When is a curve an Octahedron?

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Abstract

We consider complex curves of genus zero and answer the above riddle. Namely, the lemniscate of Bernoulli, which has obvious four-fold symmetry, actually has the octahedral group as its symmetry group, and may in fact be characterized by this symmetry.

1 Introduction

Our goal is to fully describe the remarkable hidden symmetries of the *lemniscate of Bernoulli*, and to demonstrate just how exceptional these symmetries are. This famous plane curve has equation $(x^2 + y^2)^2 + A(y^2 - x^2) = 0$ and looks like ∞ (for $A > 0$). The lemniscate has a celebrated history: it was described by Jacob Bernoulli in 1694, who determined the polar coordinate equation and the radius of curvature for the curve. The curve was hidden among a larger family of curves, the *Cassinian ovals*, which had been proposed in 1680 as planetary trajectories by the astronomer Giovanni Cassini. In this context it can be described as a locus of points the product of whose distances from two fixed points is a constant. This was not noticed until it was pointed out by Pietro Ferroni in 1782, and again by G. Saladini in 1806 ([5, p. 221]). One could argue, however, that the curve was known to Perseus two thousand years earlier, who studied the *spiric sections* derived from slicing a torus with a plane. Here the lemniscate hides among a family of curves called *hippopedes*, about which we have more to say later. (It can also be found lurking among the family of curves whose curvature is proportional to distance from the origin [10].) But most significantly, in studying the arc length of the lemniscate, Bernoulli gave us the lemniscatic integral, which played an important role in the development of elliptic integrals and elliptic functions. (See [11, pp. 215–224].)

The lemniscate is an elegant curve, with obvious fourfold symmetry. But we want to consider all the complex (and infinite) points on the lemniscate, not just the "visible" (real) ones. When we do so, the full lemniscate looks topologically like a sphere with certain pairs of points glued together; in other words, as a complex algebraic curve, the lemniscate has *genus zero.* In fact, it is the image under an immersion of the sphere in complex projective space $\mathbb{C}P^2$, with three double points. The full curve is unexpectedly more symmetric than it might appear to be; to our knowledge, the following fact, which is the main result of this work, has not been observed until now:

Theorem 1.1 (Main Theorem)**.** *The group of symmetries of the lemniscate, thought of as a subset of complex projective space, is the octahedral group. Up to projective equivalence, it is the unique genus zero curve of degree less than or equal to four with this property.*

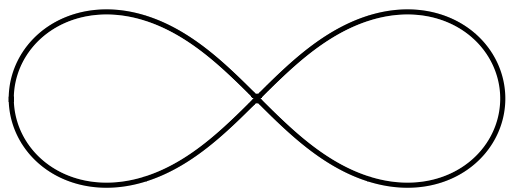

Figure 1: The Bernoulli Lemniscate

To explain this theorem, let us survey some ideas from projective geometry. Points in the *real projective line* are lines through the origin in $\mathbb{R}^2$. A point in the projective line can be represented by the *homogeneous coordinates* $[v, w] \neq [0, 0]$, where $[v, w]$ and $[\lambda v, \lambda w]$ correspond to the same point for any $\lambda \neq 0$. An ordinary real number u corresponds to the point with coordinates $[u, 1]$. We can recover the ordinary coordinate of the number u by taking the ratio $u = v/w$. However, there is one extra point, the 'point at infinity'; it has coordinates $[v, 0]$. Thus the projective line arises from the real line by adding a single point, and it closes up the line to form a circle. In exactly the same way, using complex numbers instead of real numbers, the complex plane can be completed by adding one point, forming the *Riemann sphere* $S^2 = \mathbb{C}P^1$. This is also called the extended complex plane or the complex projective line.

In the same way, points in the complex projective plane $\mathbb{C}P^2$ are (complex) lines through the origin in $\mathbb{C}^3$. That is, they can be represented by triples $[x, y, z] \neq [0, 0, 0]$ of complex numbers, where $[x, y, z]$ and $[\lambda x, \lambda y, \lambda z]$ correspond to the same point for any $\lambda \neq 0$. A point (x, y) in the (ordinary) plane corresponds to the point with projective coordinates $[x, y, 1]$. A point with coordinates $[x, y, 0]$ can be thought of as a point at infinity, since it can be seen to be a limit of points $[Rx, Ry, 1]$ as R goes to infinity; such points are called *ideal points*.

An invertible linear transformation of $\mathbb{C}^3$ takes lines through the origin to lines through the origin, so it gives what is called a *projective transformation* of $\mathbb{C}P^2$; the group of such transformations is $P = PGL(3, \mathbb{C})$, the projective general linear group. It is convenient to represent an element of P as a 3×3 matrix, with the understanding that any scalar multiple of the matrix represents the same element (since it has the same effect on any line through the origin.) If the matrix has real entries, it gives a real projective transformation; for instance, linear transformations of the plane, translations, and dilations are all projective transformations. But in general, a projective transformation will take some finite points to ideal points, so it will not correspond to a transformation of the ordinary plane.

Projective transformations of the Riemann sphere are *Möbius transformations.* Such a transformation T is represented by an invertible matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Using the identification of the complex plane plus infinity with the Riemann sphere, the transformation can be viewed as

$$T(v) = \frac{av+b}{cv+d}$$

Thus the Möbius transformations are also called the *fractional linear transformations*; they form a six real dimensional group, $PGL(2,\mathbb{C})$.

As much of our work will involve fixed points of transformations, it will be useful to establish the following basic property of projective transformations.

Proposition 1.2. *A Möbius transformation of the sphere can have at most two fixed points if it is not the identity. If T is a projective transformation of $\mathbb{C}P^2$ which is not the identity, then either T has a line of fixed points, or a line of fixed points and one other fixed point, or it has at most three fixed points. If it has exactly three fixed points, they are not collinear.*

The first statement follows from the equation of a fixed point $v = \frac{av+b}{cv+d}$, which is quadratic. The remaining statements follow from linear algebra. Since a point in $\mathbb{C}P^2$ corresponds to a line through the origin in $\mathbb{C}^3$, a fixed point of T corresponds to an invariant line for the corresponding matrix M representing T. Any nonzero vector in this line is then an eigenvector. If the eigenvalues of M are distinct, then there will be three independent eigenvectors, so T will have exactly three fixed points, which are not collinear. If two eigenvalues are equal, then there will be a two-dimensional invariant subspace, which corresponds to a (projective) line in $\mathbb{C}P^2$. Either every vector in the space is an eigenvector of M, in which case every point in this line will be fixed by T and there is one other fixed point, or there is only one eigenvector, in which case T will have only two fixed points. If all three eigenvalues are equal, then T will have one fixed point or one line of fixed points, or it will be the identity. $\square$

Now we extend the notion of *curve* to the projective plane, keeping in mind that the extended notion is really two-dimensional. The term *algebraic curve* was coined by Leibniz to describe the set of solutions in the plane of a polynomial equation $f(x, y) = 0$. A polynomial equation of degree n defining a plane curve C is extended to $\mathbb{C}P^2$ by defining the homogeneous polynomial $F[x, y, z] = z^n f(x/z, y/z)$. Homogeneous polynomials of degree k satisfy the identity $F[\lambda x, \lambda y, \lambda z] = \lambda^k F[x, y, z]$; consequently, if the equation $F[x, y, z] = 0$ holds for one set of projective coordinates of a point, it will for any other. Of course, we can always recover the original polynomial by setting $z = 1$.

For example, a circle $(x-a)^2 + (y-b)^2 - r^2 = 0$ is extended to complex projective space by the equation $(x-az)^2 + (y-bz)^2 - r^2z^2 = 0$. It is important to note that all circles pass through the *circular points* $I = [1, -i, 0]$ and $J =$

$[1, i, 0]$. In the case of the lemniscate, the resulting curve is given by the equation $(x^2 + y^2)^2 + A(y^2 - x^2)z^2 = 0$.

The picture of the lemniscate suggests that the curve "passes through" the origin twice. We may make this precise by defining the multiplicity of a point on a curve. At an *ordinary* or *smooth* point (x_0, y_0) on such a curve, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are not both zero, and the curve can be approximated by its tangent line; other points on the curve are *singularities.* A singularity has order or multiplicity k if all partial derivatives of order less than k vanish, but some partial derivative of order k does not vanish. A line typically meets a curve of order n at n distinct points. If the line passes through a point of multiplicity k then it will typically meet the curve in $n - k$ other points. We have to say "typically" because the tangent line to a curve appears to meet the curve more than once at the point of tangency. If we define the number of intersections to take into account the possible tangencies, as well as the multiplicities of the singularities, then we can say the line always meets the curve n times. For example, a line meets a quadratic curve twice, though the intersections could coincide at a point of tangency and could be imaginary or infinite.

The simplest singularities are of *order 2*, where the second order partial derivatives $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial x \partial y}$, and $\frac{\partial^2 f}{\partial y^2}$ do not all vanish. For instance, every line through the origin meets the lemniscate at most twice other than at the origin. To see this, solve the simultaneous equations $ax+by = 0$ and $(x^2+y^2)^2+A(y^2-x^2)z^2 = 0$ by eliminating y (unless $b = 0$). The resulting equation always has $x = 0$ as at least a double root. But for the lines $x = y$ or $x = -y$ the equation has $x = 0$ as a quadruple root. This identifies these two lines as being tangent to the curve. Intuitively, the curve has two branches through the origin, each of which has an inflection point, and each tangent line makes *three-point contact* with one branch and crosses the other.

Typical order 2 singularities are *nodes*, where the curve has two tangent lines, one for each branch, and *cusps*, where the curve has a sharp point with one tangent line. The simplest example of the cusp is the origin for the curve $y^2 = x^3$. There are many other more exotic singularities, as we shall see later.

We can see the node at the origin of the lemniscate, but in fact there are two more nodes, one at each of the two circular points. This is the first clue that the lemniscate has hidden symmetry, for we shall see that the curve looks the same at all three of these points. Curves that have nodes at both circular points are called *bicircular quartics.* As it also has a node at the origin, i.e., at $[0, 0, 1]$, the lemniscate is an example of a *trinodal quartic.*

If we can find polynomial functions $x(t)$, $y(t)$, and $z(t)$ giving a map $X(t) = [x(t), y(t), z(t)]$ from the (extended) complex plane onto a curve C, one-to-one except at finitely many values of t, we say that C is *rationally parametrized* and is a *rational curve.* It turns out that any trinodal quartic C is a rational curve. For example, in the case of the lemniscate this can be done explicitly by the formula:

$$X(t) \;=\; [x(t), y(t), z(t)] \;=\; [\sqrt{A}(t + t^3),\ \sqrt{A}(t - t^3),\ 1 + t^4]. \tag{1.1}$$

Such a parametrization defines a map $X : \mathbb{C} \longrightarrow C \subset \mathbb{C}P^2$ from the complex plane to the curve C. The most straightforward way to include the point at infinity is to use homogeneous coordinates in the domain; then we can extend it to a map of the Riemann sphere to the curve. For instance, in the case of the lemniscate this is given by the formula

$$X[v,w] = [\sqrt{A}(vw^3 + v^3w), \sqrt{A}(vw^3 - v^3w), w^4 + v^4]$$

which we get by replacing t by $\frac{v}{w}$ and then clearing denominators. One can check that this map is one-to-one except at six points, namely the origin $[0,1]$, the point at infinity $[1,0]$, and the points $[\pm\omega, 1]$ and $[\pm\omega^3, 1]$, where $\omega^4 = -1$, which map in pairs to the three nodes. We may therefore think of the lemniscate *in abstracto* as a sphere, which sits in projective space with three pairs of points glued together.

More generally, any irreducible algebraic curve is the image of a g-holed torus under a map which is one-to-one except on a finite set of points; g is the *genus* of the curve. (By *irreducible*, we mean that the equation of the curve does not factor into a product of equations; we will assume henceforth that the curves we examine are irreducible.)

This statement can be made more precise: the g-holed torus can be given the structure of a *Riemann surface*, that is, a surface on which one can do complex analysis. (This means one can put local complex coordinates on such a surface and take complex derivatives of functions.) For curves with no singularities other than nodes and cusps, it turns out that the genus of the curve can be computed by a formula due to Clebsch (1864). If the curve has degree n, has δ cusps and τ nodes, then

$$g = \frac{1}{2}(n-1)(n-2) - \delta - \tau. \tag{1.2}$$

For example, a nonsingular quartic has genus 3, while a trinodal quartic has genus 0, as noted above. An important and basic theorem of algebraic geometry is that a curve has genus zero precisely when it can be rationally parametrized.

A projective transformation of $\mathbb{C}P^2$ which takes a rational curve C to itself can be pulled back via the parametrization to a Möbius transformation of the sphere. This is not an obvious fact, but rather is a consequence of the fact that the projective transformation preserves the complex structure of C, that is, it is *holomorphic*. The *automorphisms* or one-to-one holomorphic maps of the Riemann sphere onto itself turn out to be precisely the Möbius transformations. So given a a rational parametrization $X : \mathbb{C}P^1 \longrightarrow \mathbb{C}P^2$ of C and projective transformation T of $\mathbb{C}P^2$ taking C to itself, there is a unique Möbius transformation F such that $T \circ X = X \circ F$. For general algebraic curves, we get automorphisms of a Riemann surface. But for surfaces of genus greater than 1, it is known that there are only finitely many such automorphisms.

We define the *symmetry group* of a curve to be the group P_C of projective transformations which take the curve to itself. Such a transformation can be recognized by the fact that it leaves the equation of the curve invariant. Using a parametrization of the curve we may think of this group as acting as symmetries of the sphere (or more generally the surface of genus g). In this way we may view the group as a subgroup of the group of *intrinsic symmetries* of the curve, that is, the automorphisms of the Riemann surface that parametrizes the curve.

In determining the symmetry group of a curve, it is very useful to make use of the following:

Remark 1.3. If C is an algebraic curve and T is a projective transformation, then the symmetry group $P_{C'}$ of curve $C' = T(C)$ is isomorphic to the symmetry group P_C. In fact, each symmetry S' in $P_{C'}$ is just the conjugate $S' = STS^{-1}$ of a an element S of P_C. We may think of T either as moving the curve to a new position or as changing the coordinates of the curve. A particularly useful example is the transformation

$$[u, v, w] = T[x, y, z] = \left[\frac{x + iy}{\sqrt{2}}, \frac{x - iy}{\sqrt{2}}, z\right] \tag{1.3}$$

which fixes the origin $[0, 0, 1]$ while taking the circular points to $[1, 0, 0]$ and $[0, 1, 0]$. The new coordinates u, v, w are called *isotropic* (or *conjugate*) coordinates in the literature.

Because any Euclidean planar symmetry of a real curve extends to all of $\mathbb{C}P^2$, it is obvious that a lemniscate has at least fourfold symmetry. For instance, the horizontal mirror symmetry of the figure eight is achieved by taking real point (x, y) to $(x, -y)$. The corresponding projective transformation can be represented by the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. This flips the real points of the lemniscate, and it also moves around the nonreal points, for example exchanging the two nodes at the circular points.

A circle obviously has a continuous group of symmetries and, less obviously, an ellipse does as well; in fact any (nondegenerate) conic can be transformed to a circle by a projective transformation. (In projective geometry a hyperbola, a parabola, and an ellipse are all the same!) Not only that, every Möbius transformation of the circle (which has genus zero) extends to a projective transformation of $\mathbb{C}P^2$. Thus such curves have maximal symmetry.

2 In search of the octahedron.

The octahedron is the Platonic solid with eight faces, each of which is an equilateral triangle, and six vertices. A simple model of the octahedron is a subset of $\mathbb{R}^3$ with vertices at the six points $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, and $(0, 0, \pm 1)$. The group of rotations of $\mathbb{R}^3$ which carry this polyhedron to itself is the *octahedral group* O; it consists of 24 elements, given by the 24 matrices of determinant $+1$ having three nonzero elements each of which is ± 1. We may view O as a subgroup of the projective group $P = PGL(3, \mathbb{C})$ using this matrix representation. As an abstract group, it turns out to be isomorphic to the group S_4 of permutations of a set with four elements. (This lovely fact can be demonstrated by looking at the four line segments joining the midpoints of opposite faces.) For future reference, we observe that it has two nontrivial normal subgroups: the *alternating group* A_4 of even permutations, and the *Klein Four group* V. The latter group is made up of the identity and the three permutations which swap four elements in pairs. It is (up to isomorphism, of course) the unique noncyclic group of order four.

Call an algebraic curve $C \subset \mathbb{C}P^2$ *octahedral* if it has genus $g = 0$ and octahedral symmetry. That is: a) C is the continuous image of a sphere (*one-to-one*, except at finitely many points) and b) the subgroup P_C of the projective group P taking C to itself is isomorphic to the octahedral group.

Remark 2.1. We should note that if one does not insist on the curve being (the image of) a sphere, there are other octahedral curves. In particular, the *Bolza curve*, given by the equation $y^2 = x^5 - x$, is a curve of genus 2 (a two-holed torus) that has such symmetry. (See [7]).

Curves of degree one, that is, complex lines, have too much symmetry, P_C being isomorphic to $PGL(2, \mathbb{C})$, the full Möbius group of intrinsic symmetries of C as a Riemann sphere. Nondegenerate curves of degree 2, i.e., conics, also have the full group of symmetries as already noted.

Nonsingular cubic curves have genus one, so they are not under consideration here. It is known, however, that any such curve has nine inflection points, which must be preserved by any symmetry. These points form a *tactical configuration*, from which it can be shown that the curve admits a group of exactly 18 symmetries. (see, e.g., [3, p. 298].) A singular cubic, which has genus 0, must have exactly one singularity, which must be fixed under any symmetry. Using projective transformations such a curve can be put into one of two standard forms:

$$A)\, y^2 z = x^2(x + z), \quad \text{and} \quad B)\, y^2 z = x^3.$$

In case A, the curve has a node at the origin, and the symmetry group has order two. In case B, the singularity at the origin is a cusp, and there is a large symmetry group, as is easily seen. Namely, the curve is invariant under transformations $[x, y, z] \mapsto [ax, by, cz]$, where $a^3 = b^2 c$ and $abc \neq 0$. Interestingly, this means the group of symmetries is isomorphic (as a curve) to the curve (minus two points)!

The next simplest place to look is among quartic curves. In fact, higher degree curves present serious difficulties. As the degree of the curve grows, the number of singularites present in a curve of genus 0 grows. By the Clebsch formula 1.2, a genus zero curve of degree n with no singularities other than nodes or cusps has $\delta + \tau = (n-1)(n-2)/2$ singularities. The number $\delta + \tau = 3$ for quartics is just right, since we expect the 3 singularities to correspond to 6 points on the Riemann sphere—potential vertices of an octahedron—which would need to be permuted. By the same token, the number $\delta + \tau = 6$ for quintics would appear to be too large already, although we have not found an elementary proof of this.

Now let us consider quartic curves with three nodes and/or cusps, the main type of quartic curve of genus zero. We are seeking a copy of the octahedral group living in the projective group $P = PGL(3, \mathbb{C})$ and a curve invariant under this group. More precisely, we seek a faithful *projective representation* of the group O, or equivalently, of the symmetric group S_4.

Remark 2.2. While the machinery of representation theory will not be required to solve the stated problem, it is worth noting that there are at the outset two candidates for such a projective representation. First, since the octahedron is a subset of $\mathbb{R}^3$ with vertices at the six points $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, and $(0, 0, \pm 1)$, we obviously can choose as our representation the 24 matrices of determinant $+1$ whose entries are 0 and ± 1. This gives us a unitary representation of O. (A unitary matrix has transpose conjugate equal to its inverse. The real unitary matrices are orthogonal matrices.)

On the other hand, there is also a representation associated with the *double group* $2O$, which is the double cover of O. (For information about this group, as well as projective representations, see [1].) Namely, the linear transformations $(x, y, z) \overset{s}{\longmapsto} \left(\frac{1+i}{2}(x+y), \frac{1-i}{2}(y-x), z \right)$ and $(x, y, z) \overset{t}{\longmapsto} \left(\frac{1+i}{\sqrt{2}}x, \frac{1-i}{\sqrt{2}}y, z \right)$ satisfy $s^3 = t^4 = (st)^2 = -Id$; so the corresponding elements of the projective group satisfy $s^3 = t^4 = (st)^2 = Id$, a standard presentation of $O = S_4$. But this defines only a projective representation, not an ordinary matrix representation, since the corresponding group of matrices generated by M_s and M_t consists of 48 matrices in 24 pairs.

But we need not dwell on such subtleties, owing to the value of geometric argument. For the problem at hand, we appeal to Remark 1.3 to transform the curve into one in a 'standard form'. Specifically, we can choose a new coordinate system $[u, v, w]$ in which the nodes and cusps of C are located at the standard points $[1, 0, 0]$, $[0, 1, 0]$, and $[0, 0, 1]$. The reason is that a projective transformation may be found taking any given triple of noncollinear points to any other. (We can in fact take four points, no three of which are collinear, to any other such four points.)

Assume that we have a subgroup G of the group of projective transformations isomorphic to the octahedral group, acting as symmetries of a quartic C with nodes (and/or cusps) at the three vertices. Since a symmetry of C must take singularities to singularites, it must permute the three singularities; this means that there is a homomorphism from G, a group with 24 elements, to the permutation group S_3, a group with six elements. The kernel of this homomorphism, which must have at least four elements, consists of those transformations that fix all three vertices of the fundamental triangle. Since those transformations are represented by diagonal matrices, they form a commutative normal subgroup of G. The only such subgroup is (isomorphic to) the Klein Four group V. Since the transformations in V have order 2, the corresponding matrices are diagonal matrices whose squares are multiples of the identity. Therefore, they may be scaled to have ± 1 along the diagonal; there are precisely four such projective transformations. Thus the homomorphism from G is surjective, and the symmetries must achieve all permutations of the three vertices. This shows also that the three singularities are either all nodes or all cusps.

Now let us pause to ask what quartic polynomials give trinodal (or *tricuspidal*) curves, and which of these curves are invariant under the action of V. The fact that there is a node or cusp at $[u,v,w] = [0,0,1]$ implies that the lowest order terms in u and v must be second order, or equivalently, the highest power of w must be 2. Similarly, this must also hold for u and v. This leads to the following form:

$$F[u,v,w] = Av^2w^2 + Bu^2w^2 + Cu^2v^2 + (Du + Ev + Fw)uvw = 0. \qquad (2.1)$$

Here $ABC \neq 0$, otherwise the equation reduces to cubic.

The Klein four group acts by changing the signs of the variables, and invariance therefore implies $D = E = F = 0$. It will be seen that we are thus left with a family of projectively equivalent curves. Specifically, we may use the diagonal transformation $[u,v,w] \mapsto [au,bv,cw]$ to put the equation in the form

$$F'[u,v,w] = b^2c^2Av^2w^2 + a^2c^2Bu^2w^2 + a^2b^2Cu^2v^2 = 0, \quad ABC \neq 0$$

in which the choices

$$a^2 = \sqrt{\frac{A}{BC}} \quad b^2 = \sqrt{\frac{B}{CA}} \quad c^2 = \sqrt{\frac{C}{AB}}$$

yield the canonical form

$$\Phi(u,v,w) = u^2v^2 + (u^2+v^2)w^2 = 0.$$

The reader should not be surprised that this curve is just a version of the Bernoulli lemnisicate. In fact, moving two nodes to the circular points and leaving the third node at the origin, the above canonical form may be regarded as the equation in isotropic coordinates for the lemniscate $(x^2+y^2)^2+4(x^2-y^2)z^2 = 0$.

However, the advantage of the canonical form may now be easily appreciated: Since Φ is symmetric in the variables u,v,w and quadratic in each, it is obvious that Φ is O-invariant, where O acts in the standard way on triples $[u,v,w]$ (as in the first representation of O described in Remark 2.2).

How can we be sure that we have found the full symmetry group P_C of the curve? By looking at the corresponding symmetries of S^2. For P_C may then be regarded as one of the finite subgroups of the Möbius group $M = PSL(2,\mathbb{C})$, and it is known that the only such groups are $\mathbb{Z}_n, D_n, A_4, S_4$ or A_5.

Remark 2.3. Projective transformations give rise to holomorphic, hence orientation-preserving, symmetries of C. The full 48-element group of symmetries of the octahedron, which includes 24 mirror symmetries, can only be achieved if we extend the notion of symmetry to allow orientation reversing transformations of $\mathbb{C}P^2$. Consider the operation of complex conjugation of coordinates: $(u,v,w) \overset{\sigma}{\longmapsto} (\bar{u},\bar{v},\bar{w})$. This is an involution of $\mathbb{C}P^2$ that fixes points of the real lemniscate, swapping the two "halves" of C created by removing the former. On the standard octahedron, this involution should be pictured as reflection in one of the planes of symmetry *not* containing any octahedral edges. This kind of symmetry is characteristic of *real* curves, that is, curves given by polynomials with real coefficients.

3 Curves with degenerate singularities

We have given almost the complete proof of the Main Theorem. It remains to consider rational quartics with fewer than three singularities. There are several special cases; for each one we will show that such a curve lacks some symmetry that would be present if the automorphism group were the octahedral group O. A full discussion of the the singularities of quartic curves can be found, e.g., in [2] or [6]. A modern treatment can be found in [4, pp. 272–276].

Such a curve must have at least one singularity that is more complicated than a simple node; we may locate it at the origin. It may have two singularities, one of which is an ordinary node or cusp and the other a *tacnode* or *rhamphoid cusp*. These singularities can merge to form a single singularity, an *oscnode* or a *tacnode cusp*. Or the curve can have one of three types of *triple point*, i.e., a singularity of order three, and no other singularity.

To understand the wide variety of singularities occurring even for curves of degree four, it is helpful to keep in mind the picture of a sphere being mapped to the curve, one-to-one except at the preimages of the singularities. The curves we have considered so far have had three singularities of order two. It is possible to have two of the three merge into a single singularity, still of order two but more complicated, and then the curve will have only two singularities. Two nodes merge to form a tacnode (Figure 2 a); a node and a cusp merge into a rhamphoid cusp, which is an asymmetrical curve feature. Any symmetry of the curve would take this point to itself, and would then have to be the identity near that point, and hence everywhere.

Or all three singularities can merge into a single singularity, either of order three (a triple point) or, and this is most remarkable, order two. Classically, the order of a point is described by looking at lines through the point and seeing how many times they meet the curve *elsewhere*. Thus a line through a triple point only crosses the curve at one other place. However, the number of points on the sphere that map to a triple point may be less than three. Likewise, a node has two preimages, while a cusp has a single preimage. We will consider and illustrate the possible singularities below.

First consider quartics with two singularities A and B. For example, the curve $(x^2 + 2y^2 - 3x)^2 - 4x^2(2 - x) = 0$ has an ordinary double point and a *tacnode*—see Figure 2a. The tacnode is evidently a double point with a *double tangent*, meeting each branch of the curve twice at the double point.

Suppose a symmetry T of such a curve had order 3. Then it would need to take each of the two singularities to itself. (Otherwise, its cube would swap the two!) Let $X : S^2 \longrightarrow C$ be a parametrization of the curve C. Then the automorphism of S^2 induced by T can only swap preimages of each double point. That is, if $X(p) = A$ then $X(F(P)) = A$, where F is the induced transformation of the sphere, and likewise for B. But then T^2 must fix the preimages, of which there are more than two. (So T cannot have order 3, since a Möbius transformation which fixes at least three points must be the identity.

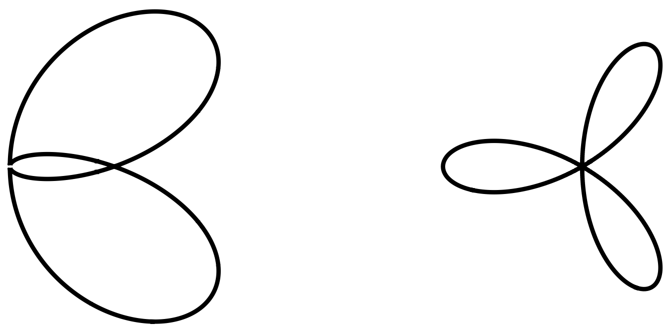

Figure 2: a) curve with tacnode; b) trifolium.

If the curve has a triple point, then it can have no other singularity. Assuming it is located at the origin, then the lowest order terms in x and y are order three. Lines through the point meet the curve in at most one other point. If it has three tangent lines, then any transformation T of even order must permute at most two of the tangent lines; then of necessity $T^2 = Id$, since all three preimages of the singular point will be fixed. So T cannot have order four. The classic example is the trifolium $(x^2 + y^2)(x^2 + y^2 + x) - 4xy^2 = 0$, shown in Figure 2 b, which has obvious S_3 symmetry.

The second type of triple point occurs when there are only two tangent lines at the point, and there are two preimages. An example of such a curve is given by $x^4 + y^4 - x^2yz = 0$ (Figure 3a), which has a triple point at the origin. The line $x = 0$ is a double tangent, while the line $y = 0$ is an ordinary tangent. This can be seen simply by ignoring the fourth order terms, or we can use the following rational parametrization of the curve: $x = t,\ y = t^2,\ z = 1 + t^4$. The origin has two preimages: $t = 0,\ t = \infty$. Note that any symmetry of such a curve would of necessity preserve the two tangent lines; hence the Mobius transformations corresponding to any symmetry fix 0 and ∞. Corresponding to the transformation $t \longmapsto it$, this curve admits the order four symmetry: $X = ix, Y = -y, Z = z$.

A more degenerate type of triple point may have just one tangent line and one preimage, resulting in a large symmetry group. An example is the curve $x^4 = y^3z$, which admits all symmetries of the form $[x, y, z] \mapsto [ax, by, cz]$, where $a^4 = b^3c$ (and $abc \neq 0$.

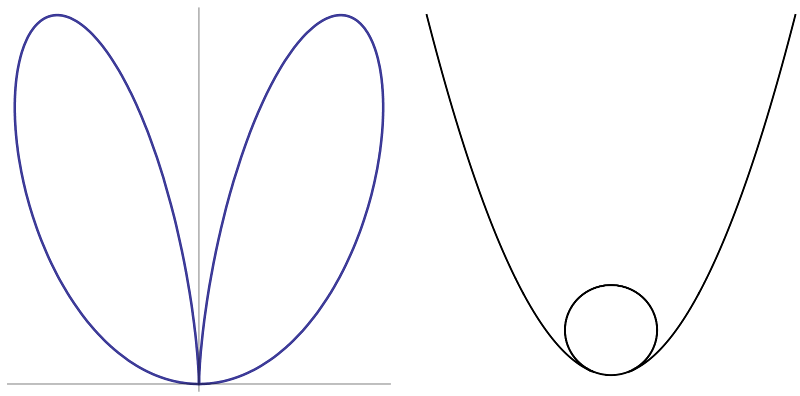

Figure 3: a) Exotic triple point   b) Curve with oscnode.

There may also be a lone double point, either a tacnode cusp (which is asymmetrical) or an oscnode. In Figure 3b, the curve $(y-x^2)^2+x^2y^2-y^3=0$ displays an oscnode. Note that lines through the singularity meet the curve in two other points. The two branches of the curve at the origin have the same osculating circle. The curve has only two-fold symmetry.

So we have seen that in each case, the symmetry group of a curve of degree four with degenerate singularities does not possess octahedral symmetry; this concludes the proof of Theorem 1.1.

4 Rational Bicircular quartics

We have seen in the previous section that if the Klein Four group V acts as symmetries of a trinodal quartic keeping the nodes fixed, then the curve has octahedral symmetry. Consider, the *Lemniscates of Booth*, also known as the *Hippopedes*, given by the equation $(x^2+y^2)^2+8y^2z^2 = 8k(x^2+y^2)z^2$ (Figure 4). For all k, these curves are trinodal, bicircular quartics. Note that for $k = .5$, the Booth Lemniscate is the Bernoulli lemniscate, while for $k = 1$ the curve is made up of two tangent circles.

This family of curves also generalizes the Bernoulli lemniscate as *Watt curves*, which are curves traced out by three-rod linkages. The Hippopedes would even appear to be as symmetrical as Bernoulli's lemniscate, but in fact they fail to have octahedral symmetry, as we may easily see. In isotropic coordinates $[u, v, w]$ (formula 1.3), the Booth lemniscates have the equation

$$4u^2v^2 - 4u^2w^2 - 4v^2w^2 + (8-16k)uvw^2 = 0$$

and thus are part of the family described in equation 2.1. They are not invariant under the octahedral symmetry $[u, v, w] \mapsto [-u, v, -w]$, for example. On the other hand, the equation has symmetries

$$\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}, \qquad \begin{bmatrix}0 & -1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}, \qquad \begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

which (together with Id) give (a homomorphic image of) the Klein four group as the symmetry group of the Booth lemniscate; there are no other projective symmetries, except in the case of the Bernoulli lemniscate. So what can we say

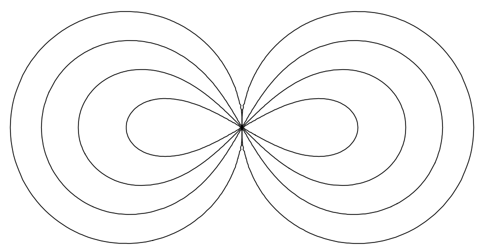

Figure 4: Booth Lemniscates, including Bernoulli's.

about the symmetry of other such curves?

Proposition 4.1. *For any p and q, consider the curve C given by $P_{p,q}(u,v,w) = p(4u^2v^2 - 4u^2w^2 - 4v^2w^2) + q(2uvw(iu - iv + w)) = 0$. Then C is invariant under the action of the matrices*

$$T_3 = \begin{bmatrix}0 & 0 & i\\ -1 & 0 & 0\\ 0 & i & 0\end{bmatrix} \qquad \textit{and } T_2 = \begin{bmatrix}0 & -1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$$

These generate a copy of the permutation group S_3.

In rectangular coordinates, the curves described above are given by the equations:

$$p\left((x^2+y^2)^2 - 4(x^2-y^2)z^2\right) + q\left((x^2+y^2)(\sqrt{2}yz + z^2)\right).$$

For special values of the parameters, the curve is degenerate: four lines or two circles. One can show that this linear family of curves is essentially unique among trinodal quartics, in that any trinodal quartic with S_3 symmetry is part of a family projectively equivalent to this. This can be seen by looking for trinodal quartics with a symmetry of order three. Among the curves in the family are the well-known limaçon and cardioid. The limaçon has two cusps and a node, while the cardioid has three cusps

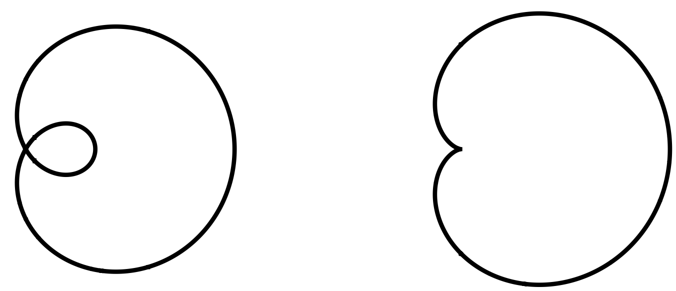

Figure 5: a) limaçon; b) cardioid.

5 Riemannian symmetry

There is another natural geometric notion of symmetry for an algebraic curve. Namely, one may use the canonical Riemannian metric defined on $\mathbb{C}P^2$, the *Fubini-Study metric*. (See, e.g., [8, p. 160].) This metric is invariant under the subgroup of P corresponding to unitary matrices. We may restrict the allowable symmetries of the curve to this subgroup of P. If we then consider the Riemannian metric (possibly with isolated singularities) that the curve inherits from the Fubini-Study metric under inclusion, the symmetry group acts by isometries.

When restricted to the real projective plane, this is a metric of constant positive curvature. To visualize it, put the plane at $z = 1$ in $\mathbb{R}^3$, then radially project from the origin onto the upper hemisphere of the unit sphere. While this is a very natural metric, it does not behave well under Euclidean translations. Circles centered at the origin look round, but as they are translated they get more and more distorted. Likewise, the complex circles centered at the origin turns out to be round spheres, while other circles are not.

Examples: Any complex line is a sphere(!) with constant Gaussian curvature 2. The unit circle $x^2 + y^2 - z^2 = 0$ has constant Gaussian curvature 1; it is "round". Likewise, the parabola $y^2 + 2xz = 0$ and the hyperbola $2xy - z^2 = 0$ have constant Gaussian curvature 1; they too are"round." Most conics are not round, however, in this sense.

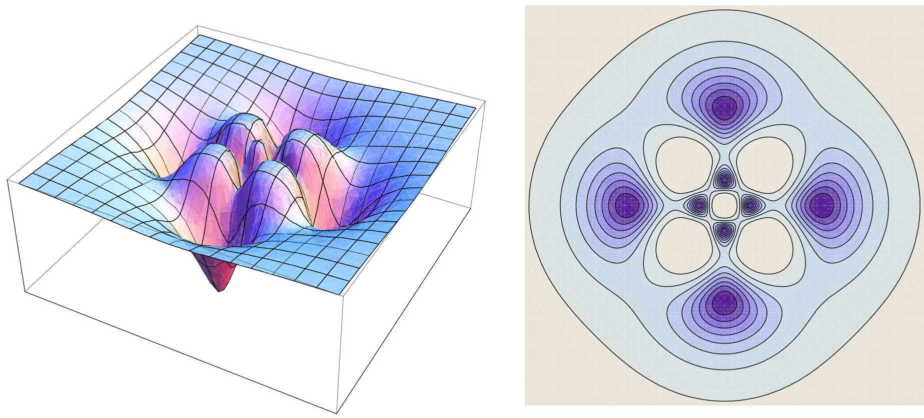

Figure 6: Left: the curvature of the lemniscate Right: level curves.

There is a general formula for the Gaussian curvature at a nonsingular point of an algebraic curve $F(x, y, z) = 0$, due to Linda Ness ([9]).

Theorem 5.1. *Let C be an algebraic curve of degree $d > 1$ defined by the homogeneous polynomial $F(x, y, z)$. Let $\| \ \|$ denote the usual norm in $\mathbb{C}$. The Gaussian curvature at a nonsingular point p of C is given by*

$$K(p) = 2 - \frac{\|p\|^6 \, |HessianF|^2}{(d-1)^6 \, \|gradF\|^6}.$$

Now we can identify a lemniscate which has octahedral Riemannian symmetry. One can check that the lemniscate $(x^2+y^2)^2 - 4(x^2-y^2)z^2 = 0$ is invariant under the transformations of $[x, y, z]$ given by the matrices:

$$S = \begin{bmatrix} \frac{1}{2} & \frac{i}{2} & \frac{i}{\sqrt{2}} \\ -\frac{i}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix} \quad T = \begin{bmatrix} -\frac{1}{2} & \frac{i}{2} & -\frac{i}{\sqrt{2}} \\ \frac{i}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \quad ST = \begin{bmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Note that $S^2 = T^3 = (ST)^4 =$Id, so these (unitary) matrices generate a copy of the octahedral group. Parametrizing the lemniscate as in formula 1.1, we can plot the Gaussian curvature in the complex t-plane; Figure 6 shows the resulting plot. The five peaks, where the curvature is exactly 2, correspond to five of the six vertices of the octahedron (the sixth being at $t = \infty$). Although the vertices are singular points of the curve, we note that the curvature is finite at such points. The curvature has eight minima, and these points together with the six maxima give the vertices of a triangulation of the lemniscate as a *tetrakis hexahedron*. Each triangle has one positively curved vertex and two negatively curved vertices. The entire curve can be assembled from 24 congruent copies of this Riemannian triangle. Question: does this triangle embed in Euclidean three-space?

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