Closed Elastic Curves and Rods

III Kirchhoff Rods

Kirchhoff Elastic Rod

We consider a thin elastic rod with circular cross-section and uniform density – the uniform symmetric (linear) Kirchhoff rod. The configurations of the rod are described abstractly using adapted framed curves:

$$\Gamma = \{\gamma(s); T, M_1, M_2\}$$

$\gamma(s)$ =(the *centerline* of the rod)

a unit speed curve in $\mathbb{R}^3$

$(T(s), M_1(s), M_2(s))$=the *material frame*

a positively oriented orthonormal frame.

$\gamma'(s) = T(s)$ the frame is *adapted* to the curve

The Darboux Vector

The rotation of the material frame can be described by the *Darboux vector*

$$\Omega = mT - m_2 M_1 + m_1 M_2$$

and the equations

$$\begin{array}{lllll} T' &= \Omega \times T &= & m_1 M_1 & + m_2 M_2 \\ M_1' &= \Omega \times M_1 &= -m_1 T & & + mc M_2 \\ M_2' &= \Omega \times M_2 &= -m_2 T & -m M_1 & \end{array}$$

Elastic Energy

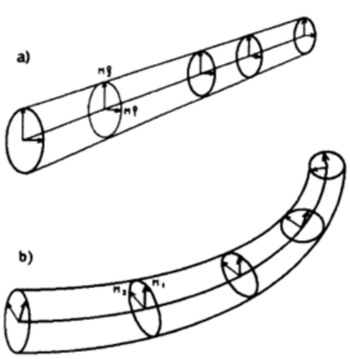
The *Total Elastic Energy* of a framed curve is given by

$$E(\Gamma) = \frac{1}{2}\int \underbrace{\alpha_1(m_1)^2 + \alpha_2(m_2)^2}_{bending} + \underbrace{\beta m^2}_{twisting}\ ds$$

where α_1, α_2 and β are material constants.

Symmetric case: $\alpha = \alpha_1 = \alpha_2$

In this case, the bending energy is $\frac{\alpha}{2}\int k^2\ ds$



(a) *undeformed rod;* (b) *deformed rod.*

Inertial Frames

Define an inertial frame along a rod by the equations:

$$\gamma'(t) = T \qquad \begin{array}{lll} T' = & & k_2 U \ + k_3 V \\ U' = & -k_2 T & \\ V' = & -k_3 T & \end{array}$$

The frame has no *twist*, i.e., U and V have no T-component to their angular velocity. Assume $U(0) = M_1(0)$. Then we can measure the twisting of the rod by looking at the angle $\theta(s)$ between $M_1(s)$ and $U(s)$.

$$m = \theta'$$

We can describe the rod by $\{\gamma, k, \theta\}$. The energy is

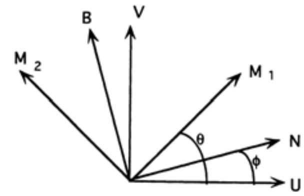
$$E = \frac{1}{2}\int \alpha k^2 + \beta(\theta')^2 \, ds$$

An elastic rod is an equilibrium configuration for the energy with appropriate boundary conditions (usually, having each end fixed in position and clamped.)

Proposition. *For a rod in equilibrium, θ' is constant. Thus $\theta(s) = ms$, for some fixed m.*

If instead of clamping the ends, we held them in collars (so the ends could not change direction but were free to twist), then the energy would be reduced by untwisting until $m = 0$. This shows that an untwisted rod is a minimizer of bending energy – an elastica.

We can also compare the inertial frame with the Frenet frame. Let $\phi(s)$ be the angle between N and U.



Torsion and Twisting

Write $N = U\cos\phi + V\sin\phi$

and $B = -U\sin\phi + V\cos\phi$

$$B' = \phi'(-U\cos\phi - V\sin\phi) + (u\sin\phi - v\cos\phi)T$$
$$= -\tau N$$

Comparing, $\phi' = \tau$

Note: $u = k\cos\phi$ and $v = k\sin\phi$

Now assume the rod is closed of length L, and (for convenience)that the material frame M satisfies $M(0) = M(L)$. That is, we take a rod, 'twist' it n times and weld the ends together. The Frenet frame automatically closes up, but the natural frame need not.

Let $\psi = \phi - \theta$ be the angle between the material frame and the Frenet frame. Then our assumption is that $\frac{\psi(L)-\psi(0)}{2\pi}$ is an integer. This leads to:

$$2\pi n = \psi(L) - \psi(0) = \int_0^L \phi'(s) - \theta'(s)ds$$
$$= \int_0^L \tau(s)ds - mL$$

The previous calculation says that an elastic rod centerline has total torsion $\int_0^L \tau(s)ds$ given by the quantity $2\pi n + mL$. Using this, it is possible to formulate a variational problem whose solutions are exactly the elastic rod centerlines.

Theorem. *For a curve $\gamma(s)$ define*

$$\mathcal{F}(\gamma) = \lambda_1 \int_\gamma ds + \lambda_2 \int_\gamma \tau ds + \lambda_3 \int_\gamma \kappa^2 ds$$

with κ and τ the curvature and torsion and $\lambda_3 \neq 0$. Then an extremal of $\mathcal{F}$ is an elastic rod centerline.

When $\lambda_2 = 0$, this is an elastic curve.

Integrating the rod equations

For a Kirchhoff elastic rod centerline γ, there are two Killing fields: for constants α and σ:

$$J = \frac{\alpha k^2 - \lambda}{2}T + \alpha k' N + k(\alpha\tau - \sigma)B$$

and

$$I = \sigma T + \alpha k B$$

Theorem. *For rod centerline γ there is an 'associated' elastic curve γ_0 whose curvature is k and torsion is $\tau - \frac{\sigma}{2\alpha}$, where k and τ are the curvature and torsion of γ.*

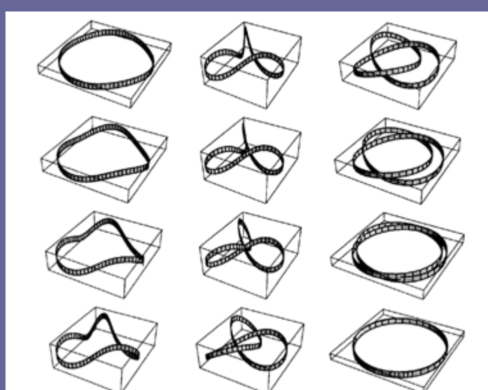
Closed Elastic Rods

Recall the result for elastic curves:

Theorem. **(Langer - Singer, 1983)** *There are infinitely many closed elastic curves which are nonplanar. All such elastica are embedded, lie on embedded tori of revolution, and represent* (m, n) *- torus knots, one for each* $m > 2n$.

In the case of rods, there is an extra parameter, allowing for many more closed curves. For rods, the result is:

Theorem. **(Ivey - Singer, 1998)** *Every torus knot type is realized by a smooth closed elastic rod centerline. For any pair of relatively prime positive integers* k *and* n *there is a one-parameter family of closed elastic rods forming a regular homotopy between the* k*-times covered circle and the* n*-times covered circle. The family includes exactly one elastic curve, one self-intersecting elastic rod, and one closed elastic rod with constant torsion.*



Closed Elastic Curves and Rods

IV Hamiltonian systems

Geometric Hamiltonian Systems*

E is an n-dimensional smooth manifold (Configuration Space).

$H : T^*E \to \mathbb{R}$ is a smooth function on the cotangent bundle (or Phase Space); H is the Hamiltonian.

In canonical coordinates $(p^1, \ldots, p^n, q^1, \ldots, q^n)$ H defines a Hamiltonian system:

$$\dot{q}^i = \frac{\partial H}{\partial p^i} \quad \dot{p}^i = -\frac{\partial H}{\partial q^i}$$

*V. Jurdjevic, Integrable Hamiltonian Systems on Complex Lie Groups, Memoirs of the A.M.S., v. 838 is an excellent reference for this lecture.

Poisson Brackets

If $F, G : T^*E \to \mathbb{R}$, then the Poisson bracket is

$$\{F, G\} = \sum_1^n \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p^i} - \frac{\partial F}{\partial p^i}\frac{\partial G}{\partial q^i}$$

$\{\bullet, H\}$ acts like a derivative:

$$\{FG, H\} = F\{G, H\} + G\{F, H\}$$

In terms of the Poisson bracket, Hamilton's equations are

$$\dot{q}^i = \{q^i, H\} \quad \dot{p}^i = \{p^i, H\}$$

If $\gamma(t) = (p^i(t), q^i(t))$ is a solution curve, then we can differentiate any quantity along γ by $\frac{dF}{dt} = \{F, H\}$. In particular, if $\{F, H\} = 0$, then F is a conserved quantity along γ, or a first integral of H.

H is Liouville integrable if there are functions $F_1, \ldots, F_n$ with all $\{F_i, F_j\} = 0, F_n = H$. The F_i are n constants of motion in involution. The Liouville-Arnol'd theorem says that the trajectories of H can be found by quadratures.

Now let $M = \mathbb{R}^3$, $\mathbb{S}^3$, or $\mathbb{H}^3$.

Let $E = FM$ be the space of positively-oriented orthonormal frames $f = (X; f_1, f_2, f_3)$ on M.

It is helpful to think of f as a linear map from $\mathbb{R}^3$ to the tangent space at X, taking the standard orthonormal basis (e_1, e_2, e_3) to the orthonormal vectors (f_1, f_2, f_3), where $f_3 = f_1 \times f_2$.

So configuration space E is a 6-dimensional manifold.

We may identify E with the Lie Group $\mathcal{G}$ of isometries of M.

M	$E \equiv \mathcal{G}$	Matrix description
$\mathbb{R}^3$	$E(3)$	$\begin{pmatrix} 1 & 0 \\ v & R \end{pmatrix}, v \in \mathbb{R}^3, R \in SO(3)$
$\mathbb{S}^3$	$SO(4)$	$A^T A = I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
$\mathbb{H}^3$	$SO(3,1)$	$A^T J A = J, J = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$\begin{array}{ccc} E & \longleftarrow & SO(3) \\ \downarrow \pi & & \\ M & & \end{array}$$

Principal Bundle

The *right* action $E \times SO(3) \to E$ (rotation of frames) defines fundamental vectorfields A_1, A_2, A_3 tangent to fibers of π (infinitesimal generators of the action)

The Lie Algebra $\mathfrak{so}(3)$

Let $\alpha_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$\alpha_3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Then

$[\alpha_1, \alpha_2] = \alpha_3$, $[\alpha_2, \alpha_3] = \alpha_1$, and $[\alpha_3, \alpha_1] = \alpha_2$

The group $SO(3)$ of rotations acts on E on the *right*: If $g : \mathbb{R}^3 \to \mathbb{R}^3$ is a rotation, then $(f, g) \mapsto f \circ g = R_g(f)$.

$$\mathbb{R}^3 \xrightarrow{g} \mathbb{R}^3 \xrightarrow{f} M_x$$

Each α_i gives rise to a one-parameter subgroup of $SO(3)$ and by the right action a one-parameter group of diffeomorphisms of E with infinitesimal generator A_i.

The vectors $A_1(X), A_2(X), A_3(X)$ span the *vertical* subspace $\mathbf{V}(X)$ of the tangent space at each point X of E.

The Riemannian metric defines the horizontal subspace $\mathbf{H}(u)$ of the tangent space at X. A basic vectorfield $B(\xi)$ is a horizontal field (using the Riemannian connection) such that $\pi_*(f)(B(\xi)) = f(\xi)$, where ξ is any vector in $\mathbb{R}^3$. In particular, let $B_i = B(e_i)$.

Equivariance property of $B(\xi)$:

$(R_g)_* B(\xi) = B(g^{-1}(\xi))$. *

FACT: A_i, B_i are left invariant vectorfields on $\mathcal{G}$.

*(Reference: Kobayashi and Nomizu, Foundations of Differential Geometry, Vol. 1)

More generally, if $\mathcal{G}$ is the isometry group of a Riemannian manifold M, then $\mathcal{G}$ acts on the space $\mathcal{E}$ of orthonormal frames on the *left*. If $\mathcal{I}$ is an isometry, then $d\mathcal{I}(x) : M_x \to M_x$ is an isometry of the tangent space at x, and takes frame f to $d\mathcal{I}(x) \circ f$.

$$\mathbb{R}^3 \xrightarrow{g} \mathbb{R}^3 \xrightarrow{f} M_x \xrightarrow{d\mathcal{I}(x)} M_x$$

This diagram shows how the isometries of M act (on the left) and the rotation group $SO(3)$ acts (on the right). The two actions commute.

Symmetries simplify!

Putting together the action of $SO(3)$ on the right and the action of the isometry group $\mathcal{G}$ on the left:

$$\mathcal{G} \times E \times SO(3) \longrightarrow E, \quad (\mathcal{I}, f, g) \longmapsto d\mathcal{I} \circ f \circ g$$

We see a nine-dimensional group $\mathcal{G} \times SO(3)$ acting on E, so there are lots of chances to reduce equations using symmetry.

Lie Bracket Formulas

1. $[A_i, A_j] = \varepsilon_{ijk} A_k \quad \longleftarrow$ Lie Algebra of $SO(3)$
2. $[A_i, B_j] = \varepsilon_{ijk} B_k$
3. $[B_i, B_j] = \varepsilon_{ijk} G A_k$

$\varepsilon_{ijk} = \pm 1$ depending on the sign of the permutation of $\{1, 2, 3\}$, 0 if two are equal.

Hamiltonians on E

If V is a vectorfield on E, then the Hamiltonian $\mathcal{H}_V : T^*E \to \mathbb{R}$ is defined by $\mathcal{H}_V(p) = p(V)$, p any covector.

This defines six *linear Hamiltonians* $\mathcal{A}_i, \mathcal{B}_i$ from A_i, B_i.

General Formula: $\{\mathcal{H}_V, \mathcal{H}_W\} = -\mathcal{H}_{[V,W]}$

In particular:

$$\{\mathcal{A}_i, \mathcal{A}_j\} = -\varepsilon_{ijk} \mathcal{A}_k$$

$$\{\mathcal{A}_i, \mathcal{B}_j\} = -\varepsilon_{ijk} \mathcal{B}_k$$

$$\{\mathcal{B}_i, \mathcal{B}_j\} = -\varepsilon_{ijk} G \mathcal{A}_k$$

The functions $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ are the generators of the algebra of left-invariant functions $\mathcal{LG}$. An element is a polynomial in six variables. We can add, multiply, and take Poisson brackets.

Key fact: 'geometric' variational problems on curves give rise to left-invariant Hamiltonian systems.

Frenet Equations

The (generalized) Frenet equations for a framed curve are

$$\gamma'(t) = T \qquad \begin{aligned} T' &= \quad k_2 U + k_3 V \\ U' &= -k_2 T \qquad\quad + k_1 V \\ V' &= -k_3 T - k_1 U \end{aligned}$$

Frenet – $k_1 = \tau \quad k_2 = k \quad k_3 = 0$

Inertial – $k_1 = 0$

If $f(t) = (\gamma(t); T, U, V)$ is a curve in E, then the Frenet equations become:

$$(FS)\ \frac{df}{dt} = B_1(f) + k_1 A_1(f) - k_3 A_2(f) + k_2 A_3(f)$$

This defines a control system: $k_i(t)$ are controls; given $f(0)$ we get a unique framed curve satisfying (FS). Then we may seek controls satisfying the condition that the "cost"

$$c = \int \mathcal{L}(k_1, k_2, k_3) ds$$

is minimal.

Example: $\frac{1}{2}\int k^2 ds (k_3 = 0)$ (Elastic curves)

$\frac{1}{2}\int c_2(k_3^2 + k_2^2) + c_1 k_1^2 ds$ (Kirkhoff rods)

Pontrjagin Maximum Principle

Given control system and cost functional, produce a left-invariant Hamiltonian system on T^*E whose trajectories project to solutions of the optimal control problem:

1. Lift (FS) to get a time-dependent Hamiltonian on T^*E (depending on control(s)) and subtract* the cost functional $\mathcal{L}$

$$(FS)\ \frac{df}{dt} = B_1(f) + k_1 A_1(f) - k_3 A_2(f) + k_2 A_3(f)$$

$$\mathcal{H}(p; k_i) = \mathcal{B}_1(f) + k_1 \mathcal{A}_1(f) - k_3 \mathcal{A}_2(f) + k_2 \mathcal{A}_3(f) - \mathcal{L}(k_1, k_2, k_3)$$

*This is a simplified description!

1. Lift (FS) to get a time-dependent Hamiltonian on T^*E (depending on control(s)) and subtract* the cost functional $\mathcal{L}$

2. Maximize $\mathcal{H}$ with respect to choice of controls $k_i(t)$ (for each fixed t. This is done by solving $\frac{\partial \mathcal{H}}{\partial k_i}$ for controls and eliminating them.

This gives a time-independent Hamiltonian.

Example: The Kirchhoff Rod

$$\mathcal{H}(p; k_1, k_2, k_3) = \mathcal{B}_1 + k_1 \mathcal{A}_1 - k_3 \mathcal{A}_2 + k_2 \mathcal{A}_3 - \frac{\alpha}{2}(k_2^2 + k_3^2) - \frac{\beta}{2} k_1^2$$

$$\frac{\partial \mathcal{H}}{\partial k_1} = 0 = \mathcal{A}_1 - \beta k_1$$

$$\frac{\partial \mathcal{H}}{\partial k_2} = 0 = \mathcal{A}_3 - \alpha k_2$$

$$\frac{\partial \mathcal{H}}{\partial k_3} = 0 = \mathcal{A}_2 - \alpha k_3$$

gives

$$\mathcal{H} = \mathcal{B}_1 + \frac{\mathcal{A}_2^2 + \mathcal{A}_3^2}{2\alpha} + \frac{\mathcal{A}_1^2}{2\beta}$$

Liouville –Arnol'd Integrability

The quadratic Hamiltonians

$$\mathcal{P} = \mathcal{A}_1\mathcal{B}_1 + \mathcal{A}_2\mathcal{B}_2 + \mathcal{A}_3\mathcal{B}_3$$

$$\mathcal{Q} = \mathcal{B}_1^2 + \mathcal{B}_2^2 + \mathcal{B}_3^2 + G(\mathcal{A}_1^2 + \mathcal{A}_2^2 + \mathcal{A}_3^2)$$

are in the center of $\mathcal{LG}$. That is, $\{\mathcal{P}, \mathcal{H}\} = 0$ and $\{\mathcal{Q}, \mathcal{H}\} = 0$ for all $\mathcal{H}$ in $\mathcal{LG}$.

(We only need check on the generators $\mathcal{A}_i$ and $\mathcal{B}_i$ because of the product rule).

Let $\mathcal{RG}$ be the algebra of *right* - invariant Hamiltonians; it is generated by the lifts of right-invariant vector fields. If $\mathcal{H} \in \mathcal{LG}$ and $\mathcal{K} \in \mathcal{RG}$, then $\{\mathcal{H}, \mathcal{K}\} = 0$. (This is because left and right actions commute, so the vector fields commute).

So if $\mathcal{H} \in \mathcal{LG}$, then by choosing $\mathcal{R}_1$ and $\mathcal{R}_2$ to be linear right-invariant Hamiltonians with $\{\mathcal{R}_1, \mathcal{R}_2\} = 0$ [which can be done in any of the three space-forms] we have five independent Hamiltonians in involution:

$\mathcal{H}$, $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}_1$, and $\mathcal{R}_2$.

To prove a given $\mathcal{H}$ is integrable, we need one more integral.

Example: $\mathcal{H} = \mathcal{B}_1 + \frac{\mathcal{A}_2^2 + \mathcal{A}_3^2}{2\alpha} + \frac{\mathcal{A}_1^2}{2\beta}$ commutes with $\mathcal{A}_1$. The other Hamiltonians also commute with it automatically. So the Kirchhoff elastic rod is Liouville integrable.

More Examples

Example: $k_1 = 0 \quad \mathcal{L}(k_2, k_3) = \frac{1}{2}(k_2^2 + k_3^2)$

Note that since $\nabla_T T = k_2 U + k_3 V = kN$, this functional gives rise to elastic curves (using inertial frames).

$\mathcal{H} = \mathcal{B}_1 + \frac{\mathcal{A}_2^2 + \mathcal{A}_3^2}{2}$ again commutes with $\mathcal{A}_1$, so the Euler elastica is integrable.

Example: $\mathcal{L}(k, \tau) = k^2 \tau$ leads to

$$\mathcal{H} = \mathcal{B}_1 + \mathcal{A}_3 \sqrt{\mathcal{A}_1}$$

Let $\mathcal{C} = \mathcal{A}_1^2 + \mathcal{A}_2^2 + \mathcal{A}_3^2 - 4\mathcal{A}_1\sqrt{\mathcal{B}_3} - 4G\mathcal{A}_1$

Then $\{\mathcal{C}, \mathcal{H}\} = 0$ (check this yourself!); so this defines an integrable system.

?

End of Part 4