Elastic Curves and Rods

Curve Straightening

The manifold of closed curves

Let

$$\Omega = \{\gamma : [0,1] \longrightarrow M \mid \left\|\gamma'\right\| \equiv \ell \neq 0, \gamma'' \in L^2\}$$

$$\Lambda = \{\gamma \in \Omega \mid \gamma(0) = \gamma(1), \gamma'(0) = \gamma'(1)\}$$

Λ is a Hilbert manifold (using, for instance,

$$\|\gamma\|_2^2 = \gamma(0)^2 + \gamma'(0)^2 + \int_0^1 \left\|\gamma''\right\|^2 ds$$

$\mathcal{F}^\lambda : \Lambda \longrightarrow \mathbb{R}$ is defined by

$$\mathcal{F}^\lambda(\gamma) = \frac{1}{2}\int_\gamma k^2 + \lambda \; ds$$

A critical point of $\mathcal{F}^0$ with contrained length is a critical point of $\mathcal{F}^\lambda$ for some λ (Lagrange multiplier). λ may be thought of as a *length penalty*. Thus if $\lambda > 0$ arbitrarily long curves will have high $\mathcal{F}^\lambda$ values. Very short curves have high $\int k^2 ds$.

$\mathcal{F}^\lambda$ is a smooth function on a Hilbert manifold, so it defines a flow via the negative gradient, called Curve-Straightening.

Curve Straightening

Theorem. *If $\lambda > 0$, then $\mathcal{F}^\lambda$ satisfies the Palais-Smale condition (C). Therefore, the trajectories of $-\nabla\mathcal{F}^\lambda$ converge to critical points (or at least have critical points as adherence points). Furthermore the minimax principle and Morse theory hold for $\mathcal{F}^\lambda$.*

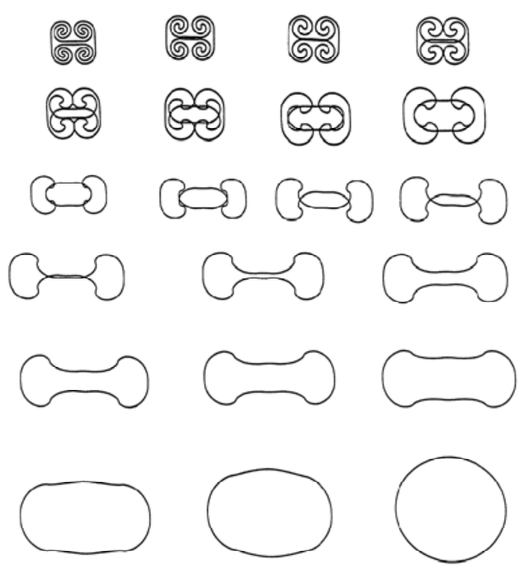
The theorem applies for M compact, and with modification for space forms.

The theorem also applies to spaces of curves of a fixed length.

Example: In $\mathbb{R}^2$, the only closed elastic curves are coverings of circles and figure-eight curves. There is precisely one critical point for $\mathcal{F}^\lambda$ ($\lambda > 0$) of rotation index $n \neq 0$; curve-straightening takes any closed curve of rotation index n to the n - fold circle. (This demonstrates the Whitney-Graustein theorem). In rotation index 0, the n - fold coverings of the figure eight curve are all critical points.

There are no critical points for $\lambda = 0$, because dilation reduces total squared curvature. Curve-straightening cannot satisfy the Palais-Smale condition in this case, and curves will expand to infinite length.

*Example from A. Linner, Some properties of the curve straightening flow in the plane, Trans. Amer. Math. Soc. **314** (1989), 605–617.



Recall the classification theorem of elastic curves in $\mathbb{R}^3$: For each pair of relatively prime integers (m, n) with $m > 2n$ there is a unique closed elastic curve (up to congruence)which lies on an embedded torus of revolution and represents an (m, n) torus knot.

Using the Palais-Smale condition one can prove more:

Theorem. *Let p and q be a pair of relatively prime integers with $0 < p \leq q$. Let $\mathcal{G}$ be the group of rotations around the z-axis generated by rotation through angle $\theta = \frac{2\pi p}{p+q}$. Then there is a non-circular closed elastica $\gamma_{p+q,p}$ which is $\mathcal{G}$-symmetric and $\mathcal{G}$-regularly homotopic to the p-fold circular elastica. It is a minimax critical point of total squared curvature (and hence unstable). It is only planar when $p = q$, in which case it is the figure-eight elastica.*

The Hyperbolic Plane

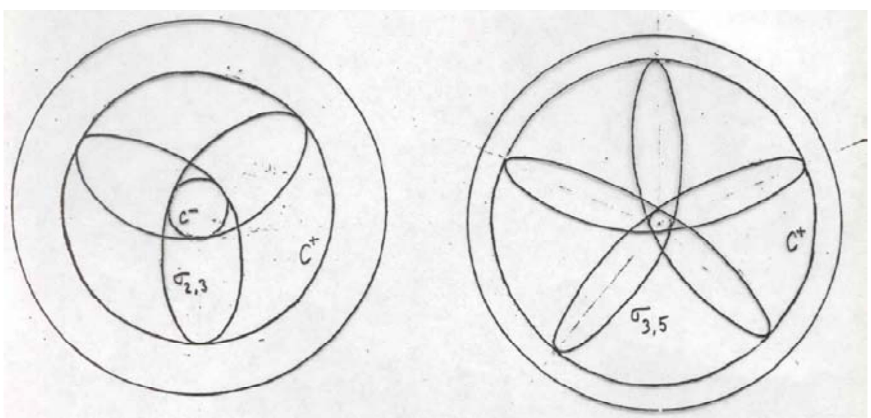
In $\mathbb{H}^2$, the closed elastic curves are much more abundant. In particular, there are critical points for $\lambda = 0$; we call such curves free elastica(e).

If the curvature of the hyperbolic plane is G, then the circle C of radius $\frac{\sinh^{-1}(1)}{\sqrt{-G}}$ is a free elastica, called the 'equator' of $\mathbb{H}^2$. (The name is by analogy to the equator of the sphere, which is a free elastica by virtue of being a geodesic.)

Free elastic curves can be classified using the Killing field $J(s) = \frac{k^2}{2}T + k'N$. Although rotation fields, translation fields, and horocycle fields all arise as examples of J, only the rotation fields are compatible with closed solutions.

Theorem. *Let γ be a free elastica in $\mathbb{H}^2$. Then either γ is C^m for some m, or γ is a member of the family of solutions $\{\sigma_{m,n}\}$ having the following description:*

if $m > 1$ and n are integers satisfying $\frac{1}{2} < \frac{m}{n} < \frac{\sqrt{2}}{2}$ there is (up to congruence) a unique curve $\{\sigma_{m,n}\}$ which closes up in n periods of its curvature $k = k_0\,\mathrm{cn}^2(\frac{k_0 s}{2}, p)$ while making m orbits about the fixed point q of the rotation field J.



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Theorem. *Let γ be a regular closed curve in $\mathbb{H}^2$, the hyperbolic plane with curvature G. Then*

$$\int_\gamma k^2 \, ds \geq 4\pi\sqrt{-G}$$

with equality precisely for the equator C.

Application: Willmore tori of revolution in $\mathbb{R}^3$.

For the Chen-Willmore problem, one considers immersions $\Psi : M^2 \longrightarrow \mathbb{R}^3$ and the total squared mean curvature functional

$$\mathcal{H}(\Psi) = \int\int_M H^2 dA$$

where H is mean curvature and dA is the area element.

The Willmore conjecture is that $\mathcal{H}(\Psi) \geq 2\pi^2$ when M is a torus. Robert Bryant and Ulrich Pinkall independently observed the following:

Theorem. *Let γ be a regular closed curve in the hyperbolic plane represented by the upper half plane above the $x-axis$. If Ψ is the torus obtained by revolving γ around the $x-axis$, then $\mathcal{H}(\Psi) = \frac{\pi}{2}\mathcal{F}(\gamma)$*

From the inequality we derive the

Corollary. *The Willmore inequality holds for tori of revolution.*

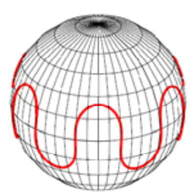
$$\mathcal{F}(\gamma) = \int_\gamma k^2 \, ds \geq 4\pi\sqrt{-G}$$

There are other ways to relate elastic curves to Willmore manifolds (critical points for total squared mean curvature). Let γ be critical for $\mathcal{F}^1$ in $\mathbb{S}^2$. U. Pinkall observed that the inverse image of γ under the Hopf map $\pi : \mathbb{S}^3 \longrightarrow \mathbb{S}^2$ is a Willmore torus; stereographic projection gives a Willmore torus in $\mathbb{R}^3$. This gives an infinite family of embedded Willmore surfaces in $\mathbb{R}^3$: recall the theorem

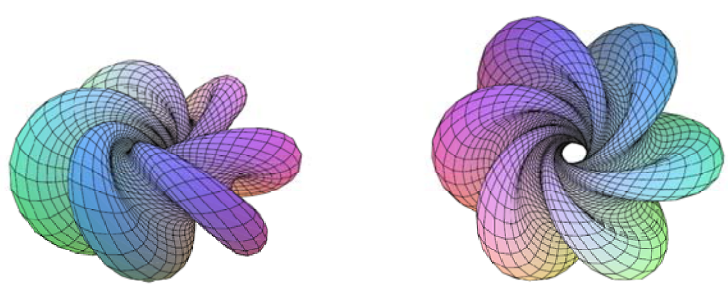
Theorem. (L-S, 1987) *Let λ be a fixed constant with $0 \leq \frac{8}{7}G$. Then for each pair of positive integers m, n with*

$$\frac{m}{2n} < 1 - \frac{\sqrt{G}}{\sqrt{4G - 2\lambda}}$$

there is a unique elastica $\gamma^{\lambda}_{m,n}$ (up to congruence) which closes up in n periods while crossing the equator m times.



A closed spherical curve p with 6 periods



Stereographic image of the Hopf torus of p

G. Preissler: Isothermic Surfaces and Hopf Cylinders

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