On unique determination of convex polytopes

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Dedication

To my daughter Masha

Geometric tomography

One of central questions in geometric tomography:
unique determination of convex bodies from some measurements

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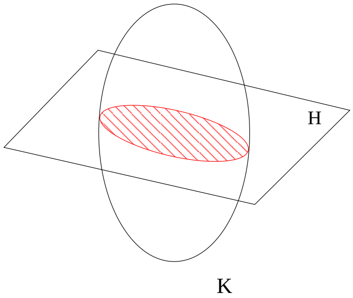
- Sections
- Projections
- Other lower dimensional data

Well-known classical result:



K, L origin-symmetric star bodies in $\mathbb{R}^n$ such that

$$\mathrm{vol}_{n-1}(K \cap H) = \mathrm{vol}_{n-1}(L \cap H)$$

for every central hyperplane H.
Then

$$K = L.$$

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Uniqueness for general (not necessarily symmetric) convex bodies:

- Groemer: half-sections
- Falconer and Gardner: hyperplane sections through two points in the interior of the body
- many others...

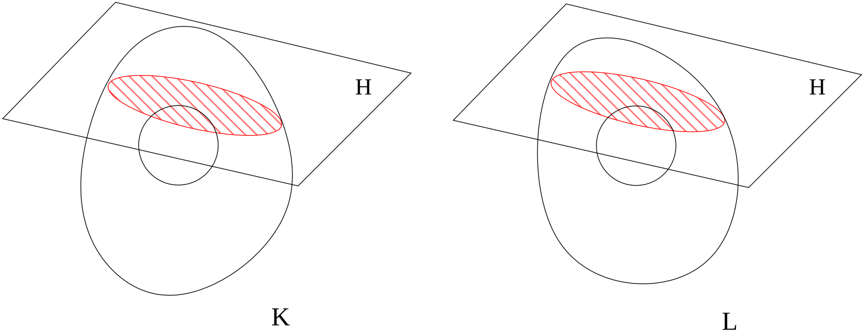
Question (Barker and Larman, 2001)
Let K and L be convex bodies in $\mathbb{R}^n$ containing a sphere of radius t in their interiors. Suppose that for every hyperplane H tangent to the sphere we have

$$\mathrm{vol}_{n-1}(K \cap H) = \mathrm{vol}_{n-1}(L \cap H).$$

Does this mean that $K = L$?

Problem # 1

$\mathbb{R}^3$



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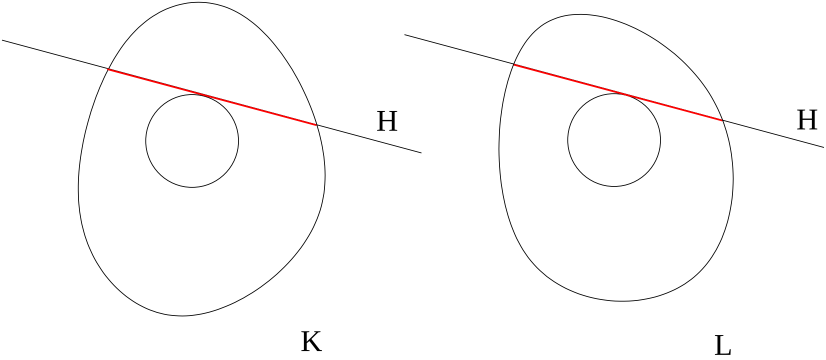
$\mathbb{R}^2$



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- Xiong, Ma and Cheung, 2008:
  - uniqueness holds for convex polygons in $\mathbb{R}^2$

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Theorem (V.Y.)

Let P and Q be convex polytopes in $\mathbb{R}^n$ containing a sphere of radius t in their interiors. If

$$\mathrm{vol}_{n-1}(P \cap H) = \mathrm{vol}_{n-1}(Q \cap H)$$

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Remark

No symmetry is assumed.

Problem # 2

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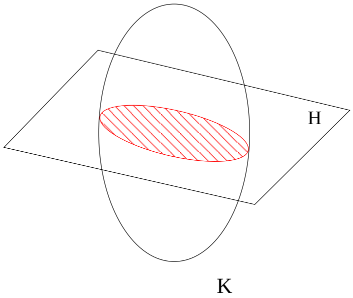

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for every central hyperplane H.
Then

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Problem # 2

Problem (Gardner, "Geometric Tomography")

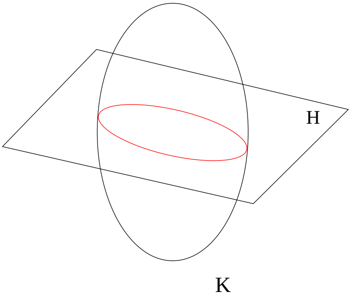
Let P and Q be origin-symmetric convex bodies in $\mathbb{R}^3$ such that

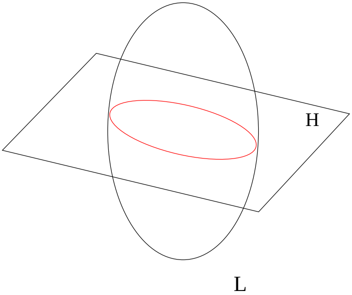
$$L(P \cap H) = L(Q \cap H)$$

for every plane H through the origin, where L is the length of the corresponding boundary curve. Is it true that

$$P = Q?$$

Problem # 2





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Some known results:

- Howard, Nazarov, Ryabogin and Zvavitch: uniqueness in the class of C^1 star bodies of revolution
- Rusu: settled an infinitesimal version of the problem, when one of the bodies is the Euclidean ball and the other is its one-parameter analytic deformation.

Problem # 2

Theorem (V.Y.)

Let $2 \leq k \leq n-1$ and suppose that P and Q are origin-symmetric convex polytopes in $\mathbb{R}^n$, $n \geq 3$, such that

$$S(P \cap H) = S(Q \cap H)$$

for every subspace $H \in G(n,k)$. Then

$$P = Q.$$

Here, $S(\cdot)$ denotes the $(k-1)$-dimensional area of the boundary surface of the corresponding k-dimensional body.

Proof

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To prove $P = Q$, we will show that P and Q have exactly the same vertices.

Suppose the contrary. Assume that P and Q are different, i.e. they have different sets of vertices.

Case 1. There is a vertex u of, say, P such that the line through the origin and the vertex u does not contain any vertices of Q.
Case 2. All vertices of P and Q lie on the same lines, i.e. if a line through the origin contains a vertex of one of the polytopes, then it also contains a vertex of the other.

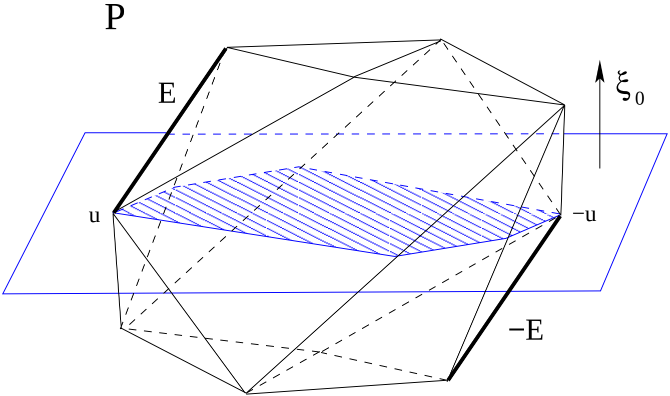
Proof

CASE 1. There is a vertex u of, say, P such that the line through the origin and the vertex u does not contain any vertices of Q.

Let E be any $(n-2)$-face of P adjacent to the vertex u. There exists $\xi_0 \in S^{n-1}$ such that

1) $\xi_0^\perp \cap E = \{u\}$,
2) $\xi_0^\perp$ contains no vertices of either P or Q (other than u, $-u$).

Proof



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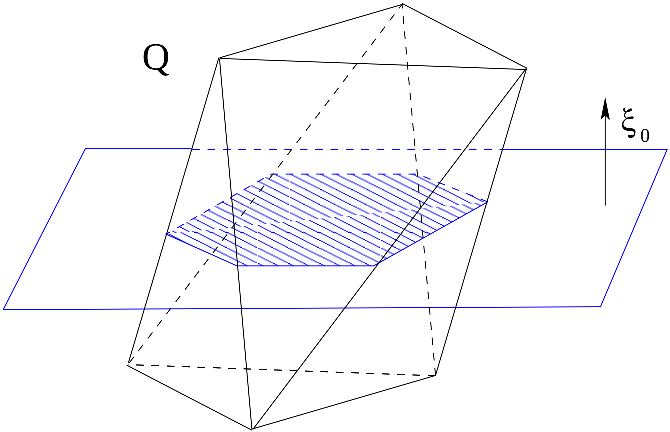

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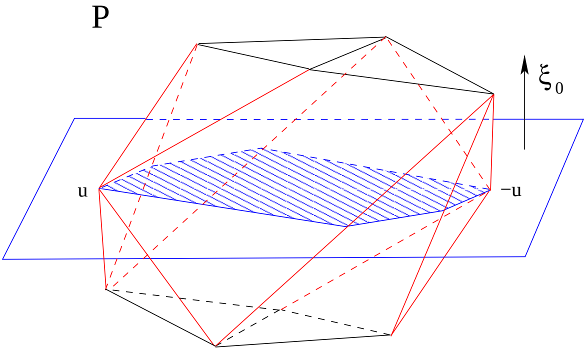
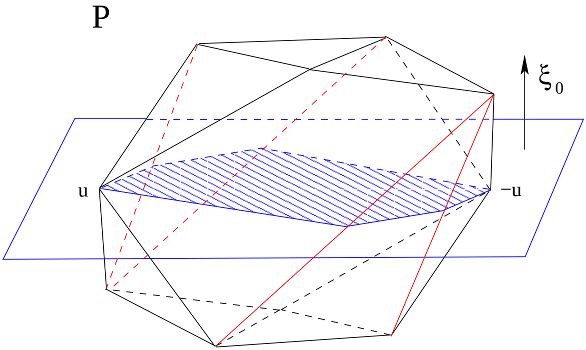


The edges of P that intersect the plane $\xi_0^\perp$ are denoted by

$$x = u_i + l_i s_i, \qquad i \in I_1 \cup I_2 \cup I_3,$$

where u_i is a vertex that belongs to the edge, l_i is the direction of the edge, s_i is a parameter.



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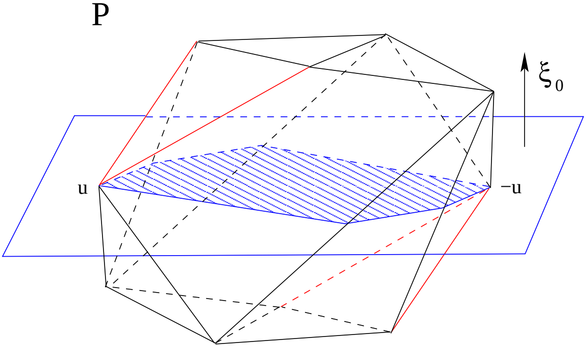

i) $i \in I_1$ if the edge is not adjacent to u or $-u$,
ii) $i \in I_2$ if the edge is adjacent to u and lies in ξ_0^+, or adjacent to $-u$ and lies in ξ_0^-,

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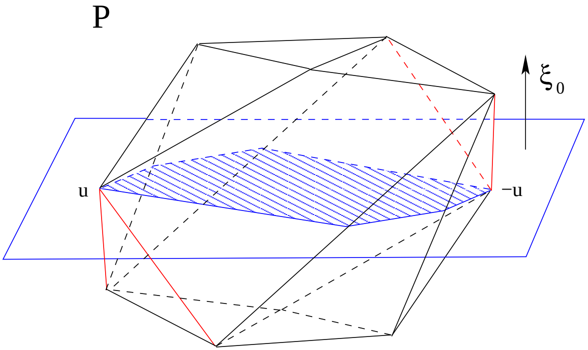

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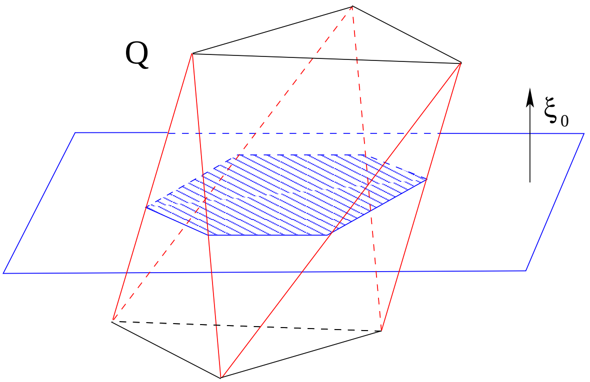


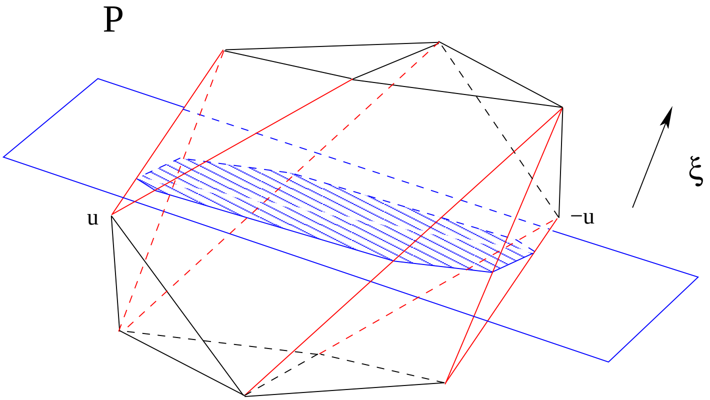
The edges of Q that intersect the plane $\xi_0^\perp$ we denote by

$$x = v_i + m_i t_i, \qquad i \in J.$$

where v_i, m_i, t_i are correspondingly a point on the edge, its direction and parameter along the edge.

Let Λ be a spherical cap centered at ξ_0. We assume that the radius of Λ is small enough to guarantee that for all $\xi \in \Lambda$ the plane $\xi^\perp$ contains no vertices of P and Q, except possibly u and $-u$.

Proof



Denote by Λ_+ the subset of those vectors $\xi \in \Lambda$ for which the plane $\xi^\perp$ lies "above" u.

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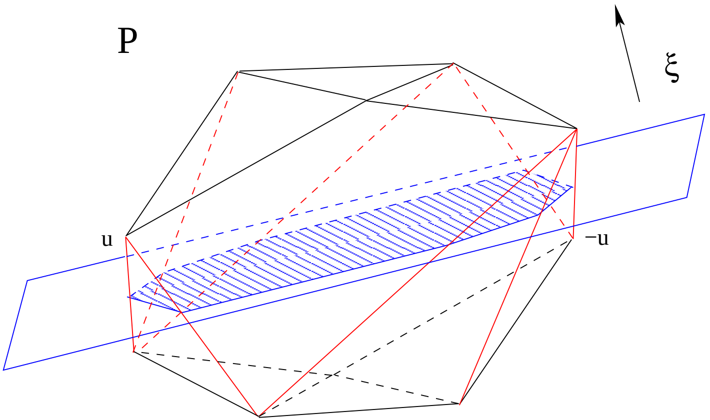

Denote by Λ_- the subset of those vectors $\xi \in \Lambda$ for which the plane $\xi^\perp$ lies "below" u.

Proof

The i-th edge of P is given by $x = u_i + l_i s_i$, so it intersects $\xi^\perp$ at the point

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The points of intersection of the edges of Q and the plane $\xi^\perp$ are given by

$$q_i = v_i - m_i \frac{\langle v_i, \xi \rangle}{\langle m_i, \xi \rangle}.$$

Proof

The $(n-2)$-dimensional surface area of $P \cap \xi^\perp$ is given by

$$S(P \cap \xi^\perp) = \sum_j \mathrm{vol}_{n-2}(F_j \cap \xi^\perp),$$

where the sum is taken over all facets F_j of P that have nonempty intersection with $\xi^\perp$.

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where the sum is taken over all facets F_j of P that have nonempty intersection with $\xi^{\perp}$.

In order to compute the latter surface area, we will fix a triangulation of each $(n-2)$-dimensional polytope $F_j \cap \xi^{\perp}$.

Proof

First, in each facet F_j consider an auxiliary segment $x = \omega_j + \nu_j \tau_j$ with the properties $\langle \nu_j, \xi \rangle \neq 0$, $\xi \in \Lambda$, and ν_j is not parallel to E.

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The point of intersection of the auxiliary segment with the plane $\xi^\perp$ is

$$z_j = \omega_j - \nu_j \frac{\langle \omega_j, \xi \rangle}{\langle \nu_j, \xi \rangle}.$$

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ii) Take convex hulls of z_j and simplices from Step (i).

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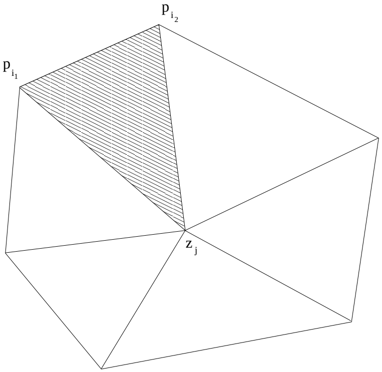

Proof

Now we write the $(n-2)$-dimensional area of $F_j \cap \xi^\perp$ as the sum of the areas of simplices in its triangulation.

If a simplex in this triangulation has vertices $z_j, p_{i_1}, \dots, p_{i_{n-2}}$, then its area is equal to the determinant

$$\frac{1}{(n-2)!\sqrt{1-\langle n_j, \xi\rangle^2}} \cdot |p_{i_1} - z_j, p_{i_2} - z_j, \dots, p_{i_{n-2}} - z_j, n_j, \xi|.$$

Here, n_j is the unit outward normal to the facet F_j.

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Similarly we triangulate the boundary of $Q \cap \xi^\perp$ and compute its surface area.

Proof

We will write

$$S(P \cap \xi^\perp) = S_+(P \cap \xi^\perp) + \tilde{S}(P \cap \xi^\perp), \text{ if } \xi \in \Lambda_+,$$

and

$$S(P \cap \xi^\perp) = S_-(P \cap \xi^\perp) + \tilde{S}(P \cap \xi^\perp), \text{ if } \xi \in \Lambda_-,$$

where S_+ (respectively, S_-) is the total area of the simplices in the boundary of $P \cap \xi^\perp$ that have at least one vertex p_i with index $i \in I_2$ (respectively, I_3), and $\tilde{S}$ is the total area of all other simplices.

Note that $\tilde{S}(P \cap \xi^\perp)$ has the same formula for both Λ_+ and Λ_-.

Proof

Since $S(P \cap \xi^\perp) = S(Q \cap \xi^\perp)$ for all $\xi \in \Lambda$, we have

$$S_+(P \cap \xi^\perp) + \tilde{S}(P \cap \xi^\perp) = S(Q \cap \xi^\perp)$$

for $\xi \in \Lambda_+$, and

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for $\xi \in \Lambda_-$.
Lemma. We can assume that these equalities hold for all $\xi \in S^{n-1}$ without finitely many great subspheres and finitely many points.

Proof

Since $S(Q \cap \xi^{\perp})$ is given by the same formula for both $\xi \in \Lambda_+$ and $\xi \in \Lambda_-$, we have

$$S_+(P \cap \xi^{\perp}) + \tilde{S}(P \cap \xi^{\perp}) = S_-(P \cap \xi^{\perp}) + \tilde{S}(P \cap \xi^{\perp}),$$

that is

$$S_+(P \cap \xi^{\perp}) = S_-(P \cap \xi^{\perp})$$

for all $\xi \in S^{n-1}$ except finitely many great subspheres and finitely many points, i.e. except the set where the denominators vanish.

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Define

$$\eta = \alpha n_1 + \beta n_2,$$

where α, $\beta > 0$, $\alpha^2 + \beta^2 = 1$, are to be chosen.

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For a small enough ϵ, consider the following curve on the sphere:

$$\xi(\epsilon) = \frac{\eta + \epsilon\lambda}{|\eta + \epsilon\lambda|},$$

where λ is a (properly chosen) vector.

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Now put $\xi(\epsilon)$ into the equality

$$S_+(P \cap \xi^\perp) = S_-(P \cap \xi^\perp),$$

multiply both sides by ϵ^{n-2}, and send $\epsilon \to 0$.

Proof

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Thus we have

$$\left(\pm\frac{\beta}{\sqrt{1-\langle n_1,\eta\rangle^2}}\pm\frac{\alpha}{\sqrt{1-\langle n_2,\eta\rangle^2}}\right) \times \sum \frac{\langle u,\eta\rangle^{n-2}}{\langle l_{i_1},\lambda\rangle\cdots\langle l_{i_{n-2}},\lambda\rangle}\left|l_{i_1},\dots,l_{i_{n-2}},n_1,n_2\right| = 0. \quad (1)$$

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Moreover, we can assume that η and λ are chosen such that
i) $\langle l_{i_k},\lambda\rangle > 0$,
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We can also show that all the determinants have the same sign.
Therefore, if we choose $\alpha \neq \beta$, then the left-hand side of (1) is nonzero.

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Moreover, we can assume that η and λ are chosen such that
i) $\langle l_{i_k},\lambda\rangle > 0$,
ii) $\langle u,\eta\rangle \neq 0$.
We can also show that all the determinants have the same sign.
Therefore, if we choose $\alpha \neq \beta$, then the left-hand side of (1) is nonzero. Contradiction.

Proof

Case 2. All vertices of P and Q come in pairs, that is if a line through the origin contains a vertex of one of the polytopes, then it also contains a vertex of the other.

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Lemma.

There exist a vertex u of P, a corresponding vertex v of Q lying on the same line and on the same side with respect to the origin, and an $(n-2)$-face E of, say, P adjacent to u that is not parallel to any $(n-2)$-face of Q adjacent to v.

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Now, having fixed the face E, proceed as in Case 1.

THANK YOU!!!