Closed Elastic Curves and Rods

I: Euler's Elastica

Acknowledgements

- Joel Langer, Case Western Reserve Univ.
- Manuel Barros, Univ. of Granada, Spain
- Oscar Garay, Univ. del Pais Vasco, Spain
- Rongpei Huang, East China Normal Univ.
- Thomas Ivey, College of Charleston, S.C.
- Vel Jurdjevic, University of Toronto, Canada
- Anders Linner, Northern Illinois Univ.
- Daniel Steinberg, Dim Sum Thinking

Classical Bernoulli-Euler Elastica

Consider regular curves

$$\gamma : [a_1, a_2] \longrightarrow \mathbb{R}^3 \qquad \|\gamma'\| \neq 0$$

Assume (geodesic) curvature $k \neq 0$

The Frenet frame $\{T, N, B\}$ is orthonormal and satisfies

$$\gamma' = vT \quad T' = kN \quad N' = -kT + \tau B \quad B' = \tau N$$

The elastica minimizes the bending energy

$$\mathcal{F}(X) = \int_\gamma k^2 ds$$

With fixed length and boundary conditions.

Bending Energy Functionals

Let

$$\Omega = \{\gamma | \gamma(a_i) = \alpha_i, \gamma'(a_1) = \alpha_i',\}$$

$$\Omega_u = \{\gamma \in \Omega | \|\gamma'\| \equiv 1\}$$

$\mathcal{F}^\lambda : \Omega \longrightarrow \mathbb{R}$ is defined by

$$\mathcal{F}^\lambda(\gamma) = \frac{1}{2}\int_\gamma \|\gamma''\|^2 + \Lambda(s)\left(\|\gamma'\|^2 - 1\right) ds$$

Lagrange multiplier principle says a minimum of $\mathcal{F}$ on Ω_u is a stationary point for $\mathcal{F}^\lambda$ for some $\Lambda(s)$. ($\Lambda(s)$ is a *pointwise* multiplier, constraining speed.)

Deriving the Euler Equations

If W is a vector field along γ

$$\partial\mathcal{F}^\lambda(W) = \frac{\partial}{\partial\epsilon}\mathcal{F}^\lambda(\gamma + \epsilon W)|_{\epsilon=0}$$

$$0 = \frac{1}{2}\frac{\partial}{\partial\epsilon}\int_{a_1}^{a_2} \|(\gamma + \epsilon W)''\|^2 + \Lambda\|(\gamma + \epsilon W)'\|^2 ds$$

$$= \int_{a_1}^{a_2} \gamma'' \cdot W'' + \Lambda(s)\gamma' \cdot W' ds$$

Integrating by parts, $0 =$

$$\int_{a_1}^{a_2} -\gamma''' \cdot W' - (\Lambda\gamma')' \cdot W ds + (\gamma'' \cdot W' + \Lambda\gamma' \cdot W)\Big|_{a_1}^{a_2}$$

Integrating by parts, $0 =$

$$\int_{a_1}^{a_2} -\gamma''' \cdot W' - (\Lambda\gamma')' \cdot W ds + (\gamma'' \cdot W' + \Lambda\gamma' \cdot W)\Big|_{a_1}^{a_2}$$

$$= \int_{a_1}^{a_2} \left[\gamma'''' - (\Lambda\gamma')'\right] \cdot W ds \quad + (\gamma'' \cdot W' + (\Lambda\gamma' - \gamma''') \cdot W)\Big|_{a_1}^{a_2}$$

$$= \int_{a_1}^{a_2} E(\gamma) \cdot W ds + (\gamma'' \cdot W' + (\Lambda\gamma' - \gamma''') \cdot W)\Big|_{a_1}^{a_2}$$

where $E(\gamma) = \gamma'''' - \frac{d}{ds}(\Lambda\gamma')$

The Euler-Lagrange Equations

The elastica must satisfy

$$E(\gamma) = \gamma'''' - \frac{d}{ds}(\Lambda\gamma') \equiv 0$$

for some function $\Lambda(s)$.

Integrating,

$$\gamma''' - \Lambda(s)\gamma' \equiv J$$

for J a constant vector.

$$0 = \int_{a_1}^{a_2} E(\gamma) \cdot W ds + (\gamma'' \cdot W' + (\Lambda\gamma' - \gamma''') \cdot W)\Big|_{a_1}^{a_2}$$

This can also be derived from Nöether's Theorem: If γ is a solution curve and W is an infinitesimal symmetry, then $\gamma'' \cdot W' + (\Lambda\gamma' - \gamma''') \cdot W$ is constant. Letting W range over all translations (i.e. W is constant), we get

$$\Lambda\gamma' - \gamma''' = C$$

for C some constant field.

$$\gamma''' - \Lambda(s)\gamma' \equiv J$$

$$0 = \int_{a_1}^{a_2} E(\gamma) \cdot W ds + (\gamma'' \cdot W' - J \cdot W)\Big|_{a_1}^{a_2}$$

Use the Frenet Equations

$\gamma' = T, \gamma'' = kN, \gamma''' = -k^2 T + k'N + k\tau B$, so

$\gamma''' - \Lambda(s)\gamma' = (-k^2 - \Lambda(s))T + k'N + k\tau B = J$

Differentiate J to get $0 = J' =$

$(-3kk' - \Lambda')T + (k'' - k^3 - \Lambda k - k\tau^2)N + (k\tau' + 2k'\tau)B$

From this it follows that $\Lambda(s) = -\frac{3}{2}k^2 + \frac{\lambda}{2}$

for some constant λ.

$$J(s) = \frac{k^2 - \lambda}{2}T + k'N + k\tau B$$

The Vector field $J(s) = \frac{k^2-\lambda}{2}T + k'N + k\tau B$ is constant along the curve. Thus it is the restriction of a translation field to the curve.

From $J' = 0$ we get the equations

$$k'' + \frac{1}{2}k^3 - k\tau^2 - \frac{\lambda k}{2} = 0$$

and

$$k\tau' + 2k'\tau = 0$$

Use Rotational Symmetry

$$0 = \int_{a_1}^{a_2} E(\gamma) \cdot W ds + (\gamma'' \cdot W' - J \cdot W)\Big|_{a_1}^{a_2}$$

If W is a symmetry, we have

$$\gamma'' \cdot W' - J \cdot W = \text{constant}$$

Now let W be the restriction of a rotation field:

$$W = \gamma \times W_0, \quad W_0' = 0$$

$$kN \cdot T \times W_0 - J \cdot \gamma \times W_0 = \text{const}$$

$$(kN \times T - J \times \gamma) \cdot W_0 = \text{const}$$

$$(kN \times T - J \times \gamma) \cdot W_0 = \text{const}$$

Since this works for any W_0, we get

$$kB + J \times \gamma = A$$

for A a constant vector. So the vector field

$$I = kB = A + \gamma \times J$$

is the restriction of an isometry to the curve.

Killing Fields

A *Killing field* along a curve is a vector field along the curve which is the restriction of an infinitesimal isometry of the ambient space. If γ is an elastica in $\mathbb{R}^3$, then we have two Killing fields along γ:

$$J(s) = \tfrac{k^2-\lambda}{2}T + k'N + k\tau B$$

and

$$I = kB = A + \gamma \times J$$

where A and J are constant fields.

Conserved Quantities

$$k^2\tau = I \cdot J = A \cdot J = c$$

is constant, as is

$$4\,\|J\|^2 = (k^2-\lambda)^2 + 4k'^2 + 4k^2\tau^2 = a^2$$

Eliminating τ and replacing k^2 by u:

$$(u-\lambda)^2 + \frac{(u')^2}{u} + \frac{4c^2}{u} = a^2$$

or

$$(u')^2 = P(u)$$

for P a cubic polynomial.

Solving this differential equation gives

$$k^2 = u = k_0^2\left(1 - \frac{p^2}{w^2}\,\mathrm{sn}^2\left(\frac{k_0}{2w}s + \delta, p\right)\right) \quad k^2\tau = c$$

With $\mathrm{sn}(x,p)$ the *elliptic sine with parameter* p, and with p, w, and k_0 parameters. $0 \le p \le w \le 1$

The parameters p, w, and k_0 are related to the constants λ and c by

$$2\lambda = \frac{k_0^2}{w^2}(3w^2 - p^2 - 1)$$

$$4c^2 = \frac{k_0^6}{w^4}(1-w^2)(w^2-p^2)$$

Planar Elastic Curves

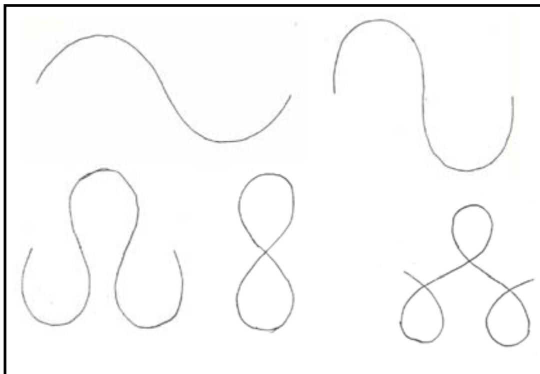
$$k^2\tau = c \qquad 4c^2 = \frac{k_0^6}{w^4}(1-w^2)(w^2-p^2)$$

A planar curve has $\tau = 0$, so $c = 0$. Thus either $w = 1$ or $w = p$. The parameter k_0 determines the maximum curvature.

When $w = p$, $k^2 = k_0^2\left(1 - \mathrm{sn}^2\left(\frac{k_0}{2p}s + \delta, p\right)\right)$

so $k = k_0\,\mathrm{cn}\left(\frac{k_0}{2p}s + \delta, p\right)$

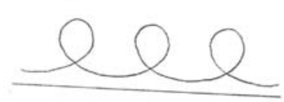
The curvature oscillates between $-k_0$ and $+k_0$. We call such a curve a "wavelike" elastica.



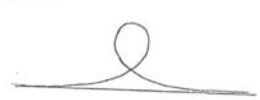
"wavelike" elastica

When $w = 1$, $k^2 = k_0^2\left(1 - p^2\,\mathrm{sn}^2\left(\frac{k_0}{2p}s + \delta, p\right)\right)$

so $k = k_0\,\mathrm{dn}\left(\frac{k_0}{2p}s + \delta, p\right)$ (elliptic delta), and k is nonvanishing. We call such a curve "orbitlike".



The borderline case, $p = w = 1$, has nonperiodic curvature:



Cylindrical Coordinates

Choose coordinates in $\mathbb{R}^3$ so that

$$J = \begin{pmatrix} 0 \\ 0 \\ \frac{a}{2} \end{pmatrix}, \quad A = \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix}, b \geq 0$$

The second equation is achieved by replacing γ by $\gamma - B$ for some B (translation).

Since $I = \gamma \times J + A$, we have

$$c = I \cdot J = A \cdot J = \frac{ab}{2}$$

$$I - \frac{I \cdot J}{J \cdot J} J = I - A = \gamma \times J$$

We have the coordinate fields

$$\partial z = \frac{2}{a} J$$

$$\partial \theta = \frac{2}{a} \gamma \times J = \frac{2}{a}(I - \frac{4c}{a^2} J)$$

$$\partial r = \frac{\partial z \times \partial \theta}{\|\partial z \times \partial \theta\|} = \frac{J \times B}{\|J \times B\|}$$

Writing $T = \frac{dr}{ds}\partial r + \frac{dz}{ds}\partial z + \frac{d\theta}{ds}\partial\theta$, we get

$$\frac{dr}{ds} = T \cdot \partial r \quad \frac{dz}{ds} = T \cdot \partial z \quad \frac{d\theta}{ds} = \frac{T \cdot \partial \theta}{\|\partial \theta\|^2}$$

and the equations for γ can be integrated explicitly.

The equation for r is

$$\frac{dr}{ds} = \frac{2k_s}{\sqrt{4k_s^2 + (k^2 - \lambda)^2}} = \frac{2kk_s}{\sqrt{k^2 a^2 - 4c^2}}$$

This integrates to $r = r_0 + \frac{2}{a^2}\sqrt{a^2 k^2 - 4c^2}$

So r has the same periodicity and critical points as k. The elastica lies between two concentric cylinders (the inner one perhaps degenerating to a line) around the z - axis. the maxima of the curvature occur on the outer cylinder and the minima on the inner cylinder.

In the 2-dimensional case ($c = 0$), the curve lies in a strip parallel to the $z - axis$. In this case the formula for r, the distance from the axis, simplifies to

$$r - r_0 = \frac{k}{a}$$

Non-planar elastic curves

The non-planar solutions satisfy $0 < p < w < 1$. For $p = 0$ we get

$$k \equiv k_0$$

the helices.

In other cases, $0 < k < k_0$ and the curve has nonvanishing curvature and torsion.

When is the curve closed? Its curvature is always periodic (except for straight lines and the borderline elastic curves ($p = 1$). We need the coordinates of γ to be periodic.

A little bit of elliptic integrals

The complete elliptic integrals $E(p)$ and $K(p)$ are given by:

$$K(p) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - p^2 \sin^2 \phi}}\, d\phi$$

and

$$E(p) = \int_0^{\frac{\pi}{2}} \sqrt{1 - p^2 \sin^2 \phi}\, d\phi$$

The elliptic function $\mathrm{sn}(x,p)$ is an odd function with $\mathrm{sn}(x + 2K(p), p) = -\mathrm{sn}(x,p)$. So the curvature of an elastica is periodic with period $4wK(p)/k_0$.

If $\triangle z$ and $\triangle \theta$ represent the change in z and θ over one period of k, then γ is a smooth closed curve if and only if $\triangle z = 0$ and $\triangle \theta$ is rationally related to 2π.

$$\triangle z = \int_0^{\frac{4w}{k_0}K(p)} \langle \partial z, T \rangle\, ds = \int_0^{\frac{4w}{k_0}K(p)} k^2 - \lambda ds$$

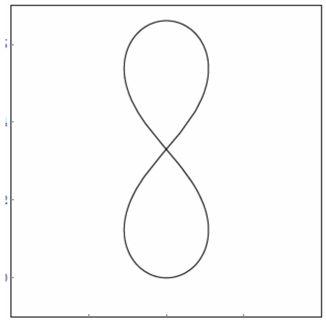
The closure condition may be written:

$$\triangle z = 0 \Longleftrightarrow 1 + w^2 - p^2 - 2\frac{E(p)}{K(p)} = 0$$

There is one closed planar curve (besides the circle): It requires

$$w = p \quad 2E(p) = K(p) \Longleftrightarrow p \approx .82$$

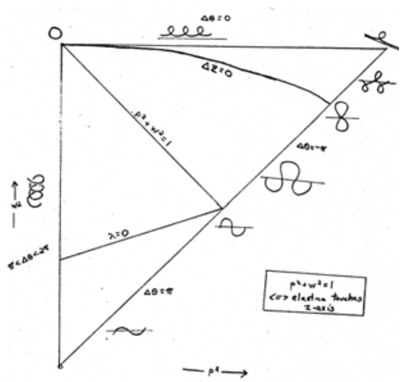
The Figure-Eight elastica



Finding Closed Curves

The second condition for closure is that the θ coordinate be periodic. Let $\triangle\theta$ denote the increase in the θ coordinate in one period of the curvature function. Then it is necessary that $\triangle\theta$ be a rational multiple of 2π.

Theorem. **(L - S., 1983)** *$\triangle\theta$ is monotonically decreasing from π to 0 along $\triangle z = 0$. Thus there are infinitely many closed elastic curves which are nonplanar. All such elastica are embedded, lie on embedded tori of revolution, and represent (m, n) - torus knots, one for each $m > 2n$.*



Closed Elastic Curves and Rods

II: Elastica in a Space Form

Riemannian Geometry

M is a smooth Riemannian manifold

metric $g(X, Y) = < X, Y >$ covariant derivative $\nabla_X Y$

For vector fields X and Y

$$\nabla_X Y - \nabla_Y X = [X, Y] = XY - YX$$

Ex: If $X = \frac{\partial}{\partial x}$, and $Y = x\frac{\partial}{\partial y}$ in local coordinates, then

$$\nabla_X Y - \nabla_Y X = \frac{\partial}{\partial y} - 0 = \left[\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}\right]$$

The curvature tensor R is given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

Curves in M

$\gamma(t)$ is an immersed curve in M with velocity vector $V = vT$ and squared curvature

$$k^2 = \|\nabla_T T\|^2$$

For a family of curves $\gamma_w(t) = \gamma(w, t)$ we will write

$$W = W(w, t) = \frac{\partial\gamma}{\partial w}$$

$$V(w, t) = \frac{\partial\gamma}{\partial t} = v(w, t)T(w, t)$$

(So V is velocity and v is speed)

Some Useful Formulas

1. $0 = [W, V] = [W, vT] = W(v)T + v[W, T]$

   So $[W, T] = -\frac{W(v)}{v}T = gT$

2. $2vW(v) = W(v^2) = 2\langle\nabla_W V, V\rangle$

   $= 2\langle\nabla_V W, V\rangle = 2v^2\langle\nabla_T W, T\rangle$

   So $W(v) = -gv$, $g = -\langle\nabla_T W, T\rangle$

3. $W(k^2) = 2\langle\nabla_T\nabla_T W, \nabla_T T\rangle$
   $+ 4gk^2 + 2\langle R(W, T)T, \nabla_T T\rangle$

Proof of (3)

$$W(k^2) = 2\langle\nabla_W\nabla_T T, \nabla_T T\rangle$$

$$= 2\left\langle\nabla_T\nabla_W T + \nabla_{[W,T]}T + R(W, T)T, \nabla_T T\right\rangle$$

$$= 2\left\langle\nabla_T\nabla_T W + \nabla_T(gT) + \nabla_{gT}T + R(W, T)T, \nabla_T T\right\rangle$$

$$= 2\langle\nabla_T\nabla_T W, \nabla_T T\rangle + 2\langle R(W, T)T, \nabla_T T\rangle + 4g\langle\nabla_T T, \nabla_T T\rangle$$

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

$\gamma : [0, 1] \to M$ is a curve of length L. Now for fixed constant λ let

$$\mathcal{F}^\lambda(\gamma) = \frac{1}{2}\int_0^L k^2 + \lambda ds = \frac{1}{2}\left(\int_0^L k^2 ds + \lambda L\right)$$

$$= \frac{1}{2}\int_0^1 (\|\nabla_T T\|^2 + \lambda)v(t)dt$$

For a variation γ_w with variation field W we compute

$$\frac{d}{dw}\mathcal{F}^\lambda(\gamma_w) = \frac{1}{2}\int_0^1 W(k^2) - (k^2 + \lambda)gds$$

$$= \int_0^1 \langle\nabla_T\nabla_T W, \nabla_T T\rangle + 2gk^2 + \langle R(W, T)T, \nabla_T T\rangle - \frac{1}{2}(k^2 + \lambda)gds$$

Now integrate by parts, using $g = -\langle\nabla_T W, T\rangle$

$$\frac{d}{dw}\mathcal{F}^\lambda(\gamma_w) = \int_0^1 \langle\nabla_T\nabla_T W, \nabla_T T\rangle - \left\langle\nabla_T W, 2k^2 T\right\rangle + \langle R(W, T)T, \nabla_T T\rangle + \frac{1}{2}\left\langle\nabla_T W, (k^2 + \lambda)T\right\rangle ds$$

$$= \int_0^L \langle E, W\rangle ds + \left[\langle\nabla_T W, \nabla_T T\rangle + \left\langle W, -(\nabla_T)T^2 + \Lambda T\right\rangle\right]_0^L$$

where

$$E = (\nabla_T)^3 T - \nabla_T(\Lambda T) + R(\nabla_T T, T)T$$

and $\Lambda = \frac{\lambda - 3k^2}{2}$

When M is a manifold of constant sectional curvature G, the formula for E can be simplified to

$$E = (\nabla_T)^3 T - \nabla_T(\Lambda_G T)$$

where

$$\Lambda_G = \frac{\lambda - 2G - 3k^2}{2}$$

$$E = \nabla_T\left(\nabla_T kN - \frac{\lambda - 2G - 3k^2}{2}T\right)$$

Euler-Lagrange Equations

$$E = \nabla_T\left(\nabla_T kN - \frac{\lambda - 2G - 3k^2}{2}T\right)$$

$$= \nabla_T\left(\frac{k^2 - \lambda + 2G}{2}T + k_s N + k\tau B\right)$$

$$= \frac{2k_{ss} + k^3 - \lambda k + 2Gk - k\tau^2}{2}N + (2k_s\tau + k\tau_s)B$$

The equation $E = 0$ for the elastica becomes:

$$2k_{ss} + k^3 - \lambda k + 2Gk - k\tau^2 = 0 \quad \text{and} \quad 2k_s\tau + k\tau_s = 0$$

$$2k_{ss} + k^3 - \lambda k + 2Gk - k\tau^2 = 0 \quad \text{and} \quad 2k_s\tau + k\tau_s = 0$$

The second equation integrates to

$$k^2\tau = c$$

Eliminating τ from the first equation and integrating:

$$k_s^2 + \frac{k^4}{4} + (G - \frac{\lambda}{2})k^2 + \frac{c^2}{k^2} = A$$

Letting $u = k^2$ this becomes

$$u_s^2 + u^3 + 4(G - \frac{\lambda}{2})u^2 - 4Au + 4c^2 = 0$$

Solutions

1. $u = k^2$=constant, τ=constant
2. $k = k_0$ sech $\left(\frac{k_0}{2w}s\right), \tau = 0$
3. $k = k_0 \,\mathrm{cn}\left(\frac{k_0}{2w}s, p\right), \tau = 0$
4. $k = k_0 \,\mathrm{dn}\left(\frac{k_0}{2w}s, p\right), \tau = 0$
5. $k^2 = k_0^2\left(1 - \frac{p^2}{w^2}\,\mathrm{sn}^2\left(\frac{k_0}{2w}s, p\right)\right)$

   where $4G - 2\lambda = \frac{k_0^2(1+p^2-3w^2)}{w^2}$

   and $0 \le p \le w \le 1$

Killing Fields

Prop.: Let M be a (simply-connected) manifold with constant sectional curvature G, and let γ be an elastica in M. Then the vectorfields $J = \frac{k^2-\lambda}{2}T + k_sN + k\tau B$ and $I = kB$ along γ extend to Killing fields (infinitesimal isometries) on M.

Idea of proof: Verify that when $W = I$ or $W = J$, then W preserves arclength parameter, curvature, and torsion of γ. For arclength, one checks that $\langle \nabla_T W, T\rangle = 0$. For curvature, use the formula for $W(k)$. For torsion, use the formula:

$$W(\tau^2) = 2\left\langle \frac{1}{k}(\nabla_T)^3 W - \frac{k_s}{k^2}(\nabla_T)^2 W + \left(\frac{G}{k} + k\right)\nabla_T W - \frac{k_s}{k^2}GW, \tau B\right\rangle$$

Elastica on the 2-sphere

The Killing field $J = \frac{k^2-\lambda}{2}T + k_sN$ is the restriction to γ of a rotation field. By choosing coordinates x, y of longitude and latitude on the sphere, we may assume that

$$\frac{\partial}{\partial x} = aJ$$

where a is a constant chosen so that J has unit length on the equator.

$$\|J\|^2 = \frac{(k^2-\lambda)^2}{4} + k_s^2 = A + \frac{\lambda^2}{4} - Gk^2$$

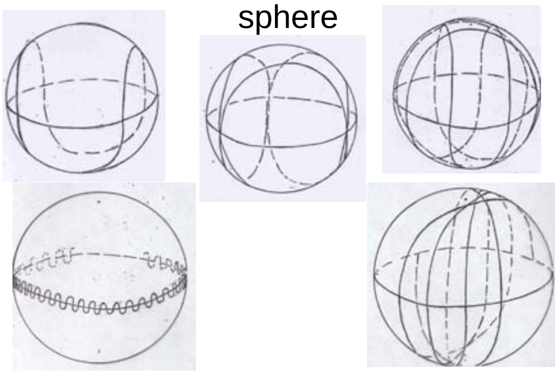
$$k_s^2 + \frac{k^4}{4} + (G - \frac{\lambda}{2})k^2 + \frac{c^2}{k^2} = A$$

$$\|J\|^2 = \frac{(k^2-\lambda)^2}{4} + k_s^2 = A + \frac{\lambda^2}{4} - Gk^2$$

The norm of J is maximized where k is minimized. So if $k = k_0 \,\mathrm{cn}\left(\frac{k_0}{2w}s, p\right)$, then k vanishes at the maxima. Since $\langle N, J\rangle = k_s \neq 0$ when $k = 0$, the curve γ is transverse to the coordinate curves y = const. at these points. It follows that the curve is crossing the equatorial curve $y = 0$ at the inflection points. The normalizing constant a is precisely $\sqrt{A + \frac{\lambda^2}{4}}$.

Theorem. *If γ is a wavelike elastica on a two-dimensional space-form, then the inflection points of γ all lie on a geodesic (the "axis" of the curve).*

Wavelike elastic curves on the sphere



Classifying wavelike elastic curves on the sphere

The wavelength Λ of an elastica is the amount of progress it makes along its axis in one complete period of k, as measured by arclength along the geodesic.

$$\frac{\Lambda(p)}{2} = \frac{\varepsilon\pi\Lambda_0(\Psi, p)}{\sqrt{G}} - \frac{(4G - 4Gp^2 - \lambda)\sqrt{1-(1-p^2)\sin^2\Psi}\,\sin\Psi K(p)}{(2G-\lambda)\sqrt{G}},$$

where $K(p)$ is the *complete elliptic integral of the first kind*, $\Lambda_0(\Psi, p)$ is the *Heumann lambda function*, and Ψ and ε are given by

$$\sin\Psi = 2\sqrt{4G^2 - 2G\lambda}\,\sqrt{1-2p^2}/[4G(1-p^2)-\lambda]\,,$$
$$\varepsilon = (4Gp^2 - \lambda)/|4Gp^2 - \lambda|\,.$$

Theorem. **(L-S, 1987)** *Let λ be a fixed constant with $0 \leq \frac{8}{7}G$. Then for each pair of positive integers m, n with*

$$\frac{m}{2n} < 1 - \frac{\sqrt{G}}{\sqrt{4G - 2\lambda}}$$

there is a unique elastica $\gamma^{\lambda}_{m,n}$ (up to congruence) which closes up in n periods while crossing the equator m times.