On the comparison of volumes of quantum states

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August 6, 2010

"Perspectives in High Dimensions"
at Case Western Reserve University, Cleveland, OH

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What is the relative size of the set of separable and entangled quantum states within the set of quantum states in terms of some measure?

Density Matrix

Complex Hilbert space $\mathcal{H} = \mathbb{C}^{D_1} \otimes \mathbb{C}^{D_2} \cdots \otimes \mathbb{C}^{D_n}$ with complex dimension $N = D_1 \cdots D_n$. Each factor of $\mathcal{H}$ system corresponds to a subsystem of $\mathcal{H}$.

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The dimension of $\mathcal{D}$ is $d = N^2 - 1$.

Separable quantum states

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$$\mathcal{S} = conv\{\rho_1 \otimes \cdots \otimes \rho_n, \rho_i \in \mathcal{D}(\mathbb{C}^{D_i})\}.$$

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What is the probability of $\mathcal{S}$ and $\mathcal{E}$ in $\mathcal{D}$?

Let $\mathcal{H} = \mathbb{C}^{D_1} \otimes \mathbb{C}^{D_2}$.

Peres-Horodecki Positive Partial Transpose Criterion

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Peres-Horodecki PPT Criterion

Let $\mathcal{PPT} = \{\rho \in \mathcal{D}(\mathcal{H}) : \ s.t. \ T_1(\rho) \geq 0\}$. Then

$$\mathcal{S} \subset \mathcal{PPT} \subset \mathcal{D}.$$

That is, a separable state must satisfy positive partial transpose criterion. Equivalently, a non-PPT state must be entangled.

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How precise is the Peres-Horodecki PPT criterion (as a tool to detect the separability)?

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where $d\nu$ is some measure on Δ_1: the chamber of the simplex (of eigenvalues) $\{(\lambda_1, \cdots, \lambda_N) : \sum_i \lambda_i = 1, \lambda_i \geq 0\}$ with order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$, and $d\gamma$ is the measure on the flag manifold (of eigenvectors) $\mathcal{F}^N = \mathcal{U}(N)/[\mathcal{U}(1)]^N$:

$$d\gamma = \prod_{i<j}^{1\cdots N} 2Re(U^{-1}dU)_{ij} Im(U^{-1}dU)_{ij}$$

where dU is the variation of unitary matrix U such that $U + dU$ is also an unitary matrix.

- The Hilbert-Schmidt measure:

$$dV_{HS} = \sqrt{N} \prod_{i<j}^{1\cdots N} (\lambda_i - \lambda_j)^2 \delta_0(\sum_i \lambda_i - 1) \prod_{i=1}^{N} d\lambda_i \, d\gamma;$$

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- Bures measure:

$$dV_B = \frac{2^{\frac{2-N-N^2}{2}}}{\sqrt{\lambda_1 \cdots \lambda_N}} \prod_{i<j}^{1\cdots N} \frac{(\lambda_i - \lambda_j)^2}{\lambda_i + \lambda_j} \delta_0(\sum_i \lambda_i - 1) \prod_{i=1}^{N} d\lambda_i \, d\gamma;$$

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- Induced measure by partial trace on $\mathcal{H}_N \otimes \mathcal{H}_K$ $(K \geq N)$:

$$dV_{N,K} = \prod_{i=1}^{N} \lambda_i^{K-N} \prod_{i<j}^{1\cdots N} (\lambda_i - \lambda_j)^2 \delta_0(\sum_i \lambda_i - 1) \prod_{i=1}^{N} d\lambda_i \, d\gamma;$$

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- the α-volume $(\alpha > 0)$:

$$dV_\alpha = \prod_{i=1}^{N} \lambda_i^{\alpha-1} \prod_{i<j}^{1\cdots N} (\lambda_i - \lambda_j)^2 \delta_0(\sum_i \lambda_i - 1) \prod_{i=1}^{N} d\lambda_i \, d\gamma.$$

Measure induced by Metric on $\mathcal{D}$

dV_{HS} is induced by Hilbert-Schmidt distance

$$d_{HS}(\rho, \sigma) = \|\rho - \sigma\|_2 = \sqrt{tr(\rho - \sigma)^\dagger(\rho - \sigma)}.$$

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dV_B is induced by Bures distance

$$d_B(\rho,\sigma) = \sup_{\{\mathbb{E}_i\}} \left(\sum_i \left[\sqrt{tr(\mathbb{E}_i\rho)} - \sqrt{tr(\mathbb{E}_i\sigma)} \right]^2 \right)^{\frac{1}{2}}$$

where the supremum runs over all the POVM (positive operator valued measurement) $\{\mathbb{E}_i\}$, that is, for some $1 \le k \le N$,

$$\sum_{i=1}^{k} \mathbb{E}_i = Id_N, \text{ and } \mathbb{E}_i = \mathbb{E}_i^\dagger, \ \mathbb{E}_i \ge 0, \ i = 1, \cdots, k.$$

Induced measure by partial trace

Let $K \geq N$, and ρ be a density matrix on $\mathcal{H}_N \otimes \mathcal{H}_K$, then

$$\rho = \begin{pmatrix} A_{11} & \cdots & A_{1N} \\ \vdots & \ddots & \vdots \\ A_{N1} & \cdots & A_{NN} \end{pmatrix}.$$

Define the partial trace over $\mathcal{H}_K$ as

$$\rho^A = Tr_B(\rho), \text{ where } (\rho^A)_{ij} = tr(A_{ij}) \text{ for } i, j = 1, \cdots, N.$$

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The partial trace process allows us to view states on $\mathcal{H}_N$ as a (pure) state on (much higher) dimensional space $\mathcal{H}_N \otimes \mathcal{H}_K$. Then the measures induced by partial trace may be considered as a projection of the $(NK - 1)$ dimensional simplex of eigenvalues into simplex of $(N - 1)$ dimension.

Comparison of α-volume with Hilbert-Schmidt volume

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$$\mathrm{VR}_\alpha(\mathcal{K}, \mathcal{D}) = \left(\frac{V_\alpha(\mathcal{K})}{V_\alpha(\mathcal{D})} \right)^{1/d}$$

and the Hilbert-Schmidt volume radii ratio of $\mathcal{K}$ to $\mathcal{D}$ is $\xi = \mathrm{VR}_{HS}(\mathcal{K}, \mathcal{D}) = \mathrm{VR}_1(\mathcal{K}, \mathcal{D})$.

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D. Ye, (J. Phys. A: Math. Theor. 2010)

Let $\alpha > 0$ be a (fixed) constant, $\alpha_{\max} = \max\{1, \alpha\}$, and $\alpha_{\min} = \min\{1, \alpha\}$. There exist universal constants $c_1, C_1 > 0$, s.t.,

$$c_1 \mathrm{VR}_{HS}(\mathcal{K}, \mathcal{D})^{\alpha_{\max}} \exp\left(\frac{(1 - \alpha_{\max}) \ln \ln(e/\xi)}{N^2 - 1} \right) \leq \mathrm{VR}_\alpha(\mathcal{K}, \mathcal{D})$$
$$\leq C_1 \, \mathrm{VR}_{HS}(\mathcal{K}, \mathcal{D})^{\alpha_{\min}} \, \exp\left(\frac{(1 - \alpha_{\min}) \ln \ln(e/\xi)}{N^2 - 1} \right).$$

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There exist universal constants $c_2, C_2 > 0$, such that

$$c_2 \ \zeta^{\max\{1, \frac{1}{2\alpha}\}} \exp\left(\frac{(1 - \max\{1, \frac{1}{2\alpha}\}) \ln\ln(e/\zeta)}{N^2 - 1} \right) \leq \mathrm{VR}_B(\mathcal{K}, \mathcal{D})$$

$$\leq C_2 \ \zeta^{\min\{\frac{1}{2}, \frac{1}{2\alpha}\}} \exp\left(\frac{\ln\ln(e/\zeta)}{2N} \right).$$

- $\alpha = 1$: D. Ye, (Journal of Mathematical Physics (JMP), 2009).

Large number of small subsystems

Let $\mathcal{H} = (\mathbb{C}^D)^{\otimes n}$ with (small) D, and $\alpha_D = \frac{\log_D(1+\frac{1}{D})}{2} - \frac{\log_D(D+1)}{2D^2}$.

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There exist universal constants $c_3, c_3', C_3, C_3' > 0$, s.t., for $\alpha > 0$,

$$\frac{c_3}{N^{1/2+\alpha_D}} \leq \mathrm{VR}_B(\mathcal{S},\mathcal{D}) \leq C_3 \sqrt{\frac{(Dn\ln n)^{1/2}}{N^{1/2+\alpha_D}}};$$

$$\left(\frac{c_3'}{N^{1/2+\alpha_D}}\right)^{\max\{1,\alpha\}} \leq \mathrm{VR}_\alpha(\mathcal{S},\mathcal{D}) \leq C_3' \left(\frac{(Dn\ln n)^{1/2}}{N^{1/2+\alpha_D}}\right)^{\min\{1,\alpha\}}.$$

G. Aubrun and S. J. Szarek proved:

$$\frac{\tilde{c}_3}{N^{1/2+\alpha_D}} \leq \mathrm{VR}_{HS}(\mathcal{S},\mathcal{D}) \leq \tilde{C}_3 \frac{(Dn\ln n)^{1/2}}{N^{1/2+\alpha_D}}.$$

Small number of large subsystems

Let $\mathcal{H} = (\mathbb{C}^D)^{\otimes n}$ with (small) n.

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There exist universal constants $c_4, c_4', C_4, C_4' > 0$, s.t., for $\alpha > 0$,

$$\frac{c_4^n}{N^{1/2-1/(2n)}} \leq \mathrm{VR}_B(\mathcal{S}, \mathcal{D}) \leq C_4 \sqrt{\frac{(n \ln n)^{1/2}}{N^{1/2-1/(2n)}}};$$

$$\left(\frac{c_4'^n}{N^{1/2-1/(2n)}}\right)^{\max\{1,\alpha\}} \leq \mathrm{VR}_\alpha(\mathcal{S}, \mathcal{D}) \leq C_4' \left(\frac{(n \ln n)^{1/2}}{N^{1/2-1/(2n)}}\right)^{\min\{1,\alpha\}}.$$

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Peres-Horodecki PPT becoming imprecise as a tool to detect separability for large N

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There exist absolute constants $c_0, c_0' > 0$ such that for any bipartite system $\mathcal{H} = \mathbb{C}^D \otimes \mathbb{C}^D$,

$$c_0 \leq \mathrm{VR}_B(\mathcal{PPT}, \mathcal{D}) \leq 1;$$
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Conclusion: $\mathrm{VR}_B(\mathcal{S}, \mathcal{PPT})$ and $\mathrm{VR}_\alpha(\mathcal{S}, \mathcal{PPT})$ go to 0, hence, the Peres-Horodecki PPT criterion is not precise as a tool to detect separability for large N.

Let $\mathcal{H} = (\mathbb{C}^D)^{\otimes n}$. The α-probability of $\mathcal{S}$ in $\mathcal{D}$ is defined as

$$\mathbb{P}_\alpha(\mathcal{S}, n, D) =: \frac{V_\alpha(\mathcal{S})}{V_\alpha(\mathcal{D})}.$$

$\mathbb{P}_2(\mathcal{S}, 8, 2) \leq 2.1 \times 10^{-1595}$, $\mathbb{P}_2(\mathcal{S}, 5, 3) \leq 1.52 \times 10^{-5301}$,
$\mathbb{P}_{0.5}(\mathcal{S}, 8, 2) \leq 8.8 \times 10^{-479}$, $\mathbb{P}_{0.5}(\mathcal{S}, 5, 3) \leq 9.5 \times 10^{-2351}$.

The conditional α-probability of $\mathcal{S}$ given $\mathcal{PPT}$ is

$$\mathbb{P}_\alpha(\mathcal{S}|\mathcal{PPT}, n, D) =: \frac{V_\alpha(\mathcal{S})}{V_\alpha(\mathcal{PPT})}.$$

$\mathbb{P}_{1.1}(\mathcal{S}|\mathcal{PPT}, 12, 2) \leq 2.5 \times 10^{-2721940}$,
$\mathbb{P}_{1.1}(\mathcal{S}|\mathcal{PPT}, 8, 3) \leq 1.82 \times 10^{-12248770}$.

1. D. Ye, *On the Bures volume of separable quantum states.* J. Math. Phys. 50 (2009) 083502;
2. D. Ye, *On the comparison of volumes of quantum states.* J. Phys. A: Math. Theor., 43 (2010) 315301 (17pp).

Thank you!