PL STRUCTURAL STABILITY

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If K^k is a finite polyhedron and $f,g:K \longrightarrow R^n$ are PL maps, then we say g is a <u>C'</u> δ-approximation to f if the following two conditions are satisfied:

1) $\|f(x)-g(x)\| < \delta$ for all $x \in K$, $\| \;\|$ the usual metric on R^n.

2) If K' subdivides K such that $f,g:K' \longrightarrow R^n$ are both linear, and if $b \in K$, then $\|(f(x)-f(b))-(g(x)-g(b))\| < \delta\|x-b\|$ for all $x \in St(b,K') = \cup\{cl(\sigma)\,|\,b \in cl(\sigma)\}$. Here K is assumed to inherit a linear metric as an embedded subcomplex of some Euclidean space.

If $PL(K,R^n)$ denotes the space of PL maps $f:K \longrightarrow R^n$ with the topology derived from this notion of approximation, it is easily seen that this topology is independent of the particular choices of metrics. (See Munkres [2] for details.)

Two maps, f and g, are <u>structurally equivalent</u> if there is a commutative square

$$\begin{array}{ccc} K & \xrightarrow{\quad f \quad} & R^n \\ P\downarrow & & \downarrow Q \\ K & \xrightarrow{\quad g \quad} & R^n \end{array}$$

where P and Q are PL homeomorphisms. A map $f \in PL(K,R^n)$ is <u>structurally stable</u> if there is a neighborhood N of f in $PL(K,R^n)$ such that each map $g \in N$ is structurally equivalent to f and the equivalence is canonical, i.e., there is a diagram

$$\begin{array}{ccc} N \times K & \xrightarrow{1_N \times f} & N \times R^n \\ P & & Q \\ N \times K & \xrightarrow{E} & N \times R^n \end{array}$$

where $E(g,x) = (g,g(x))$, $Q(g,y) = (g,Qg(y))$, $P(g,x) = (g,Pg(x))$.

The result I wish to indicate in this paper is that the set of structurally stable maps is dense and open in $PL(K,R^n)$ and such maps are of a particularly simple geometric nature.

The proofs of the following theorems may be found in [3].

THEOREM 1. For any positive integer S, there exist points $x_1,\ldots,x_s$ in R^n with the following property: Let $P_1,\ldots,P_t$ be affine subspaces of R^n each of which is spanned by some subset of $\{x_1,\ldots,x_s\}$. Let $\mathring{P}_i$ denote the subset of P_i whose complement is the union of all affine subspaces of P_i spanned by points in the spanning set of P_i. Then $\dim(\mathring{P}_1 \cap \ldots \cap \mathring{P}_t) \leq \sum_{i=1}^{t} \dim P_i - (t-1)n$.

Geometrically, this condition for planes is that of minimal intersection; note that $P_1 \cap \ldots \cap P_t$ may have larger dimension due to the overlap of spanning sets.

THEOREM 2. The set of s-tuples of points $(x_1,\ldots,x_3) \in (R^n)^s$ satisfying Theorem 1 is open and dense.

COROLLARY 3. Let $f:K \longrightarrow R^n$ be a PL map and let K' subdivide K such that $f:K' \longrightarrow R^n$ is linear. Then there is a linear map $f^*:K' \longrightarrow R^n$ such that

1. If $\sigma_1,\ldots,\sigma_t$ are simplices of K', $f(\mathring{\sigma}_1) \cap \ldots \cap f(\mathring{\sigma}_t)$ is either empty or a convex linear cell of dimension

$(\dim \sigma_1 + \ldots + \dim \sigma_t - (t-1)n)$

2. f^* is a δ-approximation to f, $\delta > 0$ arbitrarily small.

Proof. Choose points $x_1, \ldots, x_s$ in R^n arbitrarily close to the points $\{f(v_i), 1 \leq i \leq s\}$, where $\{v_1, \ldots, v_s\}$ are the vertices of K'. Define f^* to be the unique linear extension of the map $v_i \longrightarrow x_i$. Then Theorem 1 assures the desired geometrical result.

Remark. f^* may be thought of as a map in general position, and the geometrical properties of the map are stronger than the usual ones for general position, since simplices meet "nicely".

Main Theorem. f^* is structurally stable.

I will present here an outline of the proof, which is conceptually simple; the main omission will be the deluge of epsilontics required at every stage of every induction argument.

Step 1: Assume K' is the (n-1)-skeleton of an s-simplex.

Let K' be linearly independent. Thus K' lies on the boundary of an s-simplex, s the number of vertices of K'. There is a canonical extension of f^* to the n-1 skeleton $\bar{K}$ of this simplex.

Lemma. If K' is a subcomplex of $\bar{K}$ and $f: \bar{K} \longrightarrow R^n$ is a PL map, and if $\epsilon > 0$, $\exists \delta > 0$ such that if $g: K' \longrightarrow R^n$ δ-approximates $f|K'$, g can be extended to an ϵ-approximation $\bar{g}$ to f, furthermore, the process is continuous in g.

This lemma allows us to assume $\bar{K} = K'$.

Step 2: Any small perturbation of a simplex in R^n can be extended to a small perturbation of the plane spanned by this simplex in a canonical way (after appropriate preparation). If we view f^* as mapping a family of codimension 1 planes into R^n (corresponding to the (n-1)-simplices of K'), then any approximation g to f^* also can be viewed as mapping these planes into R^n. In order to construct the equivalence maps for f^* and g, we try to find a map $R^n \longrightarrow R^n$ which matches these configurations of planes. This will in particular match $f^*(K')$ and g(K') simplex for simplex.

Thus we have the following geometric problem: if a family of planes arising in the above manner is perturbed slightly (in the C^1 sense - in fact, we are able to assume everything happens in a compact portion of R^n), does the new configuration of planes resembel the old configuration?

Here the crucial lemma is needed.

Lemma 4. If two planes meeting transversely in R^n are each perturbed slightly, the intersection is also perturbed slightly.

Proof. The idea of this proof is as follows: by perturbing all of R^n slightly, we may bring one of the planes back to its original position. This allows us to assume one plane is kept pointwise fixed. Then projection in a direction parallel to the fixed plane can be made to project the other plane to its original position (setwise) with the new intersection automatically projecting to the old intersection.

Step 3: Inductive application of the preceding lemma shows that each intersection of a family of perturbed planes can be matched up with the original intersection of the planes before perturbation. The collection of codimension-one planes gives a cell-decomposition of R^n. After perturbing these planes, the new cell decomposition of R^n is exactly the same, and although the new cells are not convex, they will be starlike about appropriately chosen interior points for small enough perturbation. The homeomorphism of R^n may now be construced skeletally, matching up corresponding cells by taking the join of the map inductively defined on the boundary of the cell with the identity map on the center point.

This inductive process depends on a conjugation process for maps. When the (k-1)-skeleta have been matched up, the k-skeleta are matched up in the following way: let P be a k-plane, $\bar{P}$ be the perturbed version. $\mathcal{U}$ is the intersection of the (k-1)-skeleton with P, and $\mathcal{U}'$ is the intersection of $\bar{P}$ with the perturbed (k-1)-skeleton. We have a map $\mathcal{U} \longrightarrow \mathcal{U}'$. Projecting $\bar{P}$ back onto P gives a map of $\mathcal{U} \longrightarrow P$ which is close to the inclusion. Extend this map to a homeomorphism $P \longrightarrow P$ and follow by the inverse $P \longrightarrow \bar{P}$ of the projection. It is the construction of the map $P \longrightarrow P$ which uses the technique of matching up cells.

Step 4: The final map from R^n to R^n automatically carries $f^*(\sigma)$ to $g(\sigma)$ setwise, so the map $K \longrightarrow K$ is constructed by the composition $\sigma \xrightarrow{f^*} f^*(\sigma) \longrightarrow g(\sigma) \xrightarrow{g-1} \sigma$.

The set of structurally stable maps of the type described above has been shown to be dense in $PL(K,R^n)$. Structural stability is by definition an open condition.

If h is a structurally stable map, then we may find a structurally equivalent map h^* of the type constructed in the Main Theorem. (since <u>any</u> approximation of h will be structurally equivalent to h.) Therefore, every structurally stable map has the geometric character of a map which embeds simplices linearly with minimal intersections.

REFERENCES

1. Levine, H.I., Singularities of differentiable mappings, Lecture notes for R. Thom, 1959.

2. Munkres, J.R., Elementary Differential Topology, Princeton University Press, 1966.

3. Singer, D.A., General Position and Structural Stability in Piecewise Linear Topology, Ph.D. Thesis, University of Pennsylvania, 1970.