Foci of Algebraic Curves

November 7, 2008

With thanks to my mentor

Outline of Talk

- Historical overview, Definition of foci
- Example: k-ellipses
- Example: lemniscates
- Example: critical points of polynomials
- Example: critical points of non-Euclidean polynomials
- Example: numerical range and eigenvalues
- Example: Schwarz reflection

What are Foci?

Apollonius of Perga introduced foci of ellipse and hyperbola (3rd century B.C.E.)

Pappus found the focus of a parabola (4th century C.E.)

Kepler named foci and developed their properties (17th century)

Plücker (1832) defined foci of higher order curves

Plücker Formulation

γ is an algebraic curve, given by $P(x, y) = 0$ where P is a polynomial of degree n (and assume real coefficients)

Example: The circle $(x-a)^2+(y-b)^2-r^2=0$

We think of $\mathbb{R}^2 \subset \mathbb{C}^2 \subset \mathbb{C}P^2$

Replace $P(x, y)$ with homogeneous polynomial $P[x, y, z] = 0$ and extend to a complex curve Γ in $\mathbb{C}P^2$

Example: $(x-az)^2+(y-bz)^2-r^2z^2=0$

Note that every circle passes through the "*circular points*" $[1, i, 0]$ and $[1, -i, 0]$.

A line through a circular point tangent to Γ is an "*isotropic tangent*", with equation $x+iy=(a+ib)z$ (or $x-iy=(a-ib)z$)

The line meets the "real" plane at the focus (a, b)

In the case of the circle, this actually gives the center of the circle.

The example of the circle is special, because the isotropic tangent is actually tangent *at the circular point*. This kind of focus is special and may be called a *singular focus*. A real curve which passes through the circular points is a "circular curve".

A quadratic curve which is not a circle is not circular, so it has ordinary foci.

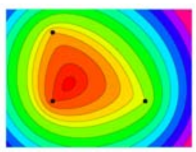
A curve Γ has *class* m if there are m tangent lines from an arbitrary point P to it. Such a curve will have at most m foci; a circular curve will have fewer.

Example 1: The k-ellipse

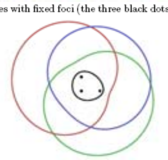
Given points $P_1, P_2, \ldots, P_k$ in the plane and a positive number r, the set of points the sum of whose distances from the P_i is r is called a k-*ellipse* or *poly-ellipse*. They were first studied by Tschirnhaus (1695).

The k-ellipse is given by a polynomial of degree 2^k if k is odd or degree $2^k - \binom{k}{k/2}$ if k is even.

The points P_i are foci (though in general there are other foci).

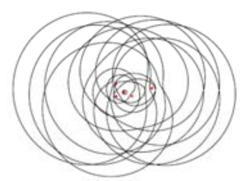


A pencil of 3-ellipses with fixed foci (the three black dots) and different radii.



The Zariski closure of the 3-ellipse is an algebraic curve of degree eight

A 3-ellipse, a 4-ellipse, and a 5-ellipse, each with its foci.



The Zariski closure of the 5-ellipse is an algebraic curve of degree 32

Semidefinite Representation of the k-Ellipse

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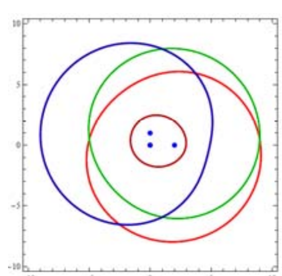
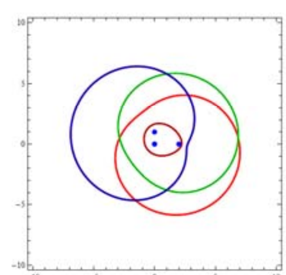
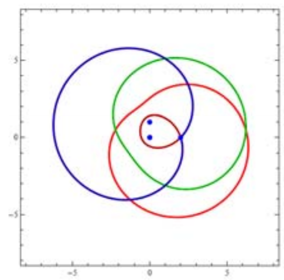
The 3-ellipse



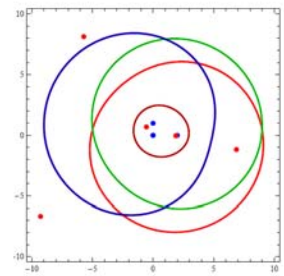
The innermost curve is convex



Non-smooth example

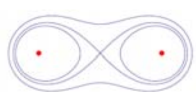


A 3-ellipse with 8 (of nine) foci



Example 2: Lemniscates

Given points $P_1, P_2, \ldots, P_k$ in the plane and a positive number r, the set of points the product of whose distances from the P_i is r is called a *(Polynomial) lemniscate* or, when $k = 2$, a *Cassini oval*.



The lemniscate is given by a polynomial of degree $2k$: If

$$P(z) = \prod_{1}^{k}(z - P_i) = u(x, y) + iv(x, y)$$

then the equation is $u^2 + v^2 = r^2$.

The points P_i are foci (though in general there are other foci).

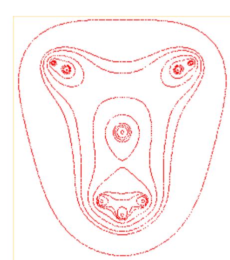
Lemniscates can be anything

(a) An original bird-shape. (b) Reconstructed shape using 7 foci and 3-pass of thinning; p=60746955193331.2 and image dimension is 256x256.

LEMNISCATE TRANSFORM: A NEW EFFICIENT TECHNIQUE FOR SHAPE CODING AND REPRESENTATION

Amlan Kundu

E.T.: Phone Home!



des cassiniennes alien … par Alain Esculier

Example 3: Siebeck's Theorem

Theorem. *The zeros of the function*

$$F(Z) = \sum_{1}^{p} \frac{m_i}{Z - Z_i}$$

are the foci of the curve of class $p-1$ which touches each line segment (Z_i, Z_j) in a point dividing the line segment in the ratio $m_i : m_j$.

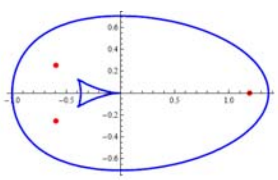
In particular, the critical points of a polynomial $P(Z)$ of degree p are the foci of a curve of class $p-1$ which is tangent to the lines joining pairs of roots of P.

Bôcher-Grace Theorem

If $P(z) = z^3 + a_2 z^2 + a_1 z + a_0$ has roots R_1, R_2, and R_3, then its derivative $P'(z) = 3z^2 + 2a_2 z + a_1$ has its roots at the foci of an ellipse tangent at the midpoints to the three sides of the triangle with vertices at R_1, R_2, and R_3.

Class Three

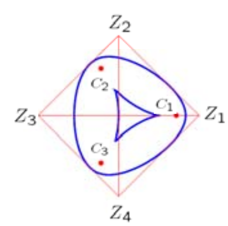
A curve of class three has the property that there are three tangent lines from any point to the curve. Such a curve must look something like this:



Siebeck's theorem says that given four points the resulting curve of class three will be tangent to the line joining any two of the points. Here is a sample picture of this phenomenon:

$$F(Z) = \sum_{1}^{4} \frac{m_i}{Z - Z_i}$$

$$F(C_i) = 0$$



The corners of the quadrilateral are the poles of

$$F(Z) = \sum_{1}^{4} \frac{m_i}{Z - Z_i}$$

and the foci are the zeroes.

Application: View

$$\sum_{1}^{p} \frac{m_i}{Z - Z_i} = \sum_{1}^{p} \frac{m_i}{|Z - Z_i|^2}(Z - Z_i)$$

as velocity vector field of an incompressible fluid flow. Then Z_i is a source of strength m_i (or sink if $m_i < 0$), and the foci are the stagnation points of the flow.

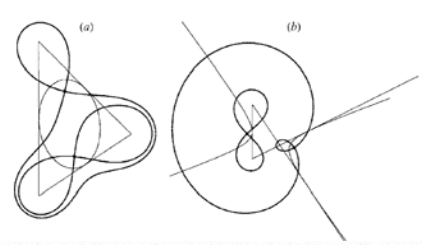


FIG. 5. Stagnation points for three-vortex flow as foci of the Siebeck conic. (a) Identical vortices; (b) vortices with strength ratio 1:1:−1.

$$u - iv = \frac{1}{2\pi i} \sum_{\alpha=1}^{N} \frac{\Gamma_\alpha}{z - z_\alpha}.$$

Hassan Aref

JOURNAL OF MATHEMATICAL PHYSICS 48, 065401 (2007)

Example 4: Non-Euclidean polynomials

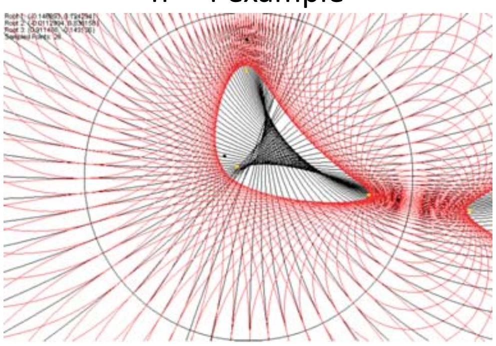
A *non-Euclidean polynomial* (Walsh, 1952),is a function of the form

$$B(Z) = \lambda \prod_{1}^{n} \frac{Z - A_k}{1 - \overline{A}_k Z}, \quad |\lambda| = 1, \quad |A_k| < 1$$

This is an n-to-one map from the closed unit disc $\overline{D}$ to itself (a *finite Blaschke product.*) with n zeros in the interior D, and has modulus unity on $C: |Z| = 1$.

Theorem. *Let $B(Z) = \lambda \prod_1^n \frac{Z-A_k}{1-\overline{A}_k Z}$, $|\lambda| = 1$, $|A_k| < 1, n > 2$ be a non-Euclidean polynomial, and let γ be the curve in D which is the envelope of the* **non-Euclidean geodesics** *(with respect to the Poincaré metric) joining pairs of points W_i, W_j on C satisfying $B(W_i) = B(W_j)$. Then γ is (part of) an algebraic curve whose real foci are the critical points of $B(Z)$ in D together with their inverses with respect to D.*

n = 4 example

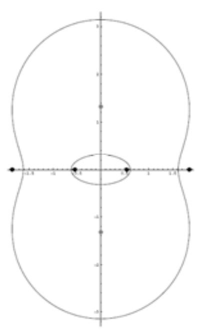


In the case $n = 3$ the curve is a *non-Euclidean ellipse* (together with its reflected image). The sum of the hyperbolic distances from the two foci to a point on the curve is constant. It is inscribed in ideal triangles whose vertices W_i satisfy $B(W_1) = B(W_2) = B(W_3)$.

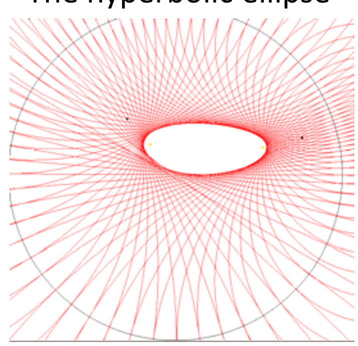
The equation for this curve, after a Möbius transformation, can be given as

$$\Gamma : \frac{4x^2}{\alpha^2} + \frac{4y^2}{\beta^2} - (x^2 + y^2 + 1)^2 = 0$$

It is a curve of class 8 and genus 1.



The hyperbolic ellipse



Example 5: Eigenvalues

The field of values $W(A)$ of an $n \times n$ matrix A is defined by

$$W(A) = \{x^*Ax : \|x\| = 1\}$$
$$= \{\frac{x^*Ax}{x^*x} : \|x\| \neq 0\}$$

It is a compact convex subset of the plane containing the eigenvalues of A. (Toeplitz - Hausdorff)

Kippenhann's Theorem

Theorem. *The boundary Γ of the numerical range of an $n \times n$ matrix A is the convex hull of the curve whose equation in line coordinates (u, v, w) is given by*

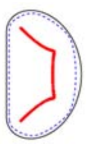
$$\Phi(u, v, w) = det(uH + vK + wI) = 0$$

where $A = H + iK$ and H and K are Hermitian matrices. Thus, Γ is the dual curve of the curve $\Phi(u, v, w) = 0$

Corollary. *The eigenvalues of A are the foci of the curve Γ.*

Proof: The line $ax + by + cz = 0$ is tangent to Γ if and only if $\phi(a, b, c) = 0$. The isotropic line $x + iy = (z_0)z$ is tangent to Γ iff

$$det(H + iK - z_0I) = det(A - z_0I) = 0$$

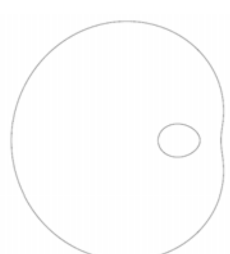


Eigenvalues (red) and Γ (solid) for 100×100 grcar matrix

Note: The dual curve $\phi(x, y, z) = 0$ is given by a determinant; it is an "RZ curve", important in applications to nonlinear optimization. The solution by Helton and Vinnikov of the Lax Conjecture is the converse: the dual of an RZ curve of degree n is the boundary curve for the numerical range of an $n \times n$ matrix.

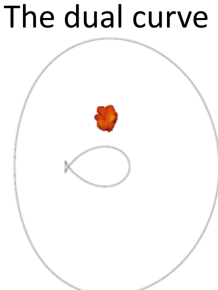
An RZ curve of order n has the property that every line through the origin meets the curve in n *real* points.

Example of an RZ curve



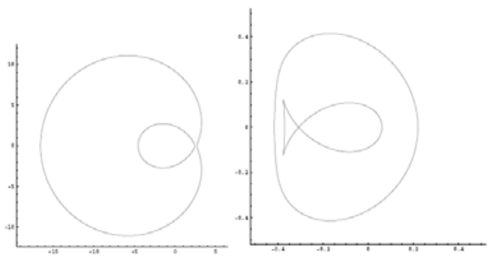
$$q(u, v) = (u^2 + 6u + v^2 - 52)^2 + 144(4u - 17) = 0$$

The dual curve

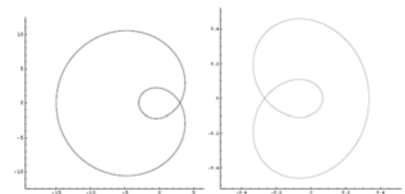


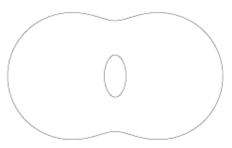
$$11025u^6 + u^4(838 + 37444v + 79393v^2) + u^2(-2087 - 13628v - 7326v^2$$
$$+89804v^3 + 139112v^4) + (4 + 17v)^2(1 - 12v - 68v^2 + 48v^3 + 256v^4) = 0$$

RZ curves and their duals

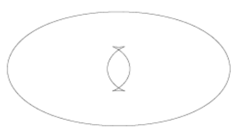


The dual of this RZ curve is not convex, but its convex hull is a numerical range





hyperbolic ellipse



its dual curve

Example 6: Schwarz Reflection

Let Γ be a curve in $\mathbb{R}^2$ given by the algebraic equation $\phi(x, y) = 0$. (Edouard) Study defined reflection of points across Γ as follows:

If $P = (x_1, y_1)$ is any point in the plane, we identify it with the point in $\mathbb{C}P^2$ with homogeneous coordinates $[x_1, y_1, 1]$. There are two lines, R_1 and B_1 given in homogeneous coordinates $[X, Y, Z]$ by

$$R_1 : X + iY = (x_1 + iy_1)Z$$

$$B_1 : X - iY = (x_1 - iy_1)Z$$

Let (X_2, Y_2) be a point in $\mathbb{C}^2$ lying on the line R_1 and satisfying the equation $\phi(X_2, Y_2) = 0$. There will be n choices of such a point if ϕ has degree n.

(This reveals the fact that Schwarz reflection is only locally defined in a $1-1$ manner.)

Now take the line

$$B_2 : X - iY = (X_2 - iY_2)Z$$

Let $Q = (x_3, y_3)$ be the point of intersection of B_2 with the real plane. $Q = \mathcal{R}(P)$ is the Schwarz reflection of P.

From this description we see: Schwarz reflection fails to be 1 to n precisely at the foci of Γ. For instance, Schwarz reflection through an ellipse gives a two-fold covering of the plane branched at the foci.

The *Schwarz function* of Γ is the analytic function such that the curve is the analytic function locally defined (near a given point $(x_0, y_0) \in \Gamma$) by $\bar{z} = S(z)$.

$S(z) = z$ for the real line $y = 0$

$S(z) = r^2/z$ for the circle $x^2 + y^2 = r^2$

Schwarzian reflection in Γ is locally characterized as the unique antiholomorphic map $\mathcal{R}$ fixing points of Γ.

It is given by the formula $\mathcal{R}(z) = \overline{S(z)}$ for z near $z_0 = x_0 + iy_0$.

The branch points of $S(z)$ are the foci of γ.

Application: Quadrature Domains

Ω is a *quadrature domain* If there are finitely many points $a_1, a_2, \ldots, a_m \in \Omega$ and coefficients $c_{j,k}$ such that for any integrable analytic function f

$$\int_\Omega f\,dA = \sum_{k=1}^{m} \sum_{j=0}^{n_k - 1} c_{kj} f^{(j)}(a_k)$$

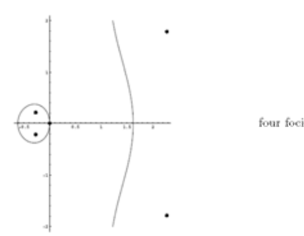
Theorem: [Davis, Aharonov - Shapiro] A bounded domain Ω is a quadrature domain if and only if the Schwartz function is meromorphic on all of Ω.

Theorem: Let $\mathcal{N} \subset \mathbb{C}$ be the numerical range of an $n \times n$ complex matrix A. Assume all eigenvalues of A lie in the interior of $\mathcal{N}$, and the boundary of $\mathcal{N}$ is a nonsingular, algebraic curve γ. Assume also that $0 \in \mathcal{N}$. Then γ^{-1}, obtained by inversion in the plane, bounds a quadrature domain $\mathcal{D}$.

Why? Inversion carries the ordinary foci of γ to the foci of γ^{-1} (and vice-versa). Since γ contains all of its foci, its inverse contains **no foci!** So $S(z)$ has no branch points inside γ^{-1}.

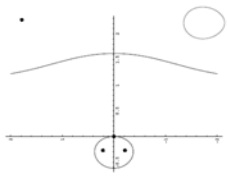
Example: The ellipse is not a quadrature domain. But the inverted ellipse is a quadrature domain: the Neumann quadrature domain, the first historical example of such a region.

Conclusion: Foci are interesting



$P(x, y) = (x - 1)(x^2 + y^2) - x$

a circular cubic whose singular focus is at the origin and lies on the curve



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