THE RECTILINEAR CROSSING NUMBER OF CERTAIN GRAPHS

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1. Prologue

The attached paper was originally written in 1971, when the author was at Cornell University. It grew out of discussions with Professor Herb Wilf of the University of Pennsylvania, who brought the problem to the attention of the author. Due to controversy at the time of its submission, it was never published, although the results have been cited in various places over the years; see, e.g., [3]. See also http://www.mathsoft.com/asolve/constant/crss/crss.html. Recently, the question of the exact value of $\bar{c}(K_{10})$ was resolved [1] and other improvements to the results in this paper were made [2]. In response to several requests in recent times, this paper is now being made available to interested readers.

References

[1] Alex Brodsky, Stephane Durocher, and Ellen Gethner, The Rectilinear Crossing Number of K_{10} is 62, UBC Computer Science Technical Report, TR-00-10, August 10, 2000.

[2] Alex Brodsky, Stephane Durocher, and Ellen Gethner, Toward the Rectilinear Crossing Number of Kn: New Drawings, Upper Bounds, and Asymptotics, 2000.

[3] Martin Gardner, *Knotted Doughnuts and Other Mathematical Entertainments*, W. H. Freeman and Company, New York, 1986.

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The Rectilinear Crossing Number of Certain Graphs*

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*Work supported by NSF Postdoctoral Fellowship.

ABSTRACT

The geometric character of rectilinear drawings of complete graphs K_n is examined and estimates of the rectilinear crossing numbers of such graphs are obtained. It is shown that the rectilinear crossing number is distinct from the crossing number at least for the cases $n = 8$ and $n = 10$, thus verifying a conjecture of F. Harary and A. Hill. In particular, $\overline{c}(K_8) = 19$ and $60 < \overline{c}(K_{10}) < 63$, the last inequality representing a new drawing of K_{10}. Finally, a construction of a rectilinear drawing of K_n, n large, is given and a new asymptotic upper bounds for $\overline{c}(K_n)/n^4$ is derived, namely $\lim \overline{c}(K_n)/n^4 \leq 5/312$.

If G is a graph, the crossing number $c(G)$ is the minimum number of crossings that occurs in a drawing of G in the plane $\mathbb{R}^2$. The rectilinear crossing number $\overline{c}(G)$ is the minimum number of crossings in a rectilinear drawing of G. It is immediate that $c(G) \leq \overline{c}(G)$ for any graph G. In this paper, we will discuss the geometric nature of rectilinear drawings of G and describe some estimates for $\overline{c}(G)$ in the case $G = K_n$, the complete graph on n vertices. In particular, we will show that $c(K_n) \neq \overline{c}(K_n)$ at least for $n = 8$ and $n = 10$, and we give an estimate for $\overline{\lim_{n\to\infty}} \overline{c}(K_n)/n^4$.

The author wishes to thank Professors Herbert Wilf and Albert Nijenhuis and Mr. Paul Vickers for many helpful discussions.

Definitions:

A drawing of G is a map $D: G \to \mathbb{R}^2$ such that no point in $\mathbb{R}^2$ is the image of more than two points of G, which must not belong to the same edge of G and neither of which may be a vertex. A point of $\mathbb{R}^2$ which is the image of two points of G is called a crossing. $k(D)$ is the number of crossings of the drawing D.

A drawing is rectilinear if each edge of G is mapped linearly to a straight line segment. It is clear that every such drawing is determined by its values at the vertices of G.

If S is a set in $\mathbb{R}^2$, the convex hull of S, denoted $[S]$ is the smallest convex set containing S. For example, $[pq]$ will denote the line segment joining p to q. $|S|$ will denote the cardinality of S. If D is a drawing, a vertex of D is a point in $\mathbb{R}^2$ which is the image of a vertex of G. Edges of D are defined similarly.

Estimation Techniques for Crossings

In this section we consider a fixed rectilinear drawing D of K_n. Let V be the set of vertices of D. $[V]$ is a convex linear cell, and there is a unique subset A of V such that $[A] = [V]$ and every vertex of A is on the boundary of $[V]$. (Note that no three vertices are collinear.) Of the remaining vertices, we may similarly choose B such that $[B] = [V \setminus A]$ and every vertex of B is on the boundary of $[V \setminus A]$. Let $C = V \setminus (A \cup B)$. Let $|A| = a$, $|B| = b$, and $|C| = c$. We must have $a \geq 3$ and $a+b+c = n$.

A set of points $P \subset V$ is of <u>type</u> $(\underline{p,q})$ if $|P \cap A| = P$ and $|P| = p+q$. Similarly, P is of type (p,r,s) if $|P \cap A| = p$, $|P \cap B| = r$ and $|P \cap C| = s$.

We will be concerned with sets P with $|P| = 4$ and estimate the number of such sets which must contribute crossings in the drawing D.

Suppose P is a set of type $(4,0)$. Then $[P]$ must be a quadrilateral and must therefore have a crossing.

If $k(4,0)$ denotes the number of crossings of type $(4,0)$ then we have the formula

(1) $$k(4,0) = a(a-1)(a-2)(a-3)/24.$$

Next we consider sets P of type $(3,1)$. Let v be a vertex of $V \setminus A$ and v_i, v_j vertices of A. Then the number of crossings involving the edge $[v_i v_j]$ and an edge $[v\ v_k]$ is determined by counting the number of vertices of A which do not lie in the same side of $[v_i v_j]$ as v. If we assume that v lies on the "favorable" side of every edge $[v_i v_j]$, it is easily seen that the number of crossings is given by the formula:

$$\begin{cases} a(a-1)(a-3)/8 & \text{if } a \text{ is odd} \\ a(a-2)^2/8 & \text{if } a \text{ is even.} \end{cases}$$

In fact, this minimum can be achieved by placing v in the most central region of [A] if a is odd; if a is even, this region is not unique due to the symmetric behavior of the main diagonals of this polygon. [Figure 1]. In any event, we have the inequality

$$(2) \qquad k(3,1) \geq \begin{cases} (n-a)a(a-1)(a-3)/8 & \text{if } a \text{ is odd} \\ (n-a)a(a-2)^2/8 & \text{if } a \text{ is even.} \end{cases}$$

There are two ways a crossing of type (2.2) can arise: one is if an edge of type (2,0) crosses an edge of type (0,2). However, since no such crossings need occur, we have no lower bound for the crossings of this type. To compute the crossings of two edges of type (1,1), consider a polygon with a sides and two points v and w inside the polygon, each connected to the vertices of the polygon by edges. We must determine how the stars of v and w intersect, where the star of v means the edges emanating from v. It can easily be shown that the minimum number of such crossings is $(a-1)^2/4$ if a is odd or $a(a-2)/4$ if a is even. [Figure 2]. This gives the formula

(3) $$k(2,2) \geq \begin{cases} (a-1)^2\,(n-a)(n-a-1)/8 & \text{if } a \text{ is odd} \\ a(a-2)(n-a)(n-a-1)/8 & \text{if } a \text{ is even.} \end{cases}$$

Crossings of type (1,3) are the most complex type, and we will only be able to give a partial analysis of them. First we analyze the crossings of type (1,3,0). This problem may be viewed in the following way: a convex cell [B] whose boundary has b sides is contained in a convex cell [A] having a sides. How many edges connecting a vertex of B to a vertex of A must pass through [B]?

By constructing examples one can see that if $b = 3$ there need be no such edges, and thus there are no crossings of type (1,3,0). For example, there is essentially one drawing of K_6 for which $a = 3 = b$ and there are no crossings of type (1,3). (Figure 3).

Definition: Two triangles are <u>concentric</u> if the vertices of one lie in the convex cell determined by the vertices of the other cell and there are no crossings of type (1,3).

Now suppose $b > 3$. Suppose the drawing D minimizes the number of edges of type (1,1) which pass through [B]. Then there must be at least (b-3) such edges. For suppose three of the vertices of B can be connected to the vertices of A by edges none of which passes through [B]. Call them b_1, b_2, b_3. There is a vertex a_1 of A which lies on the same side of the line through b_2 and b_3 as b_1. Similarly choose a_2 and a_3. The

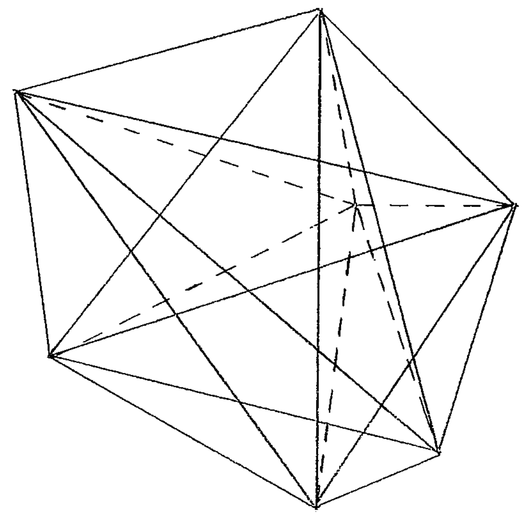


Figure 1a

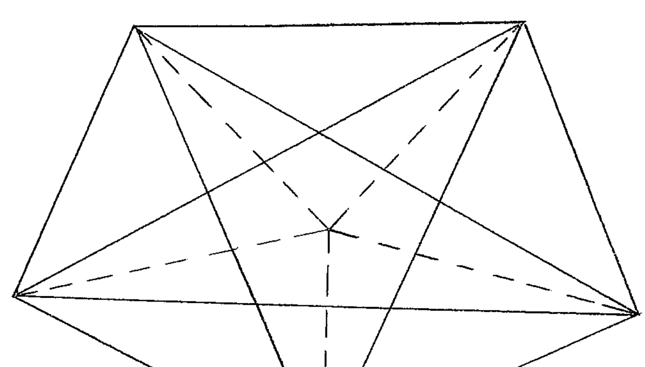


Figure 1b

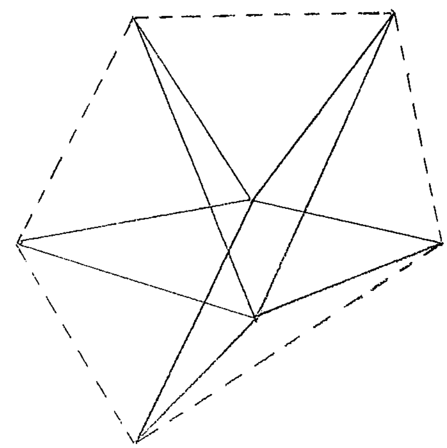


Figure 2a

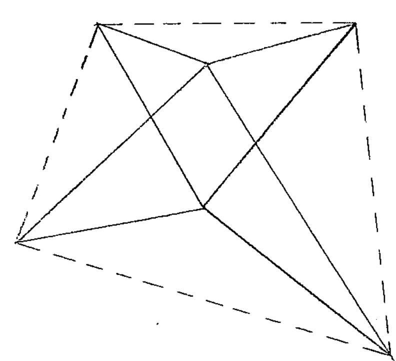
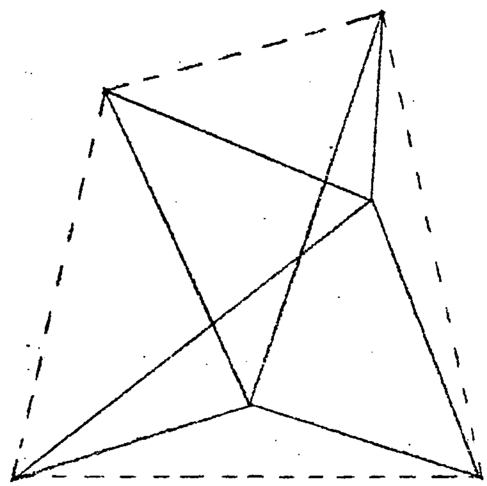
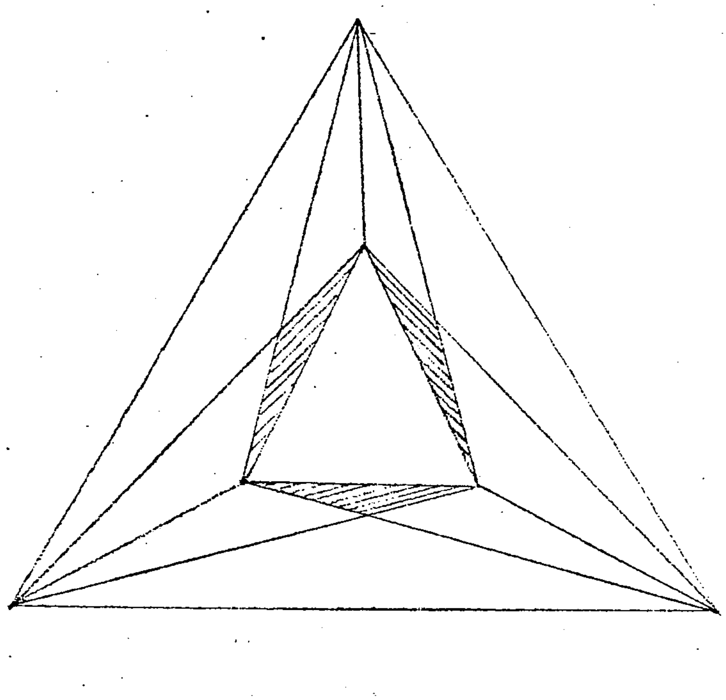
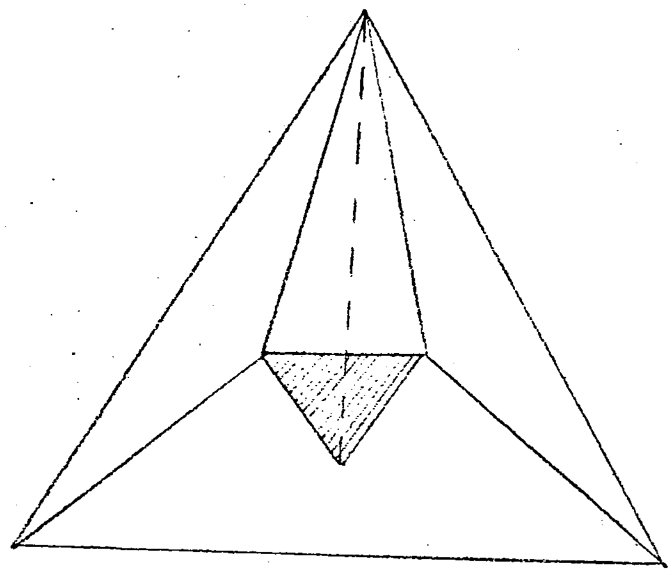
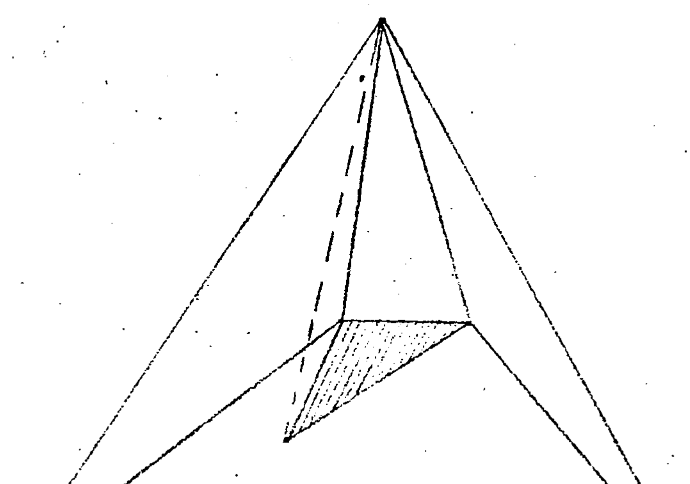


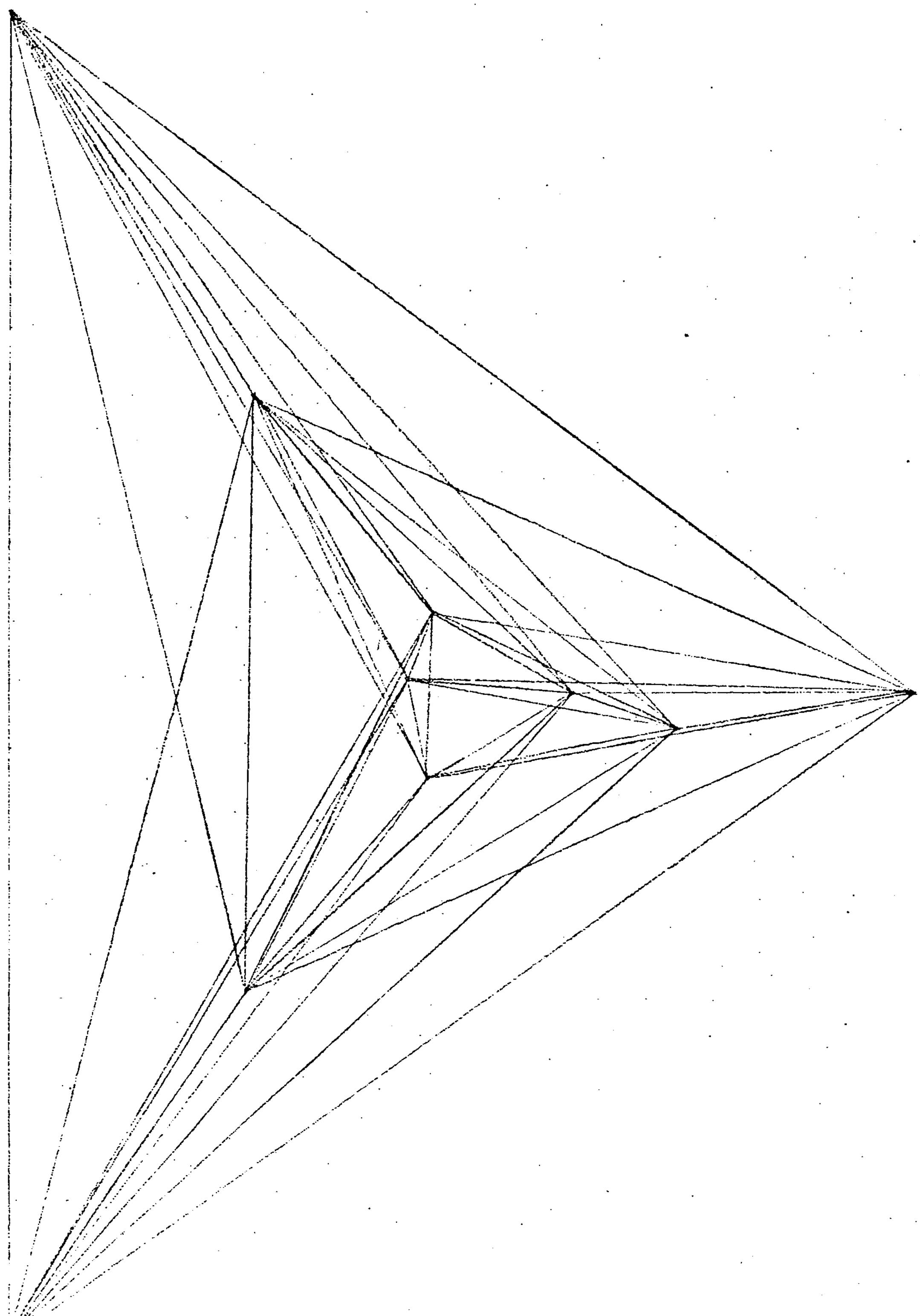
Figure 2b











six points a_1, a_2, a_3, b_1, b_2, b_3 form concentric triangles, since no edge $a_i b_j$ passes through [B]. Now if b_4 is any other point of B, it must lie in the annular region $[a_1a_2a_3] \setminus [b_1b_2b_3]$, in one of the three triangular regions bordering $[b_1b_2b_3]$ (see Figure 3). Then some edge $[a_i b_4]$ crosses [B].

Suppose an edge of type (1,1) passes through [B]. Then it must cross at least (b-2) edges of type (0,2) (and in general many more). This yields the formula

$$(4) \qquad k(1,3,0) \geq (b-3)(b-2).$$

There are two types of crossings that arise from sets of points of type (1,2,1). First, an edge of type (1,1,0) can cross an edge of type (0,1,1). This will occur whenever the edge of type (1,1,0) passes through [B]. By the last argument, this occurs at least (b-3) times, giving us an estimate:

$$(5) \qquad k(1,2,1)_1 \geq (b-3)c.$$

The second type of crossing involves an edge of type (1,0,1) and an edge of type (0,2,0). Given a vertex c_1 of C, every edge joining c_1 to a vertex of A must intersect the boundary of [B], giving at least one crossing. Suppose a_1 and a_2 are two vertices of A. Then the curve $[a_1c_1] \cup [c_1a_2]$ partitions B into B_1 and B_2, such that if b_1 is in B_1 and b_2 is in B_2, the edge $[b_1b_2]$ crosses the curve and conversely, provided only that $[a_1c_1]$ and $[c_1a_2]$ do not cross the same edge of the boundary of [B]. In that case the number of such crossings is $(B_1)(B_2) \geq (b-1)$. Since there always exist vertices a_1 and a_2 satisfying this condition, it follows that:

(6) $$k(1,2,1)_2 \geq (a+b-3)c.$$

We will later see that for $b \geq 4$, this minimum is achieved at the expense of minimizing crossings of type (0,4), for the number of crossings of type (0,3,1) is minimized by "centralizing" the inner vertex, while the opposite is true for crossings of type (1,2,1).

Finally we consider crossings of type (1,1,2). There need not be any crossings involving edges of types (1,1,0) and (0,0,2). On the other hand, for every pair of vertices c_1,c_2 of C, there must be at least two crossings of edges of type (1,0,1) and (0,1,1). For there must be two vertices b_1,b_2 of B such that $c_2 \in [b_1b_2c_1]$. Now c_2 can not be joined to every vertex of A by an edge which crosses $[b_1b_2]$. Therefore, some edge $[c_2a_1]$ crosses an edge $[b\ c_1]$. Reversing the roles of c_1 and c_2 gives the desired result.

(7) $$k(1,1,2) \geq c(c-1).$$

All other types of crossings are included in the previous considerations by deleting some of the vertices of D.

It must be noted that on the one hand, these estimates are crude, but on the other hand, each one individually is the best possible, since any one estimate may always be achieved.

Now using these estimates we derive some statements about $\overline{c}(K_n)$.

Facts About Certain Minimal Drawings

Proposition 1: Up to symmetry, there are exactly three minimal rectilinear drawings of K_7.

Proof: Let D be a drawing, and A, B, C as in the previous section. Then $|A| = a =$ either 7, 6, 5, 4, or 3. Using Formulas (1), (2), and (3), if $a \geq 4$ it follows that $k(D) \geq 13$. Therefore, we may assume $a = 3$. Then $k(4,0) = k(3,1) = 0$, and $k(2,2) = 6$ exactly.

Suppose $b = 4$. Then $k(0,4) = 1$. By (4), $k(1,3) \geq 2$. Thus $k(D) \geq 9$. Furthermore, in order to achieve the minimum $k(1,3) = 2$ and $k(D) = 9$, there must be two concentric triangles and the seventh vertex must lie in one of the three triangular regions bordering the inner triangle in Figure 3. Thus by symmetry there is a unique minimal drawing with $b = 4$.

Suppose $b = 3$. Then $k(0,4) = 0$ and $k(1,2,1)_2 = 3$ by (6). Therefore, if $k(D) = 9$, it follows that $k(1,3,0) = 0$, which means that A and B form concentric triangles, with the seventh vertex in the interior of [B]. This can be done in exactly two ways: either two crossings of type (1,2,1) can involve the same edge of the triangle B, or each edge of B is involved in exactly one crossing. ||

Theorem 1: $\overline{c}(K_8) = 19 > c(K_8) = 18$.

Proof: Using Formulas (1), (2), and (3), we observe:

IF $a = 8$, $k(D) = 70$.

IF $a = 7$, $k(D) \geq 35 + 21 + 0 = 56$.

IF $a = 6$, $k(D) \geq 15 + 24 + 12 = 51$.

IF $a = 5$, $k(D) \geq 5 + 15 + 12 = 32$.

IF $a = 4$, $k(D) \geq 1 + 8 + 12 = 21$.

Thus we may assume that $a = 3$. Now $3 \leq b \leq 5$. Using (3), $k(2,2) = 10$.

If $b = 5$, then $k(0,4) = 5$ and $k(1,3) \geq 6$ by Formula (4), so $k(D) \geq 21$.

If $b = 4$, then using Formulas (1) and (2), $k(0,4) = 3$, and $k(1,3,0) \geq 2$ by Formula (4). By Formulas (5) and (6), $k(1,2,1) \geq 5$. So $k(D) \geq 20$.

Thus in order to get no more than 19 crossings we must have $a = b = 3$, $c = 2$. Now using Formulas (3), (6) and (7), we have

$$k(2,2) = 10,\ k(1,2,1)_2 = 6,\ k(1,1,2) \geq 2,$$

and $k(0,4) = k(0,2,2) = 1$. Therefore $k(D) \geq 19$, and equality will occur if and only if A and B form concentric triangles and the vertices of C are placed so as to minimize $k(1,1,2)$. In particular $\overline{c}(K_8) = 19$. Since it is well known that $c(K_8) = 18$, the theorem is proved.

Theorem 2: In a minimal rectilinear drawing of K_9, $a = b = c = 3$.

Proof: Using (1), (2), and (3) we see that in a drawing D if $a \geq 5$, $k(D) \geq 49$.

Suppose $a = 4$. Then $3 \leq b \leq 5$. Using (1), (2), and (3) $k(4,0) = 1$. $k(3,1) = 10$. $k(2,2) \geq 20$.

Suppose $b = 5$. Then $k(0,4) = 5$. By (4), $k(1,3) \geq 6$. Therefore, $k(D) \geq 42$.

Suppose $b = 4$. Then $k(0,4) = 3$, $k(1,3,0) \geq 2$. By (5) and (6), $k(1,2,1) \geq 6$. So again, $k(D) \geq 42$.

Finally, suppose $b = 3$. Then $k(0,4) = 3$, $k(1,2,1)_2 = 6$, so $k(D) \geq 40$.

Consequently, we may assume that in a minimal rectilinear drawing D, $a = 3$. Then $k(2,2) = 15$.

Suppose now $b = 6$. Then $k(0,4) = 15$ and $k(1,3) \geq 12$ by (4). Thus $k(D) \geq 42$.

Suppose $b = 5$. Then using (1) and (2), $k(0,4) = k(0,4,0) + k(0,3,1) \geq 10$. $k(1,3,0) \geq 6$, $k(1,2,1) \geq 7$ by (4), (5), and (6). Thus $k(D) \geq 38$.

Suppose $b = 4$. Then $k(0,4) = k(0,4,0) + k(0,3,1) + k(0,2,2) \geq 7$, $k(1,3,0) \geq 2$, $k(1,2,1) \geq 10$, and $k(1,1,2) \geq 2$. Thus $k(D) \geq 36$.

Suppose $k(D) = 36$. In order to have $k(0,2,2) = 2$ there can be no crossing involving edges of types $(0,2,0)$ and $(0,0,2)$. This means that both vertices c_1 and c_2 of C must lie in the same quadrant of [B] determined by the diagonal edges of the quadrilateral. Let b_1 and b_2 be the unique points of B having the property that $[c_1b_1]$, $[c_1b_2]$, $[c_2b_1]$, and $[c_2b_2]$ do not intersect the diagonals of [B]. Now since $k(D) = 36$, $k(1,2,1)_2 = 8$ exactly. Therefore there are two vertices a_1 and a_2 of A such that $[c_1a_1]$, $[c_1a_2]$, $[c_2a_1]$ and $[c_2a_2]$ all intersect the edge $[b_1b_2]$. Assume without loss of generality that c_1 is not in $[c_2b_1b_2]$ (since otherwise c_2 is not in $[c_1b_1b_2]$). Then $[c_1a_1]$ and $[c_1a_2]$ must each cross either $[c_2b_1]$ or $[c_2b_2]$, But this would imply (see the derivation of (7)) that $k(1,1,2) \geq 3$ and $k(D) > 36$, which is a contradiction.

Thus we can conclude that $a = b = c = 3$.

Remark: There are many different minimal drawings of K_9, and it is not true that the three triangles A, B and C need be pairwise concentric. However, we can show the following:

Addendum: In any minimal drawing of K_9, the outer two triangles are concentric.

Proof: The statement of the addendum is equivalent to the following: If D is a minimal drawing of K_9, a_1 any point of A, then $[V\setminus\{a_1\}]$ is bounded by a triangle.

Let the vertices of A be a_1, a_2, a_3, those of B be b_1, b_2, b_3. Let $V\setminus\{a_1\} = V^*$; replacing V with V^* we may define sets A^*, B^*, and C^*. The notation $(p,q)^*$ will mean P points chosen from A^* and q from B^*. Now we apply our estimation techniques to the drawing of K_8 with vertices V^* in order to compute $k(D)$.

Suppose $|A^*| = 5$; i.e., $A^* = \{a_2, a_3, b_1, b_2, b_3\}$. Then A^* forms a convex pentagon and $B^* = C$ a triangle. Furthermore, since the vertices of B^* lie in $[b_1, b_2, b_3]$, no point of B^* lies in the "central" region of $[A^*]$. Thus $k(3,1)^* \geq 18$ instead of 15. $k(4,0)^* = 5$, $k(2,2)^* \geq 12$. Since every edge $[a_1 c_i]$ crosses the boundary of $[B^*]$, $k(D) \geq 38$. Therefore $|A^*| \neq 5$.

Suppose $|A^*| = 4$; i.e., $A^* = \{a_2, a_3, b_2, b_3\}$. Then $k(4,0)^* = 1$, $k(3,1)^* = 8$ and $k(2,2)^* \geq 12$. If $|B^*| = 4$, then $k(1,3)^* \geq 2$ and $k(0,4)^* = 1$. If $|B^*| = 3$, then $|C^*| = 1$ and $k(1,2,1)^*_2 \geq 4$. Thus the number of crossings in D not involving a_1 is at least 24. Each edge $[a_1 v]$, v in B^* or C^*, crosses a boundary edge of $[A^*]$, accounting for 4 crossings. Now we show that there are at least 9 crossings not counted. We consider two cases, depending on whether or not $[a_1 b_1]$ crosses $[b_2\, b_3]$. [Figure 4.]

Suppose so; let v and w be any two vertices among $\{b_1, c_1, c_2, c_3\}$. Since c_1, c_2, c_3 lie in the triangle B, we know that $[a_1 v]$ and $[a_1 w]$ cross $[b_2\ b_3]$. Now the configuration of $[A^* \cup \{v,w\}]$ is either that displayed in Figure 2b or in Figure 2c. In the former case, either $[a_1 v]$ crosses two edges of the form $[a^* w]$, a^* a vertex of A^*, or $[a_1 w]$ crosses two edges of the form $[a^* v]$, or one crossing of each of these two types occurs. In the latter case, either $[a_1 v]$ crosses one edge $[a^* w]$ or $[a_1 w]$ crosses an edge $[a^* v]$. Also in the latter case, there is a crossing of type $(2,2)^*$ not counted in our estimate of $k(2,2)^*$. Thus every pair $\{v,w\}$ gives two crossings not counted. This gives 12 new crossings in all.

Suppose $[a_1 b_1]$ does not cross $[b_2 b_3]$. [Figure 4b.] Assume w.l.o.g. that $[a_1 b_1]$ crosses $[a_2 b_2]$. If v and w are as in the last paragraph, then there are two uncounted crossings involving them (jointly) provided $[a_1 v]$ and $[a_1 w]$ both cross $[a_2 b_2]$ or both cross $[b_2 b_3]$. If however they cross different edges, analysis similar to that in the last paragraph yields only one new crossing. Of the six pairs in $\{b_1, c_1, c_2, c_3\}$, at least two can be found in which the edges $[a_1 v]$ and $[a_1 w]$ both cross $[b_2 b_3]$ or both cross $[a_2 b_2]$. Therefore, at least 8 new crossings have been found. We need only worry further if exactly two such pairs occur. But in that case, let x and y be chosen from $\{b_1, c_1, c_2, c_3\}$ so that $[a_1 x]$ crosses $[b_2 b_3]$ and $[a_1 y]$ crosses $[a_2 b_2]$. Then the loop $[a_1 x] \cup [xy] \cup [a_1 y]$ must cut the edge $[b_2 a_3]$, giving one more crossing not yet counted. This completes the proof of the addendum. ||

Theorem 3: $60 < \overline{c}(K_{10}) < 63$.

Proof: There is a rectilinear drawing of K_{10} with 62 crossings, constructed in the following way: Arrange nine points $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ in three pairwise concentric triangles A, B, C, with vertices having the same subscript close together (thus a_i, b_i, c_i are almost collinear). Arrange further that the edges a_1c_1 and a_2c_2 do not cross b_1b_2. Then a tenth vertex is placed <u>outside</u> the triangle [C] close to the barycenter of the edge $[c_1c_2]$; it can be joined to the other nine vertices adding only 26 crossings to the 36 already given. [Figure 5.] This drawing provides a clear warning about the asymmetric nature of minimal rectilinear drawings.

Suppose some drawing of K_{10} had 60 crossings. Since $\overline{c}(K_9) = 36$, it follows that deleting any one vertex leaves at least 36 crossings. Since the average number of crossings deleted is $60 \times 4/10 = 24$, it follows that deleting any vertex would leave a minimal drawing of K_9. By Theorem 2, any such drawing looks like a triangle enclosing a triangle. In order for ten points to have such a property, they would have to be arranged in three triangles A, B, and C with A and B concentric and B anc C concentric. The tenth point v would have to be in the region [C] or in one of the three triangular regions bordering [C] (as in Figure 3). Now there are 36 or more crossings involving the first nine points; 6 crossings of type (2,2) involving v; 6 crossings of type (0,4) involving v. If v is in C there are at least 12 crossings of type (1,3) involving edges $[a_i\ v]$, a_i in A. If v is not in C there are at least 11 such crossings. Finally, for each vertex c_i of C there is a vertex b_j of B and a vertex a_k of A such that $[b_j\ v]$ crossed $[a_k\ c_i]$. Thus there are at least 62 crossings in any such drawing, contrary to the assumption. ||

This proof is very close to showing that $\overline{c}(K_{10}) = 62$. By the same method used in the addendum to Theorem 2, it can be shown that A and B are concentric in any minimal drawing of K_{10}. However, if B and C are not concentric the above arguments do not work. Since minimal drawings of K_9 exist in which B and C are not concentric, it is not obvious that $\overline{c}(K_{10}) = 62$.

Asymptotic Behavior

We present here a construction of a rectilinear drawing of K_n, $n = 3^k$ which has the property that, among all constructions using concentric triangles, it minimizes the number of crossings. Using these drawings, we prove

Theorem 4. $\lim \overline{c}(K_n)/n^4 \leq 5/312$.

That this limit is known to exists follows without modification from the proof that $\lim c(K_n)/n^4$ exists; this may be found in [2].

We will give a drawing $D_n: K_n \to R^2$ for $n = 3^q$, for which the sequence $k(D_n)/n^4$ converges to $5/312$.

Assume inductively that D_r has been constructed for $r = 3^{q-1}$. (Induction begins at $q = 1$). $n = 3r$. The vertices of D_n will be partitioned into sets S_1, S_2, and S_3 with $|S_i| = r$.

By an affine transformation of R^2, we may assume that D_r takes K_r to the upper half plane with no two vertices of D_r having the same y-coordinate. The linear transformation $(x,y) \to (\epsilon x, y)$ for ϵ suitably small takes D_r to a drawing having two essential properties:

1) the new drawing has the same number of crossing as D_r,
2) although no three vertices are collinear, the set of vertices is almost collinear; that is, the line determined by any two vertices is almost parallel to the y-axis.

Construct D_{3r} by arranging three copies of the above drawing, one "along" each of the half lines $\theta = 0$, $\theta = 2\pi/3$, and $\theta = 4\pi/3$ (polar coordinates). The vertices of S_1 are near the line $\theta = 2\pi/3$, etc. If two vertices are chosen in, say, S_1, the line through them separates S_2 from S_3.

If two vertices are chosen from each set S_i, the six points form two concentric triangles. The standard drawings of K_6 and K_9 have this property.

Now we compute $k(D_{3r})$. Suppose P is a set of vertices with $|P| = 4$. If there is at least one point from each set S_i, there is no crossing. Suppose two points are chosen from each of two sets. Then there is always a crossing. There are $3\binom{r}{2}^2$ such crossings.

Suppose three points are chosen from one set S_i. Then choosing the fourth point from another set will either always give a crossing or never give a crossing, depending on which side of the line spanned by the inner and outer points the middle point of the three lines. Thus there are $3r\binom{r}{3}$ crossings of this type.

Finally, there are $3k(D_r)$ crossings arising from choosing four points from the same set S_i. We have

$$
\begin{aligned}
k(D_{3r}) &= r^2(r-1)(5r-7)/4 + 3k(D_r) \\
&= 5/4\ r^4 + Q(r) + 3k(D_r)
\end{aligned}
\tag{8}
$$

where Q is a polynomial of degree 3.

Let $b_r = k(D_r) - 5r^4/312$ (r a power of 3).

Then

$$
\begin{aligned}
b_{3r} - 3b_r &= k(D_{3r}) - 3k(D_r) - 5.81/312\ r^4 \\
&\qquad + 5.3/312\ r^4 \\
&= 5/4\ r^4 + Q(r) - 5/4\ r^4 \\
&= Q(r)
\end{aligned}
$$

Thus we have the relations

$$k(D_r) = br + 5/312\ r^4 \qquad r \text{ a power of } 3.$$

(9) $$k(D_3) = 0$$

$$b_{3r} - 3_{br} = Q(r)$$

It can then easily be shown that $\lim_{n\to\infty} b_r/r^4 = 0$ and thus $\lim_{n\to\infty} k(D_r)/r^4 = 5/312$.

In fact, one can easily solve the difference equations (9) to get

(10) $$k(D_n) = 1/312\ (5n^4 - 39n^3 + 91n^2 - 57n). \qquad ||$$

The construction above may well be a minimal drawing for n a power of 3. It is reasonable to believe that a minimal drawing of K_n utilizes concentric triangles, and among drawings of K_{3^r} as a family of pairwise-concentric triangles the above drawing is optimal. If n is not a power of 3 it is likely that this technique will not work to give an explicit construction. We summarize then with two conjectures:

Conjecture 1: $\overline{c}(K_n) = 1/312(5n^4 - 39n^3 + 91n^2 - 57n)$, for n a power of 3.

Conjecture 2: $\lim \overline{c}(K_n)/n^4 = 5/132 > \lim c(K_n)/n^4$.

Conjecture 1 implies Conjecture 2 by Theorem 4. Conjecture 2 might be verified independently by showing that for large n there is a minimal drawing of K_n which employs concentric triangles except for all small number of points which may be anywhere. The addendum to Theorem 2 suggests that this may happen even if the drawings described are not minimal.

References

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