infty]}K\right)}{j!\;\mathrm{vol}_j(B_j)}$, where $Z_{[j,\infty]}$ is a $j \times \infty$ matrix of independent $N(0,1)$ variables.

Gaussian processes – isonormal indexing

$Z = (Z_1, Z_2, \ldots)$, independent $N(0,1)$ random variables.

$K \subset \ell_2$

$t \in K, \quad t \mapsto X_t = <t, Z> = \sum_{i=1}^{\infty} t_i Z_i \sim N(0, \|t\|^2)$

$\{X_t, t \in K\}$: isonormally indexed Gaussian process.

An equivalent representation for intrinsic volumes (Tsirel'son, 1985)

$$V_j(K) = \frac{(2\pi)^{j/2}\,\mathrm{E}\,\mathrm{vol}_j\left(\{(X_t^1, X_t^2, \ldots, X_t^j), t \in K\}\right)}{j!\;\mathrm{vol}_j(B_j)}.$$

Theorem $K \in \mathcal{K}_{\text{GB}} \iff \{X_t, t \in K\}$ is an almost–surely bounded Gaussian process. That is $P(\sup_{t \in K_0} X_t < \infty) = 1$ for any denumerable subset $K_0 \subset K$.

A Classification: Suppose that $\{a_n\}$ is a decreasing sequence of positive constants and that $\{e_n, \ \ n = 1, 2, \ldots\}$ is an orthonormal sequence.

Set $K = \overline{\text{conv}}\{a_n \, e_n \, , \ \ n = 1, 2, \ldots\}$. Then

$$\begin{aligned} K \in \mathcal{K} \quad &\iff \quad a_n \downarrow 0, \\ K \in \mathcal{K}_{\text{FD}} \quad &\iff \quad a_n = 0 \ \text{ eventually}, \\ K \in \mathcal{K}_{\text{GB}} \quad &\iff \quad a_n = \mathcal{O}\left[(\log n)^{-1/2}\right]. \end{aligned}$$

Example   Brownian Motion Body (BMB)

$$f : [0,1] \to \mathcal{H} \text{ (Hilbert space)}$$

- $0 \le x_1 \le x_2 \le x_3 \le x_4 \le 1 \implies$

$$[f(x_2) - f(x_1)] \perp [f(x_4) - f(x_3)]$$

- $\|f(x_2) - f(x_1)\|^2 = |x_2 - x_1|$ for all $0 \le x_1 \le x_2 \le 1$.

Call $\overline{\text{conv}}\{f([0,1])\}$ a Brownian Motion Body (BMB).

A realization in $L^2[0,1]$: $\{g : [0,1] \to \mathbb{R}^1 \mid 0 \le g \le 1,\ g \uparrow\}$.

Theorem [F. Gao and V, 2001]

$$V_j(\text{BMB}) = \frac{\text{vol}_j(B_j)}{j!} \qquad j = 1, 2, \ldots$$

Singularities

It is possible to have GB bodies $K_n \to K$, but $V_j(K_n) \not\to V_j(K)$, in particular,

$$K_n \downarrow \{p\}, \ \text{but} \ V_1(K_n) \not\downarrow V_1(\{p\}) = 0.$$

Definition $t^* \in K \in \mathcal{K}_{\mathrm{GB}}$ is a singularity of K if, as $\varepsilon \downarrow 0$,

$$V_1(K \cap B(t^*, \varepsilon)) \not\downarrow 0.$$

Definition $\mathcal{K}_{\mathrm{GC}} = \{K \in \mathcal{K}_{\mathrm{GB}} \ : \ K \text{ has no singularities}\}.$

$$\mathcal{K}_{\mathrm{FD}} \subset \mathcal{K}_{\mathrm{GC}} \subset \mathcal{K}_{\mathrm{GB}} \subset \mathcal{K}$$

Theorem $K \in \mathcal{K}_{\mathsf{GC}} \iff \{X_t, t \in K\}$ is an almost–surely continuous Gaussian process. That is $t_n \to t \implies X_{t_n} \to X_t$ almost–surely.

A Classification (continued)

$$K = \overline{\mathrm{conv}}\{a_n\, e_n, \;\; n = 1, 2, \ldots\} \in \mathcal{K}_{\mathsf{GC}} \iff a_n = o\left[(\log n)^{-1/2}\right].$$

Itô-Nisio Theory

Theorem [Itô-Nisio, 1969] Suppose that $t^* \in K \in \mathcal{K}_{\mathsf{GB}}$. Then

$$0 \le 2 \cdot \mathrm{osc}(t^*) = \lim_{\varepsilon \downarrow 0} \left[\sup_{t \in K \cap B(t^*, \varepsilon)} X_t - \inf_{t \in K \cap B(t^*, \varepsilon)} X_t \right]$$

is (a.s.) constant. Further, $\mathrm{osc}(t^*) > 0 \Leftrightarrow t^*$ is a singularity of K.

$\mathrm{osc}(t^*)$: *oscillation* of X at t^*

$$\mathrm{osc}(t^*) \overset{\text{a.s.}}{=} \frac{1}{\sqrt{2\pi}} \lim_{\varepsilon \downarrow 0} V_1(K \cap B(t^*, \varepsilon)).$$

Theorem [V, 2001] Suppose that $\operatorname{osc}(t^*) > 0$. Then

- For each j,
$$\lim_{\varepsilon \downarrow 0} V_j(K \cap B(t^*, \varepsilon)) > 0. \tag{4}$$

- $K \cap B(t^*, 0+) \approx \frac{1}{\sqrt{2\pi \cdot \infty}} B_\infty(t^*, \operatorname{osc}(t^*))$, in the sense that, for each j, the limit in (4) is equal to
$$\lim_{d \to \infty} V_j \left(\frac{1}{\sqrt{2\pi d}} \, B_d(t^*, \operatorname{osc}(t^*) \right),$$
both equating to $\dfrac{\operatorname{osc}^j(t^*)}{j!}$.

- Define $\operatorname{osc}(K) = \sup\{\mathrm{E}\operatorname{osc}(t^*) : t^* \in K\}$. Then
  - $K \in \mathcal{K}_{\mathsf{GC}} \iff \operatorname{osc}(K) = 0$.
  - $\operatorname{osc}(K) = \lim_{j\to\infty} \frac{(j+1)V_{j+1}(K)}{V_j(K)}$.
  - Open question: if $\operatorname{osc}(K) = 0$, how slowly can the previous sequence of ratios make the approach?

The Wills Functional

$$
\begin{aligned}
W(K) &= \int_{R^d} \frac{1}{(2\pi)^{d/2}}\, e^{-\frac{1}{2}\delta^2(x,K)}\, \mathrm{d}x \quad \text{(Wills, 1973)} \\
&= \frac{1}{(2\pi)^{d/2}}\, \mathrm{E}\ \mathrm{vol}(K + \Lambda B), \\
&\qquad\qquad \text{where } \ f_\Lambda(\lambda) \,=\, 1(\lambda \geq 0)\lambda e^{-(1/2)\lambda^2}. \\
&= \sum_{j=0}^{d} \frac{1}{(2\pi)^{j/2}} V_j(K). \\
&= \mathrm{E} e^{\sup_{t \in K}\left[X_t - \frac{1}{2}\sigma_t^2\right]}.
\end{aligned}
$$

The Alexandrov-Fenchel inequality implies

$$V_j(K) \le \frac{1}{j!} V_1^j(K) = \frac{1}{j!} \left[\sqrt{2\pi}\, \mathsf{E} \sup_{t\in K} X_t \right]^j .$$

$$W(rK) = \mathsf{E} e^{\sup_{t\in K}\left[rX_t - \frac{1}{2} r^2 \sigma_t^2\right]} \le e^{r\, \mathsf{E} \sup_{t\in K} X_t}$$

(Tsirel'son, 1985; V, 1996, 2001)

$$P\left(\sup_t X_t - \mathsf{E} \sup_t X_t \ge a \right) \le e^{-a^2/2\sigma^2},$$

(Maurey-Pisier, 1986; V, 1996, 2001)

Quasi-widths

For $K \in \mathcal{K}$ and $j = 0, 1, \ldots$, define *quasi-widths*

$$m_j(K) = \frac{j\, V_j(K)}{\sqrt{2\pi} V_{j-1}(K)} \qquad (0/0 = 0).$$

- $\{m_j\}_1^\infty$ is a decreasing sequence (Alexandrov-Fenchel).
- $m_1 = \mathrm{E} \sup_{t \in K} X_t$.
- $\lim_{j \to \infty} m_j = \operatorname{osc}(K)$.
- For each i, there is the bound

$$W(rK) \leq \left[\Pi_{j=1}^{i} \left(m_j / m_i\right)\right] \;\; e^{r m_i(K)}.$$

For $i = 1$, this reduces to

$$W(rK) \leq e^{r \sup_t X_t},$$

as seen before.

- Maurey-Pisier-type deviation bounds: for each i and $a > 0$,

$$P\left(\sup_t X_t - m_i \geq a\right) \leq \left[\Pi_{j=1}^{i}\left(m_j/m_i\right)\right] \quad e^{-a^2/(2\sigma^2)}.$$

- Open question: is there a relationship between quasi-widths and Dvoretsky's theorem?

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THE WILLS FUNCTIONAL AND GAUSSIAN PROCESSES[1]

BY RICHARD A. VITALE

University of Connecticut

The Wills functional from the theory of lattice point enumeration can be adapted to produce the following exponential inequality for zero-mean Gaussian processes:

$$E\exp\left[\sup_t\left(X_t-(1/2)\sigma_t^2\right)\right]\le\exp\left(E\sup_t X_t\right).$$

An application is a new proof of the deviation inequality for the supremum of a Gaussian process above its mean:

$$P\left(\sup_t X_t - E\sup_t X_t\ge a\right)\le\exp\left(-\frac{(1/2)a^2}{\sigma^2}\right),$$

where $a>0$ and $\sigma^2=\sup_t\sigma_t^2$.

1. Introduction. The use of geometric methods in the study of Gaussian processes is by now well established. Our purpose here is to identify a surprising, and apparently deep, connection in the form of the Wills functional. Originally introduced for bounding lattice point enumeration (see [23]), the Wills functional is built up from the classical quermassintegrals ("projection–measure integrals") of Minkowski, which have effectively been applied before to Gaussian processes under the name mixed volumes (or mixed widths) (e.g., [1], [3], [4], [12], [14], [16], [17], [18], [19] and [21]). Placed in our setting, the Wills functional leads naturally to an exponential moment inequality for Gaussian processes and, as a corollary, a deviation inequality for the supremum of a Gaussian process above its mean. The latter is sharp in the sense of having the best possible constant in the exponent.

In the next section, we discuss two representations for the Wills functional. Then we turn to the exponential inequality and the deviation inequality. Section 4 carries some finite-dimensional complements, leading to a second proof of the deviation inequality. We conclude with some remarks in the last section.

For Gaussian processes and bounds in particular, see [6] and [9].

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2. The Wills functional. Suppose that K is a convex body in $\mathbb{R}^d$ (compact, convex subset) and that $\delta(x, K)$ is the distance between $x \in \mathbb{R}^d$ and K. The Wills functional can be expressed as

$$(1) \qquad W(K) = \int_{\mathbb{R}^d} \exp\left[-\pi\delta^2(x, K)\right] dx, \qquad dx = \text{Lebesgue measure}.$$

It was observed by Hadwiger [8] that (1) coincides with the original definition of Wills [23] in terms of quermassintegrals:

$$(2) \qquad W(K) = \sum_{j=0}^{d} \binom{d}{j} \frac{1}{\omega_j} W_j(K).$$

Here $\omega_j = \pi^{j/2}/\Gamma(j/2+1)$ is the volume of the unit ball B_j in $\mathbb{R}^j$, and the jth quermassintegral $W_j(K)$ is equal to $(\omega_d/\omega_{d-j})E\,\mathrm{vol}_{d-j}(\Pi_{d-j}K)$, where the expectation is of the $(d-j)$-volume of the projection of K onto a random $(d-j)$-dimensional subspace. It is of interest to consider

$$V_j(K) = \binom{d}{j} \frac{1}{\omega_{d-j}} W_{d-j}(K), \qquad 0 \le j \le d,$$

which are normalized versions of the quermassintegrals that do not depend on the dimension of the ambient space $\mathbb{R}^d$ (e.g., [3] and [10]). Following McMullen [10], they are known in the geometry literature as the *intrinsic volumes* of K [in the probability literature, they have been written $h_j(K)$].

Here is a sketch of the equivalence of (1) and (2) which has a probabilistic flavor. We start with the classical Steiner formula for the volume of a parallel body (which itself can be established using a stochastic argument, [22]; for the general theory, see [15]):

$$\mathrm{vol}(K + \Lambda B_d) = \sum_{j=0}^{d} \binom{d}{j} \Lambda^j W_j(K),$$

where $\Lambda \ge 0$, or, equivalently,

$$(3) \qquad \int_{\mathbb{R}^d} 1(\delta(x, K) \le \Lambda)\, dx = \sum_{j=0}^{d} \binom{d}{j} \Lambda^j W_j(K).$$

We then regard Λ as a random variable with density

$$(4) \qquad f(\lambda) = 1(\lambda \ge 0) 2\pi\lambda \exp(-\pi\lambda^2)$$

and take expectations on both sides of (3) with the use of Fubini's theorem and the moments $E\Lambda^j = \omega_j^{-1}$, $j = 1, 2, \dots$.

We also recall the following bound (see [11]), which is a consequence of the deep Alexandrov–Fenchel inequality (see [15]):

$$(5) \qquad W(K) \le \exp V_1(K).$$

3. Bounds. Our main result is as follows.

THEOREM 1. *Suppose that* $\{X_t,\ t \in T\}$ *is a bounded, zero-mean Gaussian process. Then*

$$E\exp\left[\sup_t\left\{X_t - (1/2)\,\sigma_t^2\right\}\right] \le \exp\left[E\sup_t X_t\right], \tag{6}$$

where $\sigma_t^2 = EX_t^2$.

NOTE. Since the process is bounded, it is continuous in probability with respect to the pseudometric $d_X(t_1, t_2) = \sqrt{E(X_{t_1} - X_{t_2})^2}$. We regard $\sup_t X_t$ as over a countable, dense (under d_X) subset of T. Any other countable, dense subset of T gives the same value for $\sup_t X_t$ almost surely.

PROOF OF THEOREM 1. It is enough to show (6) for $T = \{1, 2, \dots, n\}$, a finite set, since the general case follows by approximation. By the Gram–Schmidt procedure, there is a collection $Z_1, Z_2, \dots, Z_d$, $d \le n$, of independent, standard Gaussian variables, such that, for $1 \le k \le n$ and appropriate vectors $a_1, a_2, \dots, a_n \in \mathbb{R}^d$,

$$X_k = \langle a_k, Z\rangle, \tag{7}$$

where $Z = (Z_1, Z_2, \dots, Z_d)^T \in \mathbb{R}^d$ and $\langle\cdot,\cdot\rangle$ signifies inner product. Note that $EX_k^2 = \|a_k\|^2$ and $EX_kX_l = \langle a_k, a_l\rangle$. Let $K \subset \mathbb{R}^d$ be the convex hull of $A = \{a_k/\sqrt{2\pi}\}_1^n$. Employing the natural definition of $W(A)$ (A nonconvex) and making a change of variables at one point, we have

$$\begin{aligned}
W(A) &= \int_{\mathbb{R}^d} \exp\left[-\pi\delta^2(x, A)\right] dx \\
&= \int_{\mathbb{R}^d} \exp\left[-\pi \inf_k \left\| x - a_k/\sqrt{2\pi}\right\|^2\right] dx \\
&= \int_{\mathbb{R}^d} \exp\left[\sup_k\left(\sqrt{2\pi}\,\langle a_k, x\rangle - (1/2)\|a_k\|^2\right)\right] \exp\left[-\pi\|x\|^2\right] dx \\
&= E\exp\left[\sup_k\left(\langle a_k, Z\rangle - (1/2)\|a_k\|^2\right)\right] \\
&= E\exp\left[\sup_k\left(X_k - (1/2)\,\sigma_k^2\right)\right].
\end{aligned}$$

Now $A \subseteq K$ implies $W(A) \le W(K)$, and, together with (5), we only have to recall that $V_1(K) = E\sup_k X_k$ [e.g., [16], Proposition 14, where it is written $h_1(K)$]. □

As a consequence, we have the sharp deviation inequality.

COROLLARY 1. *Under the conditions of Theorem* 1, *for any* $a > 0$,

$$P\left(\sup_t X_t - E\sup_t X_t \ge a\right) \le \exp\left[-(1/2)(a^2/\sigma^2)\right], \tag{8}$$

where $\sigma^2 = \sup_t \sigma_t^2$.

The literature contains a variety of applications and results related to (8) (see, e.g., [6] and [9]), often associated with Maurey and Pisier (13). See also the antecedent [4].

PROOF. We use the inhomogeneity of (6) in a variational argument. Consider the process $\{rX_t\}$ for fixed $r > 0$. Then (6) provides

$$E\exp\left[\sup_t\left(rX_t - (1/2)r^2\sigma_t^2\right)\right] \le \exp\left[E\sup_t rX_t\right].$$

Since $\sigma_t^2 \le \sigma^2$ for all t, some rearrangement gives

$$E\exp\left[r\sup_t\left(X_t - E\sup_t X_t\right)\right] \le \exp\left[(1/2)r^2\sigma^2\right].$$

Using Markov's inequality, we have

$$\begin{aligned}P\left(\sup_t X_t - E\sup_t X_t \ge a\right) &= P\left(r\left[\sup_t X_t - E\sup_t X_t\right] \ge ra\right)\\ &= P\left(\exp\left[\sup_t rX_t - E\sup_t rX_t\right] \ge \exp(ra)\right)\\ &\le \exp\left[(1/2)r^2\sigma^2 - ra\right].\end{aligned}$$

Minimizing the last expression over r then yields $\exp[-a^2(2\sigma^2)^{-1}]$ at $r = a/\sigma^2$. □

4. Finite-dimensional bounds. In general, bounds which are independent of dimension are the most useful. Still, we elaborate a dimensional one here since it leads to a second proof of the deviation inequality. In addition, it illustrates a different use of the Wills functional, here allied with Urysohn's inequality (e.g., [2]; in a probabilistic format, [20]), which we state as follows.

PROPOSITION 1. *Suppose that K is a convex body in $\mathbb{R}^d$. Then*

$$\operatorname{vol}(K) \le \omega_d\left(\frac{V_1(K)}{V_1(B_d)}\right)^d. \tag{9}$$

As before, B_d is the unit ball in $\mathbb{R}^d$ with volume ω_d. There is equality in (9) *if and only if K is a ball.*

The bound is as follows.

THEOREM 2. *Suppose that $\{X_t,\ t \in T\}$ is a bounded Gaussian process that can be identified with $A \subset \mathbb{R}^d$ via* (7). *That is, for each t, $X_t = \langle a_t, Z\rangle$ for some $a_t \in \sqrt{2\pi}A$. Then*

$$E\exp\left[\sup_t\left\{X_t - (1/2)\sigma_t^2\right\}\right] \le \omega_d E\left(\frac{E\sup_t X_t}{V_1(B_d)} + \Lambda\right)^d. \tag{10}$$

Here Λ has the density (4), *and there is equality if and only if the closed convex hull of A is a ball.*

PROOF. Let K be the closed convex hull of A. By Urysohn's inequality.

$$\operatorname{vol}(K + \Lambda B_d) \le \omega_d \left(\frac{V_1(K)}{V_1(B_d)} + \Lambda \right)^d .$$

We then take expectations with respect to Λ and express the left-hand side as in the proof of Theorem 1. □

COROLLARY 2. *For each $a > 0$ and $c \ge 1$,*

$$P\left[\sup_t X_t - E\sup_t X_t \ge a\right] \le c^d \exp\left[-\frac{1}{2\sigma^2}\left\{a + \left(1 - \frac{1}{c}\right) E\sup_t X_t\right\}^2\right]. \tag{11}$$

For the proof, we use the following estimate. It bounds a polynomial with an exponential, but that is sharp enough for our purpose.

LEMMA 1.

$$\psi_d(\theta) = \omega_d E\left(\frac{\theta}{V_1(B_d)} + \Lambda\right)^d \le c^d e^{\theta/c}$$

for all $\theta \ge 0$, $c \ge 1$ and $d = 1, 2, \dots$.

PROOF. Fix $c \ge 1$. We verify the property for each $\psi_d(\cdot)$ by induction on d. For $d = 1$, $\psi_1(\theta) = 1 + \theta \le ce^{\theta/c}$. Given that the claim has been shown for ψ_{d-1} and noting that $\psi_d(0) \le ce^{0/c} = c$, it is enough to compare derivatives. Using the facts that $V_1(B_{d-1}) \le V_1(B_d)$ and $\omega_d d = V_1(B_d)\omega_{d-1}$, we have

$$\begin{aligned}
\psi_d'(\theta) &= \frac{\omega_d d}{V_1(B_d)} E\left(\frac{\theta}{V_1(B_d)} + \Lambda\right)^{d-1} \\
&\le \omega_{d-1} E\left(\frac{\theta}{V_1(B_{d-1})} + \Lambda\right)^{d-1} \\
&\le c^{d-1} e^{\theta/c} = \frac{d}{d\theta} c^d e^{\theta/c},
\end{aligned}$$

which completes the proof. □

PROOF OF COROLLARY 2. Consider the process $\{rX_t\}$. Using the lemma and a variational argument similar to that for Corollary 1 leads to the assertion. □

Setting $c = 1$ gives a second proof of the deviation inequality (Corollary 1). Finally, we can tailor (11) a bit as follows: let $c = 1 + 1/d$ and using the (possibly more accessible) quantity $\operatorname{diam}(X) = \sup_{t,t'} d_X(t,t') \le \sqrt{2\pi} E \sup_t X_t$ leads to the variant

$$P\left[\sup_t X_t - E \sup_t X_t \ge a\right] \le \exp\left[1 - \frac{1}{2\sigma^2}\left\{a + \left(\frac{1}{d+1}\right)\frac{1}{\sqrt{2\pi}}\operatorname{diam}(X)\right\}^2\right].$$

5. Remarks.

REMARK 1. The connection we have drawn between Gaussian processes and the Wills functional, which can be thought of as a measure of size for convex bodies, is in the spirit of [5], which has motivated much work. For further references on the Wills functional, see the survey [7].

REMARK 2. Derived independently, the bound (6) is formally equivalent to Corollary 1 of Tsirel'son [18], which is proved by other means. The context there is different, an infinite-dimensional maximum likelihood problem, and it is presented with no reference to the bounds for Gaussian processes. In any case, the interested reader should consult this important paper together with the others in its series, [17] and [19].

REMARK 3. The corresponding left-tail bound to (8),

$$P\left(\sup_t X_t - E \sup_t X_t \le a\right) \le \exp\left(L - \frac{(1/2)a^2}{\sigma^2}\right),$$

does not seem to be accessible by the approach described here.

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Department of Statistics
Box U-120
University of Connecticut
Storrs, Connecticut 06269
E-mail: rvitale@uconnum.uconn.edu

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INTRINSIC VOLUMES AND GAUSSIAN PROCESSES

RICHARD A. VITALE,* *University of Connecticut*

Abstract

Intrinsic volumes are key functionals in convex geometry and have also appeared in several stochastic settings. Here we relate them to questions of regularity in Gaussian processes with regard to Itô–Nisio oscillation and metrization of GB/GC indexing sets. Various bounds and estimates are presented, and questions for further investigation are suggested. From alternate points of view, much of the discussion can be interpreted in terms of (i) random sets and (ii) properties of (deterministic) infinite-dimensional convex bodies.

Keywords: Convex body; deviation bound; Gaussian process; GB/GC set; intrinsic volume; Itô–Nisio oscillation; metric; quermassintegral; Steiner point; Wills functional

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1. Introduction

As functionals on convex bodies, intrinsic volumes have appeared in many stochastic settings, including geometric probability, integral geometry, and Gaussian processes (e.g. [1], [3], [4], [12], [17], [19], [20], [21], [22], [23], [24], [32], [33]). Our aim here is to elaborate some further connections with Gaussian processes that are of interest in themselves and that also hint at a deeper theory. In particular, we treat (i) geometric formulation of the Itô–Nisio oscillation function, which quantifies the 'size' of a discontinuity of a Gaussian process, (ii) metric categorization of GB and GC sets, which are classes of compact, convex subsets of Hilbert space distinguished by the behavior of Gaussian processes using them as indexing sets, and (iii) bounds and estimates associated with (i) and (ii). An important role is played throughout by the *Wills functional*, which in a parametrized form provides the generating function of the intrinsic volumes.

The plan is as follows. The next section summarizes background and preliminaries for convenient reference. Connections between Itô–Nisio oscillation and intrinsic volumes are discussed in Section 3. This is followed by a characterization of GC processes based on a new metrization of convex bodies. Section 5 carries bounds and continuity estimates for intrinsic volumes and for a vector analogue, the Steiner point. The final section outlines directions for future work toward a general theory.

Two comments: firstly, from the viewpoint of *random sets*, the paper can be read as a detailed study of random convex bodies of the form $\boldsymbol{M}K$ (see X_K^{j*} below), where K is a convex body and $\boldsymbol{M}$ is a matrix of i.i.d. Gaussian elements. Among all models for random convex bodies, this is among the most natural and bears close examination. Secondly, from the viewpoint of convex geometry, the paper is about the extension of intrinsic volumes, known classically

Received 24 October 2000; revision received 28 March 2001.
* Postal address: Department of Statistics, University of Connecticut, Storrs, CT 06269, USA.
Email address: rvitale@uconnvm.uconn.edu

as 'quermassintegrals', to convex bodies in *infinite* dimensions, a procedure that (remarkably) appears to be best described in probabilistic terms.

2. Background

In this section, we gather some preliminaries and notation. Further background on Gaussian processes and convex bodies can be found, for example, in [13] and [18] respectively.

The setting is ℓ_2 with standard basis

$$\boldsymbol{e}_j = (\underbrace{0, 0, \dots, 0}_{j-1}, 1, 0, 0, \dots), \qquad j = 1, 2, \dots,$$

inner product $\langle \cdot, \cdot \rangle$, norm $\| \cdot \|$, and closed unit ball B. The closed line segment between $\boldsymbol{x}, \boldsymbol{y}$ is $[\boldsymbol{x}, \boldsymbol{y}]$. Finite-dimensional Euclidean space $\mathbb{R}^n$ is freely identified with the span of $\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_n$ using the same notation for inner product and norm. Orthogonal projection of ℓ_2 onto the spans of $\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_n$ and $\boldsymbol{e}_{n+1}, \boldsymbol{e}_{n+2}, \dots$ are denoted by π_n and $\overline{\pi}_n$ respectively. In $\mathbb{R}^n$, the unit sphere is S^{n-1} and the closed unit ball B_n (we specify that, in either $\mathbb{R}^n$ or ℓ_2, $B(\boldsymbol{x}, r)$ stands for the *closed* ball of radius r and centered at $\boldsymbol{x}$). Volume (Lebesgue measure) in $\mathbb{R}^n$ is $\mathrm{vol}_n(\cdot)$ with $\omega_n = \mathrm{vol}_n(B_n) = \pi^{n/2}/\,\Gamma(1 + n/2)$.

We denote by $\mathcal{K}_n$ and $\mathcal{K}$ the collections of convex bodies (i.e. non-empty, compact, convex subsets) in $\mathbb{R}^n$ and ℓ_2 respectively. The collection of all finite-dimensional convex bodies in ℓ_2 is $\mathcal{K}_{\mathrm{FD}} \subset \mathcal{K}$. Minkowski addition of sets is signified by $+$. We say that K_1, K_2 are *equalized* by L if $K_1 \subseteq K_2 + L$ and $K_2 \subseteq K_1 + L$. The support function h_K of a convex body K is given by $h_K(x) = \sup\{\langle \boldsymbol{x}, \boldsymbol{t} \rangle \mid \boldsymbol{t} \in K\}$, and the *norm* of K, written $\|K\|$, or σ_K, is $\sup\{\|\boldsymbol{t}\| \mid \boldsymbol{t} \in K\}$. Distance on $\mathcal{K}$ is given by the *Hausdorff metric* $\rho_{\mathrm{H}}(K_1, K_2) = \inf\{\varepsilon > 0 \mid K_1 \subseteq K_2 + \varepsilon B, K_2 \subseteq K_1 + \varepsilon B\}$.

For Gaussian processes, let $\boldsymbol{Z} = (Z_1, Z_2, \dots)$ comprise a sequence of independent standard Gaussian variables. While $\boldsymbol{Z} \notin \ell_2$ almost surely, there is, for every $\boldsymbol{t} \in \ell_2$, almost sure (conditional) convergence of the sum $\sum_1^\infty t_i Z_i$, which represents a Gaussian variable (written below as either $\langle \boldsymbol{t}, \boldsymbol{Z} \rangle$ or X_t) having mean 0 and standard deviation $\|\boldsymbol{t}\|$. In addition, $\mathrm{cov}(X_t, X_{t'}) = \mathrm{E}\, X_t X_{t'} = \langle \boldsymbol{t}, \boldsymbol{t}' \rangle$. For $K \in \mathcal{K}$, we write X_K for the Gaussian process $\{X_t = \langle \boldsymbol{t}, \boldsymbol{Z} \rangle, \boldsymbol{t} \in K\}$. Expressions such as $\sup_{t \in K} X_t$ mean $\sup_{t \in D \subseteq K} X_t$, where D is a countable, dense subset of K (the particular subset is immaterial almost surely).

The Steiner formula for the volume of a parallel body to a convex body has, for $K \in \mathcal{K}_n$ and $\lambda \geq 0$,

$$\mathrm{vol}_n(K + \lambda B_n) = \sum_{j=0}^{n} \binom{n}{j} \lambda^j W_j(K).$$

It can be shown that the coefficients (quermassintegrals) satisfy $W_j(K) = (\omega_n/\omega_{n-j}) \times \mathrm{E}\, \mathrm{vol}_{n-j}(\Pi_{[n-j,n]} K)$, where $\Pi_{[n-j,n]}$ signifies the first $n - j$ rows of a random (normalized Haar measure) $n \times n$ orthogonal matrix (see e.g. [27]). Dependence on the dimension n is removed by re-normalization to the *intrinsic volumes* [14]:

$$V_j(K) = \binom{n}{j} \frac{W_{n-j}(K)}{\omega_{n-j}};$$

other properties can be found in [16] and [18]. In [4] and [19], intrinsic volumes were extended to $\mathcal{K}$ with the definition $V_j(K) = \sup\{V_j(\tilde{K}) \mid \tilde{K} \subseteq K, \tilde{K} \in \mathcal{K}_{\mathrm{FD}}\}$. An alternate representation (essentially appearing in [23]) considers, for each j, the vector process $\boldsymbol{X}_t^{j*} =$

$(X_t^{(1)}, X_t^{(2)}, \dots, X_t^{(j)})$, where the components are independent copies of X_t. Setting $X_K^{j*} = \overline{\mathrm{conv}}\{X_t^{j*}, t \in K\} \subseteq \mathbb{R}^j$ ($\overline{\mathrm{conv}}$ is introduced to avoid irrelevant measurability questions) leads to $V_j(K) = (2\pi)^{j/2}\,\mathrm{E}\,\mathrm{vol}_j(X_K^{j*})/\omega_j j!$. For later reference, the *intrinsic width* $V_1(\cdot)$ satisfies $V_1(K) = (2\pi)^{1/2}\,\mathrm{E}\sup_{t\in K} X_t$ with $V_1([\boldsymbol{x}, \boldsymbol{y}]) = \|\boldsymbol{x} - \boldsymbol{y}\|$; also, $V_1(B_n) = (2\pi)^{1/2}\,\mathrm{E}(Z_1^2 + Z_2^2 + \cdots + Z_n^2)^{1/2} \sim (2\pi n)^{1/2}$. The Wills functional for X_K ([9], [10], [23], [28], [31], [34]) is given by

$$W(K) = \mathrm{E}\exp\Big\{\sup_{t\in K}[X_t - \tfrac{1}{2}\sigma_t^2]\Big\}, \qquad \sigma_t = \sqrt{\mathrm{E}\,X_t^2}$$

and provides the generating function of the intrinsic volumes:

$$W(rK) = \sum_{j=0}^{\infty}\left(\frac{r}{\sqrt{2\pi}}\right)^j V_j(K), \qquad r \geq 0. \tag{1}$$

3. Oscillation

A convex body $K \in \mathcal{K}$ is said to be GB (*Gaussian bounded*), or GC (*Gaussian continuous*), if X_K has, respectively, a version with bounded or continuous sample paths. With the class $\mathcal{K}_{\mathrm{FD}}$ of finite-dimensional convex bodies, we have the respective classes $\mathcal{K}_{\mathrm{GC}}$ and $\mathcal{K}_{\mathrm{GB}}$ satisfying $\mathcal{K}_{\mathrm{FD}} \subset \mathcal{K}_{\mathrm{GC}} \subset \mathcal{K}_{\mathrm{GB}} \subset \mathcal{K}$. They have been studied from many points of view, and in particular the following features of $\mathcal{K}_{\mathrm{GB}}$ ([4], [15], [19]) are basic:

Theorem 1. *If $K \in \mathcal{K}$, then $K \in \mathcal{K}_{\mathrm{GB}}$ if and only if $V_1(K) < \infty$, in which case $V_j(K) \leq V_1^j(K)/j!$ for $j = 2, 3, \dots$.*

From now on, assume that all convex bodies are in $\mathcal{K}_{\mathrm{GB}}$.

The classic result of Itô and Nisio [11] asserts that

$$\lim_{\varepsilon\to 0}[\sup\{X_t \mid t \in K \cap B(\hat{t}, \varepsilon)\} - \inf\{X_t \mid t \in K \cap B(\hat{t}, \varepsilon)\}] \tag{2}$$

exists as an almost sure constant, which we label $2\,\mathrm{osc}_K(\hat{t})$ (we depart from convention and introduce the factor 2 for later convenience). Geometrically, (2) says that the random interval

$$[\inf\{X_t \mid t \in K \cap B(\hat{t}, \varepsilon)\}, \sup\{X_t \mid t \in K \cap B(\hat{t}, \varepsilon)\}]$$

tends almost surely to the non-random interval $[-\mathrm{osc}_K(\hat{t}), \mathrm{osc}_K(\hat{t})]$. Our first result extends this to higher dimensions and indicates further that at a point $\hat{t}$ of discontinuity, a small ball neighborhood $K \cap B(\hat{t}, \varepsilon)$ itself *generically* resembles a ball (of small radius and high dimension, irrespective of any other features of K).

Theorem 2. *As $\varepsilon \to 0$,*

(i) *the random convex body $X^{j*}_{K\cap B(\hat{t},\varepsilon)}$ tends almost surely in the Hausdorff metric to $B_j(\mathbf{0}, \mathrm{osc}(\hat{t}))$ for $j = 1, 2 \dots,$*

(ii) $W(r(K \cap B(\hat{t}, \varepsilon))) \to \exp(\mathrm{osc}(\hat{t})r)$, *which coincides with* $\lim_{n\to\infty} W(r(1/(2\pi n)^{1/2})\, B_n(\mathbf{0}, \mathrm{osc}(\hat{t})))$.

Proof. For $\boldsymbol{u} \in S^{j-1}$,

$$h_{X^{j*}_{K\cap B(\hat{t},\varepsilon)}}(u) = \sup\{\langle \boldsymbol{u}, X_t^{j*}\rangle \mid t \in K \cap B(\hat{t}, \varepsilon)\},$$

which is equal in distribution to $\sup\{X_t \mid t \in K \cap B(\hat{t}, \varepsilon)\}$ and hence almost surely convergent to $\operatorname{osc}(\hat{t})$. Recalling that pointwise convergence of support functions implies Hausdorff metric convergence yields (i). It follows that for each j, $V_j(K \cap B(\hat{t}, \varepsilon)) \to ((2\pi)^{1/2} \operatorname{osc}_K(\hat{t}))^j/j!$, so that by the dominated convergence theorem, $W(r(K \cap B(\hat{t}, \varepsilon))) \to \exp(\operatorname{osc}_K(\hat{t})r)$. For the last assertion, let Λ have density $\lambda \mapsto \mathbf{1}(\lambda \geq 0)2\pi\lambda\exp(-\pi\lambda^2)$. Then [28]

$$W\left(\frac{r}{\sqrt{2\pi n}}B_n(\mathbf{0}, \operatorname{osc}(\hat{t}))\right) = \operatorname{E}\operatorname{vol}_n\left[\left(\frac{\operatorname{osc}_K(\hat{t})r}{\sqrt{2\pi n}} + 2\pi\Lambda\right)B_n\right]$$
$$= \sum_{j=0}^{n}\binom{n}{j}\frac{\omega_n}{\omega_{n-j}}\left(\frac{\operatorname{osc}_K(\hat{t})r}{\sqrt{2\pi n}}\right)^j.$$

Standard estimates provide

$$\binom{n}{j}\frac{\omega_n}{\omega_{n-j}}\left(\frac{1}{\sqrt{2\pi n}}\right)^j \to 1/j!,$$

with a dominating sequence of the form $c^j/j!$.

An important parameter is $\operatorname{osc}(K) = \sup\{\operatorname{osc}_K(\hat{t}) \mid \hat{t} \in K\}$. For example, $K \in \mathcal{K}_{\mathrm{GC}}$ if and only if $\lim_{\delta\to 0} V_1(K \cap B(\hat{t}, \delta)) = 0$ for every $\hat{t} \in K$, so that $\operatorname{osc}(K) = 0$ if and only if $K \in \mathcal{K}_{\mathrm{GC}}$. We show next that $\operatorname{osc}(K)$ can be read off from the intrinsic volumes. At least formally, this can be done in two ways. Consider the following maps for $r \in [0, \infty)$ and $j \in \{1, 2, \dots\}$:

$$r \mapsto \ell_K(r) = \log W(rK),$$
$$j \mapsto m_j(K) = \frac{jV_j(K)}{\sqrt{2\pi}V_{j-1}(K)} \quad \text{(by convention, } 0/0 = 0).$$

Proposition 1. (i) *The function ℓ_K is positive, increasing, concave, and bounded above by $(2\pi)^{-1/2}V_1(K)r$.*

(ii) *The sequence $m_j(K)$ is positive and decreasing.*

Proof. In (i), the first two properties are clear from (1); concavity follows from [31, Lemma 2]. The exponential bound appears in [15] (see also [23], [28], [31]), as does (ii).

Theorem 3. *For $K \in \mathcal{K}_{\mathrm{GB}}$*

$$\operatorname{osc}(K) = \lim_{r\to\infty}\frac{\mathrm{d}}{\mathrm{d}r}\ell_K(r) = \lim_{j\to\infty} m_j(K).$$

Proof. By the proposition, $\lim_{r\to\infty}(\mathrm{d}/\mathrm{d}r)\ell_K(r) = \lim_{r\to\infty}\ell_K(r)/r$. Let $\hat{t} \in K$. Theorem 2 and the monotonicity of $W(\cdot)$ show that $\exp(\operatorname{osc}(\hat{t})r) \leq W(r(K \cap B(\hat{t}, 1)) \leq W(rK)$. Taking logarithms and a supremum over $\hat{t} \in K$ gives $\operatorname{osc}(K) \leq \lim_r \ell_K(r)/r$. Conversely, let $\varepsilon > 0$. There is an open cover for K of sets $\operatorname{Int}[B(\hat{t}, \varepsilon(\hat{t}))]$, where $V_1(K \cap B(\hat{t}, \delta(\hat{t}))) < (2\pi)^{1/2}[\operatorname{osc}(\hat{t}) + \varepsilon]$. Compactness of K gives a finite subcover, say indexed by $\hat{t}_1, \dots, \hat{t}_N$; let $K_i = K \cap B(\hat{t}_i, \delta(\hat{t}_i))$. Then

$$W(rK) \leq \sum_{i=1}^{N} W(rK_i) \leq \sum_{i=1}^{N}\exp\left[\frac{V_1(K_i)}{\sqrt{2\pi}}r\right],$$

so that $\ell_K(r)/r \leq \operatorname{osc}(K) + \varepsilon + o(1)$, and hence $\lim_r \ell_K(r)/r \leq \operatorname{osc}(K)$.

Next, $(\mathrm{d}/\mathrm{d}r)W(rK) = \sum_{j=0}^{\infty}(r/(2\pi)^{1/2})^j V_j(K)m_{j+1}(K)$. Monotonicity of the m_j sequence implies that this is bounded from below by $[\lim_j m_j(K)]W(rK)$, so that $\lim_j m_j(K) \leq \lim_r(\mathrm{d}/\mathrm{d}r)\ell_K(r)$. Similarly, for any $\tilde{j}$, there is an upper bound of the form $m_{\tilde{j}}(K)W(rK) + o(r^{\tilde{j}})$, which implies that $\lim_r(\mathrm{d}/\mathrm{d}r)W(rK)/W(rk) \leq m_{\tilde{j}}$ and $\lim_r(\mathrm{d}/\mathrm{d}r)\ell_K(r) \leq \lim_{\tilde{j}} m_{\tilde{j}}(K)$.

As an application, there is an alternate proof of a deviation bound ([6]; see also [13]):

Theorem 4. *For $K \in \mathcal{K}_{\mathrm{GB}}$ and $a > 0$,*

$$\log \mathrm{P}\Big(\sup_K X_t \geq a + \operatorname{osc}(K)\Big) = -\frac{a^2}{2\sigma_K^2} + o(a), \qquad \sigma_K^2 = \sup_{t\in K} X_t^2.$$

Proof. Following an argument in [28],

$$\log \mathrm{P}\Big(\sup_K X_t \geq a + \operatorname{osc}(K)\Big) \leq \frac{r^2\sigma_K^2}{2} + \ell_K(r) - r(a + \operatorname{osc}(K))$$

for $r > 0$. Letting $r = a/\sigma_K^2$ gives the bound $-(a^2/2\sigma_K^2) + \ell_K(a/\sigma_K^2) - (a/\sigma_K^2)\operatorname{osc}(K)$, and the result follows.

As a final note, we mention that the intrinsic volumes associated with Brownian motion have recently been determined [7]. This led to a conjectured phase transition in the gap between GB and GC sets:

Conjecture 1. *Either* $\lim_j m_j > 0$ *or* $m_j = O(j^{-1/2})$.

4. Metrization

Discontinuity of a Gaussian process clearly presents special problems for analysis. The sequel presents a general setting in which, on the contrary, continuity of the process is assured. This is done with a characterization of $\mathcal{K}_{\mathrm{GC}}$; the following section shows, with specific bounds, continuity of the intrinsic volumes and Steiner point on $\mathcal{K}_{\mathrm{GB}}$.

Our point of departure is that, geometrically, a discontinuity of a Gaussian process can be regarded as a *diameter/intrinsic-width anomaly* (which can occur only in infinite dimensions). This is when $K_n \downarrow \{\hat{t}\}$, that is, $\operatorname{diam}(K_n) \downarrow 0$, but in such a way that $V_1(K_n) \not\downarrow 0$. This suggests focusing first on the continuity properties of V_1. Examples show that the classic Hausdorff metric for convex bodies must be replaced with a more delicate one. The following is natural:

$$\rho_{V_1}(K_1, K_2) = \inf\{V_1(L) \mid K_1, K_2 \text{ are equalized by } L \in \mathcal{K}\}.$$

Proposition 2. (i) *On $\mathcal{K}_{\mathrm{GB}}$, ρ_{V_1} is a (finite) metric.*

(ii) *If $K_1, K_2 \in \mathbb{R}^n$, then*

$$\rho_{\mathrm{H}}(K_1, K_2) \leq \rho_{V_1}(K_1, K_2) \leq V_1(B_n)\rho_{\mathrm{H}}(K_1, K_2). \tag{3}$$

Proof. For (i), note that if $K_1, K_2 \in \mathcal{K}_{\mathrm{GB}}$, then they are equalized by $L = \operatorname{conv}[(K_1 + (-K_2)) \cup ((-K_1) + K_2)] \in \mathcal{K}_{\mathrm{GB}}$, so that $\rho_{V_1}(K_1, K_2)$ is finite. It is easy to see that if L equalizes K_1 and K_2, then $\mathbf{0} \in L$ and consequently $\sigma_L = \|L\| \leq V_1(L)$, since $\boldsymbol{x} \in L$ with $\|\boldsymbol{x}\| = \|L\|$ implies that if $[0, \boldsymbol{x}] \subseteq L$, then $\|L\| = \|\boldsymbol{x}\| = V_1([\mathbf{0}, \boldsymbol{x}]) \leq V_1(L)$ by the monotonicity of V_1. It follows that if $\rho_{V_1}(K_1, K_2) = 0$, then $K_1 = K_2$. The triangle inequality follows from the (sub)-additivity of $V_1(\cdot)$. For (ii), if L equalizes K_1, K_2, then so does $\|L\|B_n$, and thus $\rho_H(K_1, K_2) \leq \|L\| \leq V_1(L)$. On the other hand, if εB_n equalizes K_1, K_2, then $\rho_{V_1}(K_1, K_2) \leq V_1(\varepsilon B_n) = \varepsilon V_1(B_n)$, and the right inequality of (3) holds as well.

We then have the following characterization of $\mathcal{K}_{GC}$:

Theorem 5. *The completion of* $(\mathcal{K}_{FD}, \rho_{V_1})$ *is* $(\mathcal{K}_{GC}, \rho_{V_1})$.

Note that this parallels the classic observation that completing $\{\mathcal{K}_{FD}, \rho_H\}$ leads to $\{\mathcal{K}, \rho_H\}$ (e.g. [2]). The proof is based on two lemmas.

Lemma 1. *Suppose that* $K \in \mathcal{K}_{GB}$. *Then, as* $n \to \infty$, $\sup_{t\in K} \sum_n^\infty t_i Z_i$ *converges to a nonnegative constant almost surely and in* L_1.

Proof. For $m = \dots, -2, -1$, let $R_m = \sup_{t\in K} \sum_{-m}^\infty t_i Z_i$, and $\mathcal{F}_m = \sigma(Z_{-m}, Z_{-m+1}, \dots)$. Then $\mathrm{E}[R_{m+1} \mid \mathcal{F}_m] = \mathrm{E}[\sup_{t\in K} \sum_{-m-1}^\infty t_i Z_i \mid Z_{-m}, Z_{-m+1}, \dots] \geq \sup_{t\in K} \sum_{-m}^\infty t_i Z_i = R_m$, so that $\{R_m, \mathcal{F}_m\}$ is a reverse submartingale. If $\mathbf{0} \in K$, then $0 \leq \inf_m R_m$ almost surely. This suffices for almost-sure and L_1 convergence of the R_m. The limit variable is obviously tail-measurable and so must be a nonnegative constant. If $\mathbf{0} \notin K$, note that the argument holds for $K - \tilde{t}, \tilde{t} \in K$, but the shift is asymptotically negligible since $\langle \overline{\pi}_n \tilde{t}, \boldsymbol{Z}\rangle \to 0$ almost surely.

Lemma 2. *Suppose that* $K \in \mathcal{K}$ *is such that, for arbitrary* $\varepsilon > 0$, *there are* $K_1 \in \mathcal{K}_{FD}$ *and* $K_2 \in \mathcal{K}_{GB}$ *with* $K \subseteq K_1 + K_2$ *and* $V_1(K_2) < \varepsilon$. *Then* $K \in \mathcal{K}_{GC}$.

Proof. For arbitrary $t_0 \in K$, it must be shown that $V_1(K \cap B(t_0, \tilde{\varepsilon})) \to 0$ as $\tilde{\varepsilon} \to 0$. Assume that K, ε, K_1, K_2 are as described, and assume also (without loss of generality: effected by a translation if necessary) that $\boldsymbol{t}_0 = \mathbf{0} \in K \cap K_1 \cap K_2$. We orthogonalize the inclusion $K \subseteq K_1 + K_2$ as follows: let π_{K_1} stand for projection onto the subspace spanned by vectors in K_1 and $\overline{\pi}_{K_1}$ for projection onto the complementary subspace. Since $K_2 \subseteq \pi_{K_1} K_2 + \overline{\pi}_{K_1} K_2$, $K \subset [K_1 + \pi_{K_1} K_2] + \overline{\pi}_{K_1} K_2$, where again the first summand in brackets is finite-dimensional and the second satisfies $V_1(\overline{\pi}_{K_1} K_2) \leq V_1(K_2) < \varepsilon$. Additionally, the two now reside in orthogonal subspaces. This implies the following inclusion upon intersection with the ball $B(\mathbf{0}, \tilde{\varepsilon})$:

$$K \cap B(\mathbf{0}, \tilde{\varepsilon}) \subset \{[K_1 + \pi_{K_1} K_2] \cap B(\mathbf{0}, \tilde{\varepsilon})\} + \{\overline{\pi}_{K_1} K_2 \cap B(\mathbf{0}, \tilde{\varepsilon})\}.$$

Applying $V_1(\cdot)$ gives

$$\begin{aligned} V_1(K \cap B(\mathbf{0}, \tilde{\varepsilon})) &\leq V_1([K_1 + \pi_{K_1} K_2] \cap B(\mathbf{0}, \tilde{\varepsilon})) + V_1(\overline{\pi}_{K_1} K_2 \cap B(\mathbf{0}, \tilde{\varepsilon})) \\ &\leq V_1([K_1 + \pi_{K_1} K_2] \cap B(\mathbf{0}, \tilde{\varepsilon})) + \varepsilon. \end{aligned}$$

As $\tilde{\varepsilon} \to 0$, the first quantity on the right tends to zero since $\mathcal{K}_{FD} \subset \mathcal{K}_{GC}$. Since $\varepsilon > 0$ was arbitrary, it follows that $\lim_{\tilde{\varepsilon}\to 0} V_1(K \cap B(\mathbf{0}, \tilde{\varepsilon})) = 0$, and this completes the proof.

Proof of Theorem 5. The proof is in two parts, showing (i) that $\{\mathcal{K}_{FD}, \rho_{V_1}\}$ is dense in $\{\mathcal{K}_{GC}, \rho_{V_1}\}$ and (ii) that $\{\mathcal{K}_{GC}, \rho_{V_1}\}$ is complete.

(i) Suppose that $K \in \mathcal{K}_{GC}$. Without loss of generality, $\mathbf{0} \in K$. An argument by contradiction will show that $\rho_{V_1}(\pi_n K, K) \to 0$ as $n \to \infty$. Suppose that this is not the case. It is easy to see that $\pi_n K$ and K are equalized by $L_n = \overline{\mathrm{conv}}(\overline{\pi}_n K, -\overline{\pi}_n K)$. This implies that $\rho_{V_1}(\pi_n K, K) \leq V_1(L_n) \leq V_1(\overline{\pi}_n K) + V_1(-\overline{\pi}_n K) = 2V_1(\overline{\pi}_n K) < \infty$. By Proposition 1, it must be the case that, as $n \to \infty$, $V_1(\overline{\pi}_n K) = (2\pi)^{1/2} \mathrm{E} \sup_{t\in K} \sum_{n+1}^\infty t_i Z_i$ tends to a constant c, where, by hypothesis, c is strictly positive.

There is a halfspace bisection procedure for localizing this phenomenon: given a unit vector $\boldsymbol{e}$, let $\alpha = \frac{1}{2}(\min_{t\in K}\langle \boldsymbol{t}, \boldsymbol{e}\rangle + \max_{t\in K}\langle \boldsymbol{t}, \boldsymbol{e}\rangle)$, and define $K', K'' \in \mathcal{K}_{\mathrm{GC}}$ by $K' = K \cap \{\boldsymbol{x} \mid \langle \boldsymbol{x}, \boldsymbol{e}\rangle \le \alpha\}$, $K'' = K \cap \{\boldsymbol{x} \mid \alpha \le \langle \boldsymbol{x}, \boldsymbol{e}\rangle\}$. By the same reasoning as above, $V_1(\overline{\pi}_n K') \to c' \ge 0$ and $V_1(\overline{\pi}_n K'') \to c'' \ge 0$. Further, since $\overline{\pi}_n K = \overline{\mathrm{conv}}(\overline{\pi}_n K', \overline{\pi}_n K'')$, we have $\sup_{t\in\overline{\pi}_n K}\langle \boldsymbol{t}, \boldsymbol{Z}\rangle = \max\{\sup_{t\in\overline{\pi}_n K'}\langle \boldsymbol{t}, \boldsymbol{Z}\rangle, \sup_{t\in\overline{\pi}_n K''}\langle \boldsymbol{t}, \boldsymbol{Z}\rangle\}$. Lemma 1 implies that this equation has the almost sure limiting form $c = \max\{c', c''\}$, that is, $c' = c$ and/or $c'' = c$. There is a program of such bisections yielding a nested decreasing sequence of sets $K \supseteq K^{(j)} \supseteq K^{(j+1)} \supseteq \cdots$ such that, for each j, $V_1(K^{(j)}) = c > 0$ and $\mathrm{diam}(K^{(j)}) \to 0$. By the compactness of K, $\boldsymbol{t}^* = \bigcap_j K^{(j)}$ exists, and the bisection construction implies that $V_1(K \cap B(\boldsymbol{t}^*, \varepsilon)) \not\to 0$ as $\varepsilon \searrow 0$. This however contradicts the assumption that $K \in \mathcal{K}_{\mathrm{GC}}$.

(ii) To show that $\{\mathcal{K}_{\mathrm{GC}}, \rho_{V_1}\}$ is complete, assume that the sequence $\{K_j\}$ is ρ_{V_1}-Cauchy. Then it is also ρ_{H}-Cauchy and, since $(\mathcal{K}, \rho_{\mathrm{H}})$ is complete, there is convergence $K_j \overset{\rho_H}{\to} K$ for some $K \in \mathcal{K}$. We show that $K \in \mathcal{K}_{\mathrm{GC}}$ and that $K_j \overset{\rho_{V_1}}{\to} K$. Begin by extracting a subsequence of indices $j_1 < j_2 < \cdots$ increasing sufficiently fast so that $\rho_{V_1}(K_{j_i}, K_{j_{i+1}}) < 2^{-i}$. It follows that there are $L_1, L_2, \ldots \in \mathcal{K}_{\mathrm{GB}}$ with $V_1(L_i) < 2^{-i}$ and such that $K_{j_i}, K_{j_{i+1}}$ are equalized by L_i. Since $\mathbf{0} \in L_i$ for each i, $L_m \subseteq L_m^* = \sum_m^\infty L_i \in \mathcal{K}_{\mathrm{GB}}$ with $V_1(L_m^*) \le 2^{1-m}$. It follows that for $m < n$, K_{j_m}, K_{j_n} are equalized by L_m^*. Letting $n \to \infty$ then yields that K_{j_m}, K are equalized by L_m^*. Putting these facts together implies that the sequence $\{K_{j_i}\}$, and hence the original sequence $\{K_j\}$, converges ρ_{V_1} to K. By part (i), n can be taken sufficiently large so that $K_{j_m} \subseteq \pi_n K_{j_m} + \overline{\pi}_n K_{j_m}$, where $V_1(\overline{\pi}_n K_{j_m})$ is arbitrarily small. It follows that $K_{j_m} \subseteq \pi_n K_{j_m} + [\overline{\pi}_n K_{j_m} + L_m^*]$, and by Lemma 2, $K \in \mathcal{K}_{\mathrm{GC}}$.

Remarks. (i) The last argument can be modified in an obvious way to show that $\{\mathcal{K}_{\mathrm{GB}}, \rho_{V_1}\}$ is complete.

(ii) The fact that $\rho_{V_1}(\pi_n K, K) \to 0$ for $K \in \mathcal{K}_{\mathrm{GC}}$ is folklore; a proof for balanced convex bodies was given in [21]. From there, we have used the martingale idea in a slightly different fashion.

5. Continuous functionals on $\{\mathcal{K}_{\mathrm{GB}}, \rho_{V_1}\}$

In this section, we show that continuous functionals on $\{\mathcal{K}_{\mathrm{GB}}, \rho_{V_1}\}$ include the intrinsic volumes, the Wills functional, and a related vector functional, the Steiner point. A bound for the kernel of the Wills functional is the starting point:

Theorem 6. *For $K_1, K_2 \in \mathcal{K}_{\mathrm{GB}}$,*

$$\begin{aligned}\mathrm{E}\left|\exp\left(\sup_{t\in K_2}\{Y_t - \tfrac{1}{2}\sigma_t^2\}\right) - \exp\left(\sup_{s\in K_1}\{X_s - \tfrac{1}{2}\sigma_s^2\}\right)\right| \\ \le [Q(K_1) + Q(K_2)]\rho_{V_1}(K_1, K_2)\exp(2\rho_{V_1}^2(K_1, K_2)),\end{aligned} \tag{4}$$

where $Q(K) = 2[2 + \sigma_K]\exp(1 + V_1(K) + \sigma_K^2)$.

Proof. Use the isonormal representations $X_s = \langle \boldsymbol{s}, \boldsymbol{Z}\rangle$, $Y_t = \langle \boldsymbol{t}, \boldsymbol{Z}\rangle$ and, for $K \in \mathcal{K}_{\mathrm{GB}}$, set $\Psi(\boldsymbol{Z}, K) = \sup_{t\in K}[\langle \boldsymbol{t}, \boldsymbol{Z}\rangle - \frac{1}{2}\|\boldsymbol{t}\|^2]$, so that the aim is to bound $\mathrm{E}\,|e^{\Psi(\boldsymbol{Z},K_2)} - e^{\Psi(\boldsymbol{Z},K_1)}|$. Suppose that $L \in \mathcal{K}_{\mathrm{GB}}$ equalizes K_1, K_2. Then

$$e^{\Psi(\boldsymbol{Z},K_2)} - e^{\Psi(\boldsymbol{Z},K_1)} \le e^{\Psi(\boldsymbol{Z},K_1+L)} - e^{\Psi(\boldsymbol{Z},K_1)},$$
$$e^{\Psi(\boldsymbol{Z},K_1)} - e^{\Psi(\boldsymbol{Z},K_2)} \le e^{\Psi(\boldsymbol{Z},K_2+L)} - e^{\Psi(\boldsymbol{Z},K_2)},$$

so that it suffices to bound $\sum_{i=1}^{2} \mathrm{E}[\mathrm{e}^{\Psi(Z,K_i+L)} - \mathrm{e}^{\Psi(Z,K_i)}]$. Treating one of the terms and omitting subscripts, let $J = \mathrm{E}[\mathrm{e}^{\Psi(Z,K+L)} - \mathrm{e}^{\Psi(Z,K)}]$. It is elementary that $\mathrm{e}^{\Psi(Z,K+L)} - \mathrm{e}^{\Psi(Z,K)} \le [\Psi(Z, K+L) - \Psi(Z, K)]\,\mathrm{e}^{\Psi(Z,K+L)}$. An application of the Cauchy–Schwarz inequality provides

$$[\langle t + w, Z\rangle - \tfrac{1}{2}\|t + w\|^2] - [\langle t, Z\rangle - \tfrac{1}{2}\|t\|^2] \le \langle w, Z\rangle + \|t\|\,\|w\|,$$

which, upon rearrangement and taking suprema, yields $\Psi(Z, K + L) - \Psi(Z, K) \le \sup_{w\in L}\langle w, Z\rangle + \sigma_K\sigma_L$. This is bounded above by $[\sup_{w\in L}\langle w, Z\rangle - \mathrm{E}\sup_{w\in L}\langle w, Z\rangle]_+ + \mathrm{E}\sup_{w\in L}\langle w, Z\rangle + \sigma_K\sigma_L$, and in turn $[\sup_{w\in L}\langle w, Z\rangle - \mathrm{E}\sup_{w\in L}\langle w, Z\rangle]_+ + (1/(2\pi)^{1/2} + \sigma_K) \times V_1(L)$ where we have used $\sigma_L \le V_1(L)$. Applying the Cauchy–Schwarz inequality again together with Jensen's inequality twice gives

$$\begin{aligned}
J^2 &\le \mathrm{E}\bigg\{\Big[\sup_{w\in L}\langle w, Z\rangle - \mathrm{E}\sup_{w\in L}\langle w, Z\rangle\Big]_+ + \Big(\frac{1}{\sqrt{2\pi}} + \sigma_K\Big) V_1(L)\bigg\}^2 \mathrm{E}\,\mathrm{e}^{2\Psi(Z,K+L)} \\
&\le 2\,\mathrm{E}\bigg\{\Big[\sup_{w\in L}\langle w, Z\rangle - \mathrm{E}\sup_{w\in L}\langle w, Z\rangle\Big]_+^2 + \Big(\frac{1}{\sqrt{2\pi}} + \sigma_K\Big)^2 V_1^2(L)\bigg\} \mathrm{E}\,\mathrm{e}^{2\Psi(Z,K+L)} \\
&\le 2\,\mathrm{E}\bigg\{\Big[\sup_{w\in L}\langle w, Z\rangle - \mathrm{E}\sup_{w\in L}\langle w, Z\rangle\Big]_+^2 + \Big(\frac{1}{\pi} + 2\sigma_K^2\Big) V_1^2(L)\bigg\} \mathrm{E}\,\mathrm{e}^{2\Psi(Z,K+L)}. \qquad (5)
\end{aligned}$$

Then Theorem 4 gives

$$\begin{aligned}
\mathrm{E}\Big[\sup_{w\in L}\langle w, Z\rangle - \mathrm{E}\sup_{w\in L}\langle w, Z\rangle\Big]_+^2 &= \int_0^\infty \mathrm{P}\Big[\Big(\sup_{w\in L}\langle w, Z\rangle - \mathrm{E}\sup_{w\in L}\langle w, Z\rangle\Big)_+^2 > x\Big]\,\mathrm{d}x \\
&= \int_0^\infty \mathrm{P}\Big[\Big(\sup_{w\in L}\langle w, Z\rangle - \mathrm{E}\sup_{w\in L}\langle w, Z\rangle\Big)_+ > \sqrt{x}\Big]\,\mathrm{d}x \\
&\le \int_0^\infty \mathrm{e}^{-x/2\sigma_L^2}\,\mathrm{d}x = 2\sigma_L^2 \le 2V_1^2(L),
\end{aligned}$$

so that the first expectation in (5) is bounded above by $(2 + 1/\pi + 2\sigma_K^2)V_1^2(L)$, or (loosely) $4[2 + \sigma_K]^2 V_1^2(L)$. To estimate the expected exponential in (5),

$$\begin{aligned}
2\Psi(Z, K + L) &= 2 \sup_{t\in K, w\in L} [\langle t + w, Z\rangle - \tfrac{1}{2}\|t + w\|^2] \\
&= \sup_{t\in K, w\in L} [2\langle t + w, Z\rangle - 2\|t + w\|^2 + \|t + w\|^2] \\
&\le \sup_{t\in K, w\in L} [2\langle t + w, Z\rangle - 2\|t + w\|^2] + \sup_{t\in K, w\in L} \|t + w\|^2 \\
&\le \sup_{t\in K, w\in L} [2\langle t + w, Z\rangle - 2\|t + w\|^2] + [\sigma_K + \sigma_L]^2 \\
&\le \sup_{t\in K, w\in L} [2\langle t + w, Z\rangle - 2\|t + w\|^2] + 2[\sigma_K^2 + \sigma_L^2] \\
&\le \sup_{t\in K, w\in L} [2\langle t + w, Z\rangle - 2\|t + w\|^2] + 2[\sigma_K^2 + V_1^2(L)].
\end{aligned}$$

A previous estimate ([23], [28]) implies that

$$\begin{aligned}\mathrm{E}\,\mathrm{e}^{2\Psi(Z,K+L)} &\le \exp\left\{2\,\mathrm{E}\sup_{t\in K,w\in L}\langle \boldsymbol{t}+\boldsymbol{w},\boldsymbol{Z}\rangle + 2[\sigma_K^2+V_1^2(L)]\right\}\\ &\le \exp\left\{\sqrt{\frac{2}{\pi}}[V_1(K)+V_1(L)]+2[\sigma_K^2+V_1^2(L)]\right\}\\ &\le \exp\{2[1+V_1(K)+\sigma_K^2+2V_1^2(L)]\}.\end{aligned}$$

Putting these bounds together and taking a square root yields

$$J \le 2[2+\sigma_K]\,\mathrm{e}^{1+V_1(K)+\sigma_K^2}\,V_1(L)\,\mathrm{e}^{2V_1^2(L)}.$$

Summing for $K = K_1, K_2$ and minimizing the result over equalizing L yields (4).

Continuity properties are now straightforward to show (for an alternate formulation using different methods, see [8, Proposition 2.4.1]).

Theorem 7. *The Wills functional and all intrinsic volumes are continuous on* $(\mathcal{K}_{\mathrm{GB}}, \rho_{V_1})$.

Proof. Letting $K_1 = K$ and $K_2 = K_n$ in (4), note that $Q(K_n)$ remains bounded as $\rho_{V_1}(K_n, K) \to 0$ since $K_n \subseteq K + L$ implies that $V_1(K_n) \le V_1(K) + V_1(L)$ and $\sigma_{K_n} \le \sigma_K + \sigma_L \le \sigma_K + V_1(L)$. Then

$$\begin{aligned}|W(K_n)-W(K)| &= \left|\mathrm{E}\exp\left\{\sup_{t\in K_n}[Y_t-\tfrac{1}{2}\sigma_t^2]\right\} - \mathrm{E}\exp\left\{\sup_{t\in K}[X_t-\tfrac{1}{2}\sigma_t^2]\right\}\right|\\ &\le \mathrm{E}\left|\exp\left\{\sup_{t\in K_n}[Y_t-\tfrac{1}{2}\sigma_t^2]\right\} - \exp\left\{\sup_{t\in K}[X_t-\tfrac{1}{2}\sigma_t^2]\right\}\right|\end{aligned}$$

and (4) imply the continuity of $W(\cdot)$. For the intrinsic volumes, we show a somewhat stronger assertion: suppose first that $K_1, K_2 \in \mathcal{K}_{\mathrm{GB}}$ and that L equalizes them. Then, for each $j = 1, 2, \dots$, $V_j(K_2) - V_j(K_1) \le V_j(K_1 + L) - V_j(K_1)$ and the corresponding relation with K_1 and K_2 reversed. It follows that $|V_j(K_2) - V_j(K_1)| \le [V_j(K_1+L) - V_j(K_1)] + [V_j(K_2+L) - V_j(K_2)]$. Multiplying this bound by $(2\pi)^{-j/2}$, summing over j, and noting (1) yields

$$\sum_{j=0}^{\infty}(2\pi)^{-j/2}|V_j(K_2)-V_j(K_1)| \le [W(K_1+L)-W(K_1)]+[W(K_2+L)-W(K_2)].$$

As in the proof of Theorem 6, the infimum of the right-hand side over all L equalizing K_1, K_2 is majorized by the bound (4).

We conclude this section with a related result. Several years ago the author was first introduced to this topic by a question of Zvi Artstein about high dimensional behavior of the Steiner point, which is a vector analogue of intrinsic volumes and a natural centroid for finite-dimensional convex bodies. At that time, a negative result was shown: there is a sequence of (finite-dimensional) convex bodies that is ρ_{H}-convergent to an infinite-dimensional body and such that the associated sequence of Steiner points does not converge. Indeed, one can arrange to have *any* point of the limit body as the limit of such a sequence of Steiner points ([25]). In present terms, this flexibility is afforded by the limit body's not being GB. By contrast, we have now the following positive result.

Theorem 8. *The Steiner point map* $s : \mathcal{K}_{\mathrm{GB}} \to \ell^2$ *is* ρ_{V_1}*-Lipschitz continuous.*

Proof. Recall the Gaussian interpretation $s(K) = \mathrm{E}[h_K(\boldsymbol{Z})\boldsymbol{Z}]$ ([4], [25], [29], [30]). For a unit vector $\boldsymbol{u}$ and $K_1, K_2 \in \mathcal{K}_{\mathrm{GB}}$,

$$\langle s(K_2) - s(K_1), \boldsymbol{u}\rangle = \mathrm{E}[h_{K_2}(\boldsymbol{Z}) - h_{K_1}(\boldsymbol{Z})]\langle \boldsymbol{u}, \boldsymbol{Z}\rangle, \tag{6}$$

so that the Cauchy–Schwarz inequality implies that the square of (6) is bounded above by $\mathrm{E}\,|h_{K_2}(\boldsymbol{Z}) - h_{K_1}(\boldsymbol{Z})|^2$ (uniformly in $\boldsymbol{u}$). Then it is enough to show (more generally) that, for $p \geq 1$, there is a constant c_p such that

$$\mathrm{E}\,|h_{K_2}(\boldsymbol{Z}) - h_{K_1}(\boldsymbol{Z})|^p \leq c_p \rho_{V_1}^p(K_1, K_2) \tag{7}$$

(for a related metric, see [26]). This follows from a Khintchine–Kahane bound (e.g. [5, Section 3.2]): suppose that K_1, K_2 are equalized by L. Then, for all $\boldsymbol{x}$, $|h_{K_2}(\boldsymbol{x}) - h_{K_1}(\boldsymbol{x})| \leq h_L(\boldsymbol{x})$, so that $\mathrm{E}\,|h_{K_2}(\boldsymbol{Z}) - h_{K_1}(\boldsymbol{Z})|^p \leq \mathrm{E}\,h_L^p(\boldsymbol{Z})$. Now

$$\begin{aligned}
\mathrm{E}\,h_L^p(\boldsymbol{Z}) &= \int_0^\infty pr^{p-1}\,\mathrm{P}[h_L(\boldsymbol{Z}) \geq r]\,\mathrm{d}r = \int_0^{\mathrm{E}\,h_L(\boldsymbol{Z})} + \int_{\mathrm{E}\,h_L(\boldsymbol{Z})}^\infty \\
&\leq \int_0^{\mathrm{E}\,h_L(\boldsymbol{Z})} pr^{p-1}\,\mathrm{d}r + \int_{\mathrm{E}\,h_L(\boldsymbol{Z})}^\infty pr^{p-1}\,\mathrm{e}^{-r^2/(2\sigma_L^2)}\,\mathrm{d}r \\
&\leq [\mathrm{E}\,h_L(\boldsymbol{Z})]^p + \int_0^\infty p(\sigma_L s + \mathrm{E}\,h_L(\boldsymbol{Z}))^{p-1}\,\mathrm{e}^{-s^2/2}\sigma_L\,\mathrm{d}s \\
&\leq \left[\frac{V_1(L)}{\sqrt{2\pi}}\right]^p + [V_1(L)]^p \int_0^\infty p(s + 1/\sqrt{2\pi})^{p-1}\,\mathrm{e}^{-s^2/2}\,\mathrm{d}s.
\end{aligned}$$

Minimizing over L yields (7) with $c_p = (1/(2\pi)^{1/2})^p + \int_0^\infty p(s + 1/(2\pi)^{1/2})^{p-1}\,\mathrm{e}^{-s^2/2}\,\mathrm{d}s$.

6. Future directions and speculation

As the foregoing sections indicate, important properties of a Gaussian process are mirrored in its intrinsic volumes. Toward a deeper theory (as well as for a new source of estimates, bounds, etc.), we can ask how far this can be pushed, i.e. for the nature of equivalence classes of Gaussian processes reduced modulo equality of intrinsic volumes. This seems to be a difficult question at present, but, as a first step, a characterization of the class of intrinsic volume sequences is relevant; some progress along these lines will appear elsewhere. There is a sense as well in which the treatment of this paper has been 'in the mean'. There is, for example, the following intriguing combination of formulas from Section 2

$$\mathrm{E}\exp\left\{\sup_{t\in K}[X_t - \tfrac{1}{2}\sigma_t^2]\right\} = \mathrm{E}\left[\sum_{j=0}^\infty \frac{1}{\omega_j j!}\mathrm{vol}_j(X_K^{j*})\right],$$

and we can ask if there is any sense in which the expectations can be dropped.

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