# CRITICAL POINTS OF HYPERBOLIC CUBIC POLYNOMIALS

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Abstract. A theorem of Bôcher and Grace states that the critical points of a cubic polynomial are the foci of an ellipse tangent to the sides of the triangle joining the zeros. We prove an analogous result for hyperbolic cubic polynomials, that is, for Blaschke products with three roots in the unit disc.

## 1. Introduction

The well-known theorem of Lucas [6] states that the critical points of a polynomial in the complex plane lie within or on the convex hull of the zeros. But more is known about the critical points. A remarkable theorem of Bôcher [1] and Grace [4] relates the position of the zeroes of a cubic polynomial in the plane to the position of the critical points. It states:

**Theorem 1.1.** *The critical points of a cubic polynomial $P(z)$ are the foci of an ellipse $E$ which is tangent to the midpoints of the three line segments joining the roots of $P(z)$. More generally, the zeroes of the function $F(z) = \sum_1^3 m_i(z - z_i)^{-1}$ are the foci of the conic that touches the line segments $(z_1, z_2)$, $(z_2, z_3)$, and $(z_3, z_1)$ in points which divide these segments in the ratios $m_1 : m_2$, $m_2 : m_3$, and $m_3 : m_1$, respectively.*

The first assertion follows from the second when $F(z)$ is chosen to be the logarithmic derivative $\frac{P'(z)}{P(z)}$, in which case $m_i = 1,\ 1 \leq i \leq 3$. The theorem is actually a special case of a theorem first proved by Siebeck [10] which states:

**Theorem 1.2.** *The zeros of the function $F(z) = \sum_1^p \frac{m_i}{z-z_i}$ are the foci of the curve of class $p - 1$ which touches each line segment $(z_i, z_j)$ in a point dividing the line segment in the ratio $m_i : m_j$.*

For the proof of these theorems, see Marden [7], pp. 7–11.

*Remark* 1.3. Given a polynomial $F(z)$, we can form a family $F_a(z) = F(z) - a$ of polynomials with the same critical points. It follows then that one can replace "zeros" with "preimages of $a$" in Theorem (1.1). One can show that there is another ellipse consisting of sets of preimages such that the ellipse $E$ is the envelope of the family of line segments joining pairs $(z, w)$ with $P(z) = P(w)$. We have found an elementary proof of the Bôcher-Grace theorem based on this observation.

We will discuss an analogous result in the hyperbolic plane. The notion of a *non-Euclidean polynomial* seems to be due to Walsh [12]. He uses this term for a

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function of the form function of the form

$$P(z) = \lambda \prod_{1}^{n} \frac{z - \alpha_k}{1 - \overline{\alpha}_k z}, \quad |\lambda| = 1, \quad |\alpha_k| < 1 \tag{1.1}$$

This is the general form for a rational function which takes the closed unit disc $\overline{D}$ to itself, and it is usually referred to as a *finite Blaschke product.* $P(z)$ is an $n$-to-one map of $\overline{D}$ onto itself, has precisely $n$ zeros in the interior $D$, and has modulus unity on $C\colon |z| = 1$. We may regard $D$ as the hyperbolic plane and the unit circle $C$ as the set of ideal points, using either the Poincaré model or the Klein model of the hyperbolic plane. Then it is reasonable to think of $P(z)$ as a polynomial, and indeed it has exactly $n-1$ critical points (counting multiplicity) in $D$, with the remaining critical points in the exterior of the disc, symmetric with respect to inversion in the circle. There is an analogue to Lucas's theorem:

**Theorem 1.4** (Walsh, [12], p. 157)**.** *Let $P(z)$ be defined by (1.1). The critical points of $P(z)$ in the interior of the disc lie within or on the (non-Euclidean) convex hull of the zeroes of $P(z)$, with respect to the Poincaré metric.*

The goal of this paper is to describe an analogue of the theorem of Bôcher and Grace for non-Euclidean cubic polynomials. A closely related result is already known; Daepp, Gorkin and Mortini proved ([2]) that if $P(z)$ is cubic and one of the zeros of $P(z)$ is located at the origin, then the other two zeros are the foci of an ellipse which is inscribed in any (Euclidean) triangle whose vertices are the pre-images of a complex number $\lambda$ of modulus 1. We will show:

**Theorem 1.5** (Main Theorem)**.** *Let $P(z) = \beta \prod_1^3 \frac{z-\alpha_k}{1-\overline{\alpha}_k z}, \quad |\beta| = 1, \quad |\alpha_k| < 1$ be a cubic non-Euclidean polynomial, and let $\gamma$ be the curve in $D$ which is the envelope of the non-Euclidean geodesics joining pairs of points $w_i, w_j$ on $C$ satisfying $P(w_i) = P(w_j)$. Then $\gamma$ is a non-Euclidean ellipse whose foci are the critical points of $P(z)$ in $D$.*

## 2. Blaschke Products

We will work with the space $\mathcal{P}$ of NE (for non-Euclidean) cubic polynomials and the group $G$ of isometries of the hyperbolic plane. The elements of $G$ are the fractional linear transformations which preserve the unit disc. We will consider the domain and range of functions in $\mathcal{P}$ to be the closed disc. An element $P$ of $\mathcal{P}$ is given by

$$P(z) = \mu \frac{z-a}{1-\overline{a}z} \frac{z-b}{1-\overline{b}z} \frac{z-c}{1-\overline{c}z}, \quad |\mu| = 1, \quad |a|, |b|, |c| < 1 \tag{2.1}$$

An element of $G$ is given by

$$g(z) = \beta \frac{z-\alpha}{1-\overline{\alpha}z}, \quad |\beta| = 1, \quad |\alpha| < 1 \tag{2.2}$$

$G$ acts on $\mathcal{P}$ on the left by $L_g(P)(z) = g(P(z))$ and on the right by $R_g(P)(z) = P(g^{-1})(z)$.

The following propositions summarize the behavior of NE polynomials under the left and right actions.

**Proposition 2.1.** *The right action satisfies the following properties:*

1. *If $z_1$ and $z_2$ are the critical points of $P$, then the critical points of $R_g(P)$ are $g(z_1)$ and $g(z_2)$.*

(2) *If $\lambda_1$, $\lambda_2$ and $\lambda_3$ are the three pre-images of a point $z$ under $P$, then $g(\lambda_1)$, $g(\lambda_2)$ and $g(\lambda_3)$ are the preimages of $z$ under $R_g(P)$.*
(3) *In particular, if $a$, $b$ and $c$ are the zeros of $P$, then $g(a)$, $g(b)$ and $g(c)$ are the zeros of $R_g(P)$.*

**Corollary 2.2.** *Let $\gamma = \gamma_P$ be the curve in $D$ formed as the envelope of the non-Euclidean geodesics joining pairs of points $w_i, w_j$ on $C$ satisfying $P(w_i) = P(w_j)$. Then the corresponding envelope $\gamma_{R_g(P)}$ of $R_g(P)$ is the image curve $g(\gamma)$. Thus the two envelopes are isometric as curves in the Poincaré disc.*

**Proposition 2.3.** *The left action satisfies the following properties:*

(1) *If $z_1$ and $z_2$ are the critical points of $P$, then the critical points of $L_g(P)$ are $z_1$ and $z_2$.*
(2) *If $\lambda_1$, $\lambda_2$ and $\lambda_3$ are the three pre-images of a point $z$ under $P$, then $\lambda_1$, $\lambda_2$ and $\lambda_3$ are the preimages of $g(z)$ under $L_g(P)$.*
(3) *In particular, if $a$, $b$ and $c$ are the zeros of $P$, then the zeros of $L_g(P)$ are the preimages of $g^{-1}(0)$.*

**Corollary 2.4.** *The envelope curve $\gamma$ is invariant under the left action of $G$.*

We will prove that the curve $\gamma_P$ is a non-Euclidean ellipse with foci at the critical points of $P$. Corollary (2.2) shows that to prove this for a polynomial $P$ we need only prove it for some polynomial $R_g(P)$. Similarly, Corollary (2.4) ensures that we may act on the left by any element of $G$. This allows us to put $P$ in a canonical form.

Using the right action of $G$, we may move the two critical points of $P$ so that they lie on the real axis at $\pm\epsilon$, $0 \leq \epsilon < 1$. The value of $\epsilon$ is determined by the fact that $g$ is an isometry; the NE distance between the two critical points is an invariant.

The conditions on $P$ insuring that the critical points are at $\pm\epsilon$ are determined by rewriting $P$ in *un-factored* form:

$$P(z) = -\mu \frac{z^3 - r_2 z^2 + r_1 z - r_0}{\overline{r_o} z^3 - \overline{r_1} z^2 + \overline{r_2} z - 1} \tag{2.3}$$

The critical point equation $P'(z) = 0$ is the self-inversive equation:

$$\begin{aligned}(\overline{r_0} r_2 - \overline{r_1}) z^4 + (2\overline{r_2} - 2 r_1 \overline{r_0}) z^3 + (-3 - |r_2|^2 + |r_1|^2 + 3|r_0|^2) z^2 \\ + (2r_2 - 2\overline{r_1} r_0) z + (r_0 \overline{r_2} - r_1) = 0\end{aligned} \tag{2.4}$$

The four roots of this equation are supposed to be $\pm\epsilon$ and $\pm\frac{1}{\epsilon}$. This means that the coefficient of $z^3$ must vanish, or $r_2 = \overline{r_1} r_0$. Substituting this into equation (2.4) and simplifying, we get:

$$\overline{r_1} z^4 + (3 - |r_1|^2) z^2 + r_1 = 0, \qquad |r_1| < 1 \tag{2.5}$$

In order that the roots of (2.5) are real, $r_1$ must be real and negative. Letting $r_1 = -s^2$, we have:

**Proposition 2.5.** *Every NE cubic polynomial is right equivalent to one of the form:*

$$P(z) = \mu\frac{z^3 - s^2 r_0 z^2 - s^2 z + r_0}{\overline{r_0} z^3 - s^2 z^2 - s^2\overline{r_0} z + 1}, \qquad 0 \le s < 1, \quad |r_0| < 1, \quad \mu\overline{\mu} = 1$$

Next, we apply the left action. If $P(z)$ is given in the form described in proposition 2.5 and $g(z) = \frac{z-\alpha}{1-\overline{\alpha}z}$, then a straightforward (but tedious) computation shows

$$L_g(P)(z) = g(P(z)) = \nu\frac{z^3 - s^2 r z^2 - s^2 z + r}{\overline{r} z^3 - s^2 z^2 - s^2\overline{r} z + 1} \tag{2.6}$$

where

$$\nu = \frac{\mu - \alpha\overline{r_0}}{1 - \overline{\alpha}\mu r_0} = h(\mu) \text{ for } h(z) = \frac{z - \alpha\overline{r_0}}{1 - \overline{\alpha} r_0 z} \tag{2.7}$$

and

$$r = \frac{\mu r_0 - \alpha}{\mu - \alpha\overline{r_0}} = \frac{r_0 - \alpha\overline{\mu}}{1 - \overline{r_0}\alpha\overline{\mu}} = k(\alpha\overline{\mu}) \text{ for } k(z) = \frac{z - r_0}{\overline{r_0} z - 1} \tag{2.8}$$

As $\alpha$ varies over $D$, so does $\alpha\overline{\mu}$ and so, by formula (2.8), does $r = k(\alpha\overline{\mu})$. It follows that by choosing appropriate $\alpha$ we may assume that $r = 0$. This completes the proof of

**Theorem 2.6** (Normal Form). *Every element of $\mathcal{P}$ is equivalent, under the two-sided action of $G$, to an element of the form*

$$P(z) = \mu\frac{z^3 - s^2 z}{1 - s^2 z^2}, \qquad 0 \le s < 1, \quad \mu\overline{\mu} = 1$$

Note that the left action left the critical points at $\pm\epsilon$ while moving the zeros of $P$ to 0, $s$, and $-s$, where, by (2.5), $\epsilon$ and $s$ are related by the formula

$$s^2\epsilon^4 - (3 - s^4)\epsilon^2 + s^2 = 0 \tag{2.9}$$

## 3. The Non-Euclidean ellipse

The ellipse in the hyperbolic plane is not a totally familiar object. Using the projective model of the hyperbolic plane as the lines interior to the absolute conic $< X, X >= -x_0^2 + x_1^2 + x_2^2 = 0$ or the Klein model of the hyperbolic plane $x^2 + y^2 < 1$, given by $x = \frac{x_1}{x_0}, y = \frac{x_2}{x_0}$, it can be described as, respectively, a conic interior to the absolute conic (see [9]),or a compact quadratic curve given in orthogonal coordinates as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a < 1, b < 1$ (see [11]). Schilling proved that the ellipse defined in this manner is the locus of a point the sum of whose hyperbolic distances from the two foci are constant.

To see this explicitly, we may use the distance formula in the hyperbolic plane ([5], p. 75):

$$\cosh d(A, B) = \frac{1 - A\overline{B}}{\sqrt{1 - A\overline{A}}\sqrt{1 - B\overline{B}}}, \qquad A, B \in D \tag{3.1}$$

Let $A = x + iy$ and $B_\pm = \pm\delta$. Then the equation of an ellipse with foci at $B_\pm$ is

$$C = \cosh(d(A,B_+) + d(A,B_-)) \tag{3.2}$$
$$= \cosh(d(A,B_+))\cosh(d(A,B_-)) + \sinh(d(A,B_+))\sinh(d(A,B_-))$$
$$= \frac{1-\delta^2x^2}{(1-r^2)(1-\delta^2)} + \frac{\sqrt{(1+\delta^2x^2-(1-r^2)(1-\delta^2))^2-4\delta^2x^2}}{(1-r^2)(1-\delta^2)}$$

where $r^2 = x^2 + y^2$. Rearranging terms,

$$C(1-r^2)(1-\delta^2) - (1-\delta^2x^2) \tag{3.3}$$
$$= \sqrt{(1-\delta^2x^2)^2 - 2(1+\delta^2x^2)(1-r^2)(1-\delta^2) + (1-r^2)^2(1-\delta^2)^2}$$

Upon squaring, cancelling terms and dividing out the common factors, we get

$$(C^2-1)(1-r^2)(1-\delta^2) = (2C-2) - (2C+2)\delta^2x^2 \tag{3.4}$$

This can now be written in the standard form:

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1, \quad \alpha^2 = \frac{C-1}{C+1}, \quad \beta^2 = \frac{(C-1)-\delta^2(C+1)}{(C+1)(1-\delta^2)} \tag{3.5}$$

Eliminating $C$, we get the equation

$$\beta^2 = \frac{\alpha^2-\delta^2}{(1-\delta^2)} \tag{3.6}$$

Viewed as a Euclidean ellipse, this curve has foci at $\pm s$, where

$$s^2 = \alpha^2 - \beta^2 = \frac{\delta^2(1-\alpha^2)}{1-\delta^2} \tag{3.7}$$

The Euclidean foci $\pm s$ are closer to the center than the non-Euclidean foci $\pm\delta$.

Consider the NE polynomial in normal form given in Theorem 2.6. This has zeros at 0 and $\pm s$. Since one of the roots of $P$ is at 0, we may now apply this result to the theorem of Daepp, Gorkin and Mortini [2] which states that the envelope of the Euclidean line segments joining preimages of points on the circle is the ellipse

$$|z-s| + |z+s| = 1 + s^2 \tag{3.8}$$

or equivalently

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1, \quad \alpha = \frac{1+s^2}{2}, \quad \beta = \frac{1-s^2}{2} \tag{3.9}$$

Comparing equation (3.11) with equation (3.9),

$$s^2 = \frac{\delta^2(3-2s^2-s^4)}{4(1-\delta^2)} \tag{3.10}$$

and so

$$\delta^2 = \frac{4s^2}{3+2s^2-s^4} \tag{3.11}$$

Thus we have the corollary:

**Corollary 3.1.** *Let* $P(z) = \mu\frac{z^3-s^2z}{1-s^2z^2}, \qquad 0 \le s < 1, \quad \mu\overline{\mu} = 1$. *Then the curve* $\eta$ *which is the envelope of the family of (Euclidean) line segments joining preimages of points on the unit circle is a NE ellipse with NE foci at the points* $\pm\delta$ *with* $\delta^2 = \frac{4s^2}{3+2s^2-s^4}$

## 4. Proof of the Main Theorem

We have been considering the disc $D$ as having the Klein metric, for which geodesics are Euclidean straight line segments. The result we want must be formulated with respect to the Poincaré metric. The relationship can be described as follows. If we consider the complex plane as the boundary of upper half space $\mathbb{H}^3$ with the Poincaré metric $ds^2 = (dx_1^2 + dx_2^2 + dx_3^2)/x_3^2$, then the *unit hemisphere model* for the hyperbolic plane is given by the unit hemisphere $x_3 = \sqrt{1 - x_1^2 - x_2^2}$ with the induced metric. (See [8], page 191) The Klein model, which Milnor refers to as the *projective disc model*, is gotten by ignoring the $x_3$ - coordinate. On the other hand, the Poincaré model is achieved by stereographic projection of the unit hemisphere onto the disc from the south pole $(0, 0, -1)$. We define a canonical map relating these two models.

**Definition 4.1. The Klein-to-Poincaré map** $KP : D \longrightarrow D$ is defined by

$$KP(z) = \frac{z}{1 + \sqrt{1 - |z|^2}}$$

The Klein-to-Poincaré map is the unique isometry from the disc $D$ with the Klein metric to the disc with the Poincaré metric which keeps the ideal boundary pointwise fixed. It takes the straight line segment between points on the boundary to the NE geodesic between the same two points. It therefore takes the envelope described in Corollary 3.1 to the non-Euclidean envelope $\gamma$.

Applying the Klein-to-Poincaré map to the focus $\delta$ of the curve $\eta$, we get the equation for the focus $\epsilon$ of $\gamma$:

$$\epsilon = KP(\delta) = \frac{\sqrt{3 + 2s^2 - s^4} - \sqrt{3 - s^2 - s^4}}{2s} \tag{4.1}$$

Thus

$$\epsilon^2 = \frac{3 - s^4 - \sqrt{(3 - s^4)^2 - 4s^4}}{2s^2} \tag{4.2}$$

which in turn means that $\epsilon^2$ is a root of the equation $s^2x^2 - (3 - s^4)x + s^2 = 0$. But this is precisely formula (2.9), and thus the NE foci of the envelope $\gamma$ are the critical points of $P(z)$, at least for $P(z)$ in normal form. Since both the envelope and the critical points of $P(z)$ are preserved under isometries, the proof of Theorem 1.5 is complete.

Daepp, Gorkin and Martini also proved:

**Theorem 4.2.** *Let* $B(z) = \frac{z(z-a)}{1-\overline{a}z}$ *be a Blaschke product with* $a \neq 0$. *Then if* $z_1$ *and* $z_2$ *are two points on the unit circle satisfying* $B(z_1) = B(z_2)$, *then the line segment joining them passes through* $a$.

Applying the map $KP$ to such a map one can verify that the NE line segment joining $z_1$ and $z_2$ passes through the critical point of $B$, $\frac{a}{1+\sqrt{1-a^2}}$. Since any NE quadratic polynomial can be transformed into the form $B$, it follows that:

**Corollary 4.3.** *The critical point of a NE quadratic polynomial is the point of intersection of the NE geodesics joining preimages of points on the unit circle.*

It is natural to conjecture that the critical points of the general NE polynomial of degree $p$ are the "non-Euclidean foci" of curve of class $p-1$; that is, there should be a non-Euclidean analogue of Siebeck's theorem. A difficulty to overcome would be to determine what such a curve should be. Thus one must define what "foci" and "class" mean in this context.

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