GENERAL POSITION AND STRUCTURAL STABILITY

IN PIECEWISE LINEAR TOPOLOGY

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INTRODUCTION

General position for piecewise linear maps is a tool whereby the geometric character of a map is simplified in order to make a problem more susceptible to geometric methods of solution. The definition of general position varies greatly in the literature; usually a map is said to be in general position if it satisfies the conditions necessary to solve the particular problem for which the notion was defined. These ad hoc definitions leave open the question of what precisely can and should be required of maps in general position. This is the question considered in this paper, specifically for maps of polyhedra into Euclidean space.

One way to approach the problem is to list a family of conditions which should be met. First of all, in order for a definition to be useful, it must be attainable; specifically, one should require that given any PL map f there should be a nearby map f' in general position. Here the word "nearby" refers to a topology on the space of maps of the polyhedron K into $\mathbb{R}^n$, which we denote $PL(K, \mathbb{R}^n)$. This restriction puts an upper bound on the complexity of the definition.

Similarly, a lower bound for the definition comes immediately from the geometric nature of the concept. The set of singularities, or self-intersections, of a map f, denoted $S(f)$, is the closure of the double point set $\{x \varepsilon K: \ f(x) = f(x'), \text{ some } x' = x\}$. If K is a k-complex, then in general $S(f)$ is a subcomplex of dimension at least $2k-n$; this is because if s_1 and s_2 are two k-simplexes of K, f is linear on s_1

and s_2, and $f(\mathring{s}_1) \cap f(\mathring{s}_2) \neq \emptyset$, then $\dim (f(s_1) \cap f(s_2)) \geq 2k-n$ by linear algebra. Thus a minimal definition for maps in general position should include the condition that $\dim S(f) \leq 2k-n$.

There are some obvious refinements that can be made to this last condition. If $S_r(f)$, r any integer greater than 1, is the closure of the set $\{x \in K: f^{-1}f(x)$ has at least r elements$\}$ we may require that $\dim S_r(f) \leq rk-(r-1)n$. If we assume $k<n$, this immediately implies that f is <u>non-degenerate</u>, that is, that it does not reduce dimension anywhere. These conditions do not yet seem strong enough, since if K is an inhomogeneous complex, so that at some point x in K, K looks l-dimensional, $l<k$, then we would like $S_r(f)$ to be smaller near x. One way to accomplish this is to require that if s_1 and s_2 are simplexes of some given triangulation of K, a map f should satisfy $\dim(f\mathring{s}_1 \cap f\mathring{s}_2) \leq \dim s_1 + \dim s_2 - n$. Similarly, one can generalize this to consideration of r simplexes, r any integer. Thus we may define a map to be in general position if for some triangulation of K any set of simplexes of K have their images meet in minimal dimension. Of course, it may then be necessary to try to characterize such intersections.

The difficulty of all of this is that one never knows when to stop unless he has already gone too far. There is, however, another approach to the problem, and this is the notion of structural stability.

If f and g are maps from $K \to \mathbb{R}^n$, say that f is structurally equivalent to g if there are homeomorphisms $P:K \to K$ and $Q:\mathbb{R}^n \to \mathbb{R}^n$ such that $Qf=gP$. Whatever definition of general position we make, it must be true that if f is in general position and g is structurally equivalent to f then g is in general position. A map is structurally stable if it is structurally equivalent to every map near it. Since general position is dense, any such map must be in general position, since some nearby map is in general position. Therefore, if we consider all structurally stable maps, these should be our models for maps in general position.

Structural stability, however, is not easily found in PL topology. If $PL(K,\mathbb{R}^n)$ is given the usual (i.e., compact-open) topology and $\dim K \geq 1$, then no maps at all are structurally stable. This reflects the fact that the usual topology has a non-geometric character. There is however another topology on $PL(K, \mathbb{R}^n)$, namely the "smooth" or C^1-topology, which is an analogue of the smooth topology in the differentiable category. In this topology, the structurally stable maps are dense, so that general position can be achieved.

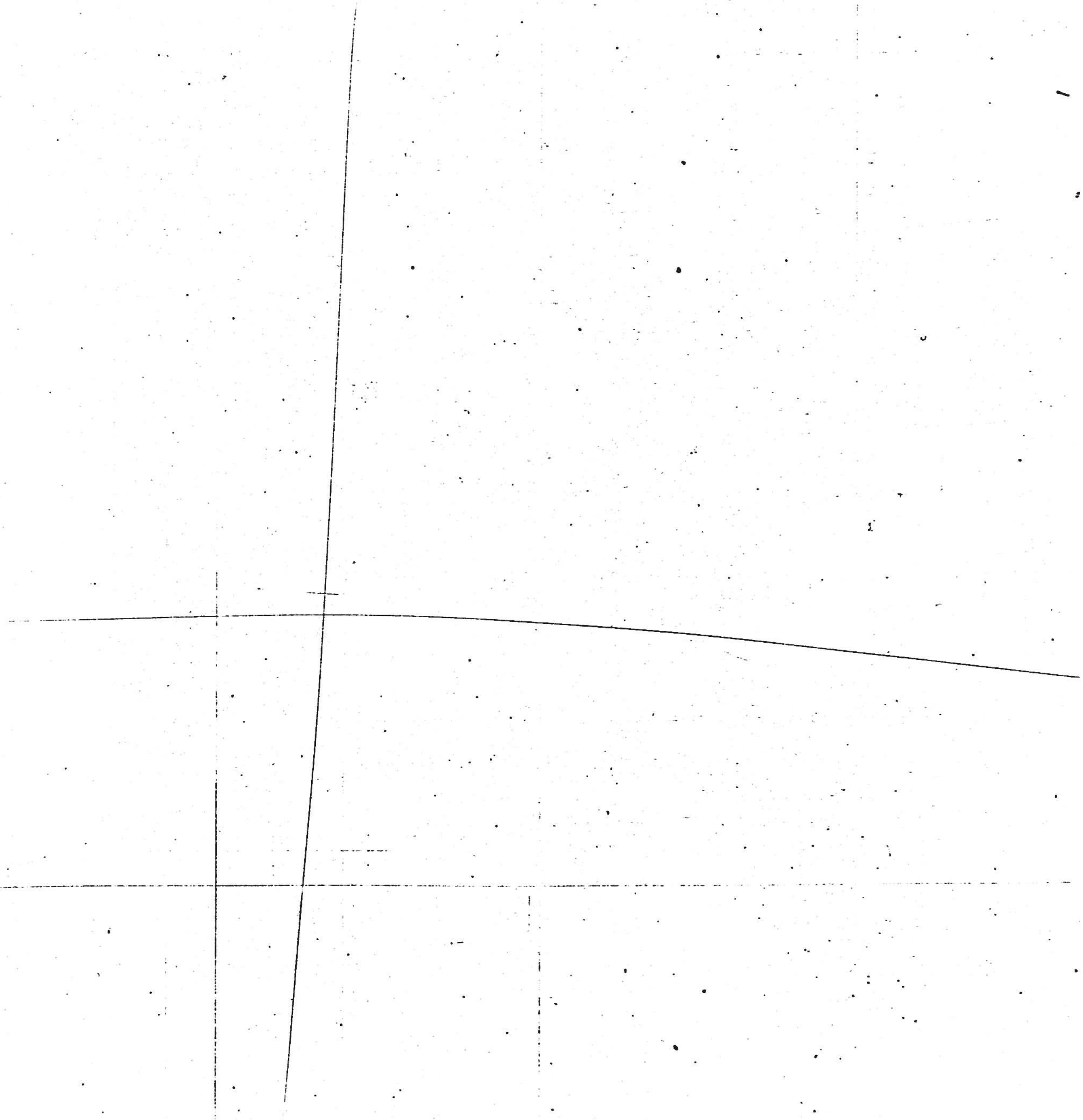
There is a danger in putting a new topology on $PL(K,\mathbb{R}^n)$, namely the danger that the topology will have so many open sets that structural stability will be meaningless. We therefore verify that structurally stable maps satisfy the geometric criteria expected of maps in "general position". Furthermore, since the topology described above depends not merely on the piecewise linear structure of $\mathbb{R}^n$ but on its linear structure, we must show that this "hybrid" topology on $PL(K,\mathbb{R}^n)$ yields results in "pure" PL topology.

In section I, some basic and necessary facts about planes in Projective space are indicated. In section II the concept of general position of points in Projective space is developed. In section III general position for linear maps is defined. Here a linear map is in general position if the images of the open simplexes meet in minimal dimension. This requirement on a map guarantees that all of the geometric requirements referred to above are satisfied. Furthermore, since the maps considered are linear, the condition is more than merely a dimensional statement, since it requires that simplexes actually meet in convex linear cells rather than arbitrary subcomplexes. The set of such maps is shown to be open, and the results of section II allow us to show that such maps are also dense, the crucial fact we need in order to know that this may be used as a criterion for general position. Here denseness is considered for the space $L(K,\mathbb{R}^n)$ of linear maps of a complex K into $\mathbb{R}^n$ with the compact-open topology.

Structural stability is defined in IV and some of its properties explored. In V we show that the structurally stable maps are dense in $L(K,\mathbb{R}^n)$. Since $L(K,\mathbb{R}^n)$ really has only one reasonable topology, we should expect any reasonable topology on $PL(K,\mathbb{R}^n)$ to induce the compact-open topology on $L(K,\mathbb{R}^n)$. Theorem 5.1 therefore acts as an encouragement that structural stability can be used as a technique in $PL(K,\mathbb{R}^n)$.

Section VI defines the C^1 Topology and section VII shows that in fact structural stability is a dense phenomenon. A corollary of this is that the general position definition of III is the correct definition of general position, that is, the most restrictive defintion possible.

A corollary of this definition is derived in VIII, using the notion of relative general position. It is shown that if X is in relative general position with respect to Y, then X meets Y nicely, i.e. transversely if X and Y are manifolds or, using the generalized defintion of transversality of Armstrong, if X and Y are polyhedra of codimension 3 in $\mathbb{R}^n$. This shows that relative structural stability is a refinement of the notion of transversality.



I PLANES IN PROJECTIVE SPACE

Real projective space $\mathbb{P}^n$ is the quotient space of $\mathbb{R}^{n+1}-\{0\}$ under the equivalence relation $(x_1,\ldots,x_{n+1}) \sim (tx_1,\ldots,tx_{n+1})$, $t\varepsilon\mathbb{R}-\{0\}$. Denote by $[x_1,\ldots,x_{n+1}]$ the equivalence class of $(x_1,\ldots,x_{n+1})$. From the observations that for $(x_1,\ldots,x_{n+1}) \sim (y_1,\ldots,y_{n+1})$, $x_i \neq 0$ iff $y_i \neq 0$ and $x_i/x_j = y_i/y_j$, we may define $\mathbb{R}^n_{(i)} = \{[x_1,\ldots x_{n+1}] \mid x_i \neq 0\}$ and a homeomorphism $\phi_i:\mathbb{R}^n_{(i)} \to \mathbb{R}^n$ given by $\phi_i[x_1,\ldots,x_{n+1}] = (x_1/x_i,\ldots,x_{i-1}/x_i,x_{i+1}/x_i,\ldots,x_{n+1}/x_i)$. The collection $\{(\mathbb{R}^n_{(i)},\phi_i)\}_1^{n+1}$ is an atlas for P^n giving it the structure of a manifold.

We will view $\mathbb{R}^n$ as a subspace of $\mathbb{P}^n$ by the embedding $\phi_{n+1}^{-1}:\mathbb{R}^n \to \mathbb{P}^n$. Although we will often view $\mathbb{R}^n$ as a PL manifold living in $\mathbb{P}^n$, $\mathbb{P}^n$ does not have a PL structure (with this atlas). Nevertheless, $\mathbb{R}^n$ is a "linear" submanifold, in the sense explicated below.

The Grassman Manifold $G_k(\mathbb{R}^n)$ of k-planes through 0 in $\mathbb{R}^n$ has the topology defined by the following neighborhood bases: $N(Q,\varepsilon) = \{Q'\varepsilon G_k\mathbb{R}^n : \measuredangle(Q,Q') < \varepsilon\}$. One can show that the map $\theta:G_1\mathbb{R}^n \to \mathbb{P}^n$ defined by $\theta(\ell) = |x_1,\ldots,x_{n+1}|$ for $\ell = \{(tx_1,\ldots,tx_{n+1}) \mid t\varepsilon\mathbb{R}\}$ is a homeomorphism.

If $A\subseteq\mathbb{P}^n$, $\hat{A}\subset\mathbb{R}^{n+1}$ is defined by $\hat{A} = \bigcup\{\ell:\ell\varepsilon\theta^{-1}(A)\}$. A subset Q of $\mathbb{P}^n$ is called a *q - plane* if $\hat{Q}$ is a (q+1)-plane in $\mathbb{R}^n$. Now suppose Q is a q-plane in $\mathbb{P}^n$.

PROPOSITION: Either $Q\cap\mathbb{R}^n = \phi$ or $Q\cap\mathbb{R}^n$ is a q-plane in $\mathbb{R}^n$.

PROOF: View $\mathbb{R}^n$ as a subspace of $\mathbb{R}^{n+1}$ by $\psi:\mathbb{R}^n \to \mathbb{R}^{n+1}$, given by

$\Psi(x_1,\ldots,x_n) = (x_1,\ldots,x_n,1)$. We show $\Psi(Q\cap\mathbb{E}^n) = \hat{Q}\cap\Psi(\mathbb{E}^n)$. If $a\in Q\cap\mathbb{E}^n$, $a = [x_1,\ldots,x_n,1]$ for appropriate choice of $x_1,\ldots,x_n$. Then $\Psi(a) = (x_1,\ldots,x_n,1)\in\Theta^{-1}(a)\subset\hat{Q}$. Therefore $\Psi(Q\cap\mathbb{E}^n)\subset\hat{Q}\cap\Psi(\mathbb{E}^n)$. Conversely, if $p\in\hat{Q}\cap\Psi(\mathbb{E}^n)$, $p = (x_1,\ldots,x_n,1)$ and $[x_1,\ldots,x_n,1]\in Q$. Therefore, $\hat{Q}\cap\Psi(\mathbb{E}^n)\subset\Psi(Q\cap\mathbb{E}^n)$. Now $\Psi(\mathbb{E}^n)$ is an n-plane in $\mathbb{E}^{n+1}$ and $\hat{Q}$ is a (q+1)-plane, so $\hat{Q}\cap\Psi(\mathbb{E}^n)$ is either empty or a q-plane. ||

Conversely, suppose S is a q-plane in $\mathbb{E}^n$. Recall that if A and B are planes in $\mathbb{E}^n$, $A\vee B$ is the smallest affine subspace of $\mathbb{E}^n$ containing both A and B. In particular, $\{o\}\vee\Psi(S)$ is a (q+1)-plane through 0 in $\mathbb{E}^{n+1}$; consequently, there is a q-plane $\tilde{S}$ in $\mathbb{P}^n$ with $\hat{\tilde{S}} = \{0\}\vee\Psi(S)$. $\tilde{S}\cap\mathbb{E}^n = S$, since $\Psi(\tilde{S}\cap\mathbb{E}^n) = \hat{\tilde{S}}\cap\Psi(\mathbb{E}^n) = \Psi(S)$. Thus we have established the notion that $\mathbb{E}^n$ is a linear submanifold of $\mathbb{P}^n$.

If Q_1 and Q_2 are planes in $\mathbb{P}^n$, define the *span* $Q_1\vee Q_2$ to be the intersection of all planes in $\mathbb{P}^n$ containing Q_1 and Q_2. It is clear that $\hat{Q}_1\vee\hat{Q}_2 = \widehat{Q_1\vee Q_2}$. Similarly $\hat{Q}_1\cap\hat{Q}_2 = \widehat{Q_1\cap Q_2}$.

DEFINITION: $\underline{G_k(\mathbb{P}^n)}$ is the Grassman Manifold of k-planes in $\mathbb{P}^n$, topologized by the set isomorphism $\wedge: G_k(\mathbb{P}^n)\to G_{k+1}(\mathbb{E}^{n+1})$ given by $Q\to\hat{Q}$. $G\mathbb{P}^n = \sum_{k=o}^{n-1} G_k\mathbb{P}^n$, the topological (disjoint) union.

We now use the homeomorphism $\wedge$, which commutes with span and intersection, to derive some facts about $G\mathbb{P}^n$.

PROPOSITION 1.1: If $Q,Q'\in G\mathbb{P}^n$, then $\dim(Q\vee Q') + \dim(Q\cap Q') = \dim Q + \dim Q'$. Here $\dim S = -1$ means $S = \Phi$.

PROOF: Since $\hat{Q}$ and $\hat{Q}'$ are vector subspaces of $\mathbb{E}^{n+1}$, it follows from linear algebra that $\dim(\hat{Q}\vee\hat{Q}') + \dim(\hat{Q}\cap\hat{Q}') = \dim\hat{Q} + \dim\hat{Q}'$. But $\dim(\hat{S}) = \dim(S) + 1\ \forall S\in G\mathbb{P}^n$, from which the result follows. ||

PROPOSITION 1.2: If $Q_1,\ldots,Q_r \in G\mathbb{P}^n$, then $\dim(Q_1 \cap \cdots \cap Q_r) \geq q_1+\cdots+q_r-(r-1)n$ for $q_i = \dim Q_i$

PROOF: True for $r = 1$. Assume true for $r - 1$. Then

$$\dim((Q_1 \cap \cdots \cap Q_{r-1}) \cap Q_r) = \dim(Q_1 \cap \cdots \cap Q_{r-1}) + \dim(Q_r) - \dim((Q_1 \cap \cdots \cap Q_{r-1}) \vee Q_r)$$

$$\geq q_1+\cdots+q_{r-1} - (r-2)n + q_r - n = q_1+\cdots+q_r - (r-1)n. \qquad \|$$

PROPOSITION 1.3: Let $Q_1,\ldots,Q_r$ be planes in $\mathbb{P}^n$. Let $Q_1',\ldots,Q_r' \in G\mathbb{P}^n$ such that Q_i' is close to Q_i. Then

(1) If $\dim(Q_1 \vee \cdots \vee Q_r) = \dim(Q_1' \vee \cdots \vee Q_r')$, then $Q_1 \vee \cdots \vee Q_r$ is close to $Q_1' \vee \cdots \vee Q_r'$.

(2) If $\dim(Q_1 \cap \cdots \cap Q_r) = \dim(Q_1' \cap \cdots \cap Q_r')$, then $Q_1 \cap \cdots \cap Q_r$ is close to $Q_1' \cap \cdots \cap Q_r'$.

In other words, if $V_r : (G\mathbb{P}^n)^r \to G\mathbb{P}^n$ $G_n\mathbb{P}^n$ and $\cap_r : (G\mathbb{P}^n)^r \to G\mathbb{P}^n$ are the span and intersection maps, then $V_r|V_r^{-1}(G\mathbb{P}^n)$ and $\cap_r|\cap_r^{-1}(G\mathbb{P}^n)$ are continuous.

PROOF: *(1)* Suppose $\dim(\hat{Q}_1 \vee \cdots \vee \hat{Q}_r) = k$. Choose vectors $v_1,\ldots,v_k$ in $\hat{Q}_1 \cup \cdots \cup \hat{Q}_r$ which span $\hat{Q}_1 \vee \cdots \vee \hat{Q}_r$. Since for $v \in \hat{Q}$ the distance $d(v,Q^1)$ is continuous in Q' and 0 at Q, it follows that for $\hat{Q}_i'$ close to $\hat{Q}_i$ we may find $\{v_1',\ldots,v_k'\}$ in $\hat{Q}_1' \cup \cdots \cup \hat{Q}_r'$ close to $\{v_1,\ldots,v_r\}$. Since $v_1,\ldots,v_r$ are linearly independent, $\exists\, \delta\epsilon$ if $d(v_i, v_i') < \delta$, $v_1',\ldots,v_k'$ are linearly independent. By hypothesis, $\{v_1',\ldots,v_k'\}$ must span $\hat{Q}_1' \vee \cdots \vee \hat{Q}_k'$. Thus, $\hat{Q}_1' \vee \cdots \vee \hat{Q}_k'$ must be close to $\hat{Q}_1 \vee \cdots \vee \hat{Q}_k$ by linear algebra.

(2) Recall that since $\mathbb{P}^n = G_1\mathbb{R}^{n+1}$ is a compact metric space, there is a Hausdorff metric on compact sets in $\mathbb{P}^n$, defined by

setting $d(C, D) = \max \{\sup\{d(x, D) \mid x \in C\}, \sup\{d(C, y) \mid y \in D\}\}$. It is easy to verify that this is the same metric as the usual metric on $G_k\mathbb{P}^n$. Thus this becomes a problem about compact sets in a metric space. The result is now a consequence of the following lemma whose proof is a straightforward exercise:

LEMMA: Let $Q_1, \ldots, Q_r$ be compact sets in a metric space M. For $\epsilon > 0$ there exists $\delta > 0$ such that if $Q_i^1 \subset N(Q_i, \delta)$, then $Q_1^1 \cap \ldots \cap Q_r^1 \subset N(Q_1 \cap \ldots \cap Q_r, \epsilon)$.

THEOREM 1.4: Let $Q_1, \ldots, Q_r \in G\mathbb{P}^n$. There exists $\delta > 0$ such that if $Q_1^1, \ldots, Q_r^1$ are planes with $d(Q_1, Q_i^1) < \delta$, then $\dim(Q_1^1 \vee \ldots \vee Q_r^1) \geq \dim(Q_1 \vee \ldots \vee Q_r)$ and $\dim(Q_1^1 \cap \ldots \cap Q_r^1) \leq \dim(Q_1 \cap \ldots \cap Q_r)$. Note that for δ small, $\dim Q_i^1 = \dim Q_i$, $\forall\, i$.

PROOF: It suffices in the first case to prove these results for $\hat{Q}_1, \ldots, \hat{Q}_r$. Suppose $\dim(\hat{Q}_1 \vee \ldots \vee \hat{Q}_r) = K$. Let $\{v_1, \ldots, v_k\}$ be vectors in $\hat{Q}_1 \cup \ldots \cup \hat{Q}_r$ which span $\hat{Q}_1 \vee \ldots \vee \hat{Q}_r$. For $\hat{Q}_1^1, \ldots, \hat{Q}_r^1$ sufficiently close, we can find $\{v_1^1, \ldots, v_k^1\}$ in $Q_1^1 \cup \ldots \cup Q_r^1$ sufficiently close to $\{v_1, \ldots, v_k\}$ as to be linearly independent (as in the proof of PROP. 1.3) Therefore $\dim Q_1^1 \vee \ldots \vee Q_r^1 \geq k$.

The second statement follows immediately from the LEMMA preceding this theorem, since a large - dimensional plane can never be in a small neighborhood of a small - dimensional plane. //

THEOREM 1.5: Let $Q_1, \ldots, Q_r \in G\mathbb{P}^n$. Given any $\epsilon > 0$ there exists Q_i^1 with $d(Q_i, Q_i^1) < \epsilon$ such that $\dim(Q_1^1 \cap \ldots \cap Q_r^1) = q_1 + \ldots + q_r - (r - 1)n$, where (dim $(Q_1 \cap \ldots \cap Q_r) < 0$ means $Q_1 \cap \ldots \cap Q_r = \emptyset$.

PROOF: By induction on r, beginning trivially at $r = 1$. Assume true for $r - 1$. Assume $Q_1^1 \dots Q_{r-1}^1$ have been chosen such that $d(Q_i, Q_1^1) < \epsilon$ and $\dim(Q_1^1 \cap \dots \cap Q_{r-1}^1) = q_1 + \dots + q_{r-1} - (r-2)\,n = q$. Let $Q = Q_1^1 \cap \dots \cap Q_{r-1}^1$. Then we must find $Q_r^1 \ni \dim(Q \cap Q_r^1) = q + q_r - n$. Thus the problem reduces to the case $r = 2$.

It suffices to show that given $\hat{Q}, \hat{Q}_r$ with $\dim \hat{Q} = \hat{q} = q + 1$ and $\dim \hat{Q}_r = \hat{q}_r = q_r + 1$, given $\epsilon > 0$, $\exists\, \hat{Q}_r^1 \ni d(\hat{Q}_r, \hat{Q}_r^1) < \epsilon$ and $\dim(\hat{Q} \cap \hat{Q}_r^1) = \dim(\widehat{Q \cap Q_r^1}) = q + q_r - n + 1 = \hat{q} + \hat{q}_r - (n+1)$.

Suppose $\dim(\hat{Q} \cap \hat{Q}_r) = \hat{q} + \hat{q}_r - (n+1) + k = \ell$. Then $\dim(\hat{Q} \vee \hat{Q}_r) = n + 1 - k$ (PROP. 1.1).

Choose vectors $v_1, \dots, v_\ell, v_{\ell+1}, \dots, v_{\hat{q}}, w_{\ell+1}, \dots w_{\hat{q}r}$ spanning $\hat{Q} \vee \hat{Q}_r$ such that $v_1, \dots, v_\ell$ span $\hat{Q} \cap \hat{Q}_r$, $v_1, \dots, v_q$ span $\hat{Q}$, and $v_1, \dots, v_\ell, w_{\ell+1}, \dots, w_{\hat{q}r}$ span $\hat{Q}_r$.

Case 1. $\ell > k$. Choose $x_\ell, x_{\ell-1}, \dots x_{\ell-k+1}$ successively such that $x_{\ell-i} \notin \hat{Q} \vee \hat{Q}_r \vee \{x_\ell\} \vee \dots \vee \{x_{\ell-i+1}\}$ and $d(x_j, v_j) < \delta$, δ small. Let $\hat{Q}_r^1$ be the plane spanned by $\{v_1, \dots, v_{\ell-k}\}$ and $\{x_{\ell-k+1}, \dots, x_\ell\}$ and $\{w_{\ell+1}, \dots, w_{\hat{q}r}\}$. Then we may make $d(Q_r, Q_r^1) < \epsilon$. Now $\dim(\hat{Q} \cap \hat{Q}_r) = \dim$ (space spanned by $v_1, \dots, v_{-k}) = \ell - k$ $= \hat{q} + \hat{q}_r - (n+1)$.

Case 2. $k \geq \ell$. Choose $x_\ell, \dots, x_1$ as in Case 1 and define $\hat{Q}_r^1$ by the plane spanned by $\{x_1, \dots, x_\ell, w_{\ell+1}, \dots, w_{\hat{q}r}\}$. Then we have $\hat{Q} \cap \hat{Q}^1 = \phi$.

COROLLARY 1.6: If $\dim(Q_1 \cap \dots \cap Q_r) = q_1 + \dots + q_r - (r-1)\,n$, then $\exists\, \delta > 0$ $\ni$ if $d(Q_i^1, Q_i) < \delta$, $\dim(Q_1^1 \cap \dots \cap Q_r^1) = q_1 + \dots + q_r - (r-1)n$.

PROOF: By PROP. 1.2, the dimension cannot decrease. By THEOREM 1.4, the dimension will not increase for δ small. ||

II GENERAL POSITION IN $\mathbb{P}^n$

Let $Q_1, \ldots, Q_r \in G\,\mathbb{P}^n$, the set of planes in $\mathbb{P}^n$. Let $\dim Q_i = q_i$, $1 \leq i \leq r$. For any $x \in \mathbb{P}^n - \bigcup\limits^r Q_i$ we may define a *transversal* through $x, Q_1, \ldots, Q_r$ as a line ℓ in $\mathbb{P}^n$ which meets $x, Q_1, \ldots, Q_r$. A *proper transversal*[1] is a line which meets $x, Q_1, \ldots, Q_r$ in $r + 1$ distinct points.

Let Π_r = the set of partitions of the set $\{1, \ldots, r\}$. The symbol $\underline{i \overset{\sim}{\pi} j}$, for $1 \leq i, j \leq r$, means i, j are in the same class of the partition π. Using this notation we may classify all transversals through $x, Q_1, \ldots, Q_r$.

DEFINITION: *The transversal set corresponding to* π, denoted by $T_\pi(x; Q_1, \ldots, Q_r; \mathbb{P}^n)$, is the union of all transversals ℓ through $x, Q_1, \ldots, Q_r$ such that $\ell \cap Q_i = \ell \cap Q_j$ iff $i \overset{\sim}{\pi} j$. The *proper transversal set* $T(x; Q_1, \ldots, Q_r; \mathbb{P}^n)$ is the union of all proper transversals through $x; Q_1, \ldots, Q_r$. Thus, if π is the discrete partition $\{\{1\}, \{2\}, \ldots, \{r\}\}$, $T_\pi(x; Q_1, \ldots, Q_r: \mathbb{P}^n) = T(x; Q_1, \ldots, Q_r; \mathbb{P}^n)$.

The following facts are simple consequences of the above definitions:

1. For any $\pi \in \Pi_r$, either $T_\pi(x; Q_1, \ldots, Q_r; \mathbb{P}^n) = \phi$ or it is at least one-dimensional.
2. If $\pi \neq \pi' \in \Pi_r$, $T_\pi(x; Q_1, \ldots, Q_r; \mathbb{P}^n) \cap T_{\pi'}(x; Q_1, \ldots, Q_r; \mathbb{P}^n) \subset \{x\}$. Equality occurs when both are non-empty.
3. For $c \in \pi$, let $Q_c = \bigcap\limits_{i \in c} Q_i$. Then $T_\pi(x; Q_1, \ldots, Q_r; \mathbb{P}^n) = T(x; Q_{c_1}, \ldots, Q_{c_s}; \mathbb{P}^n)$, for $\pi = \{c_1, \ldots, c_s\}$.

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[1] E. C. Zeeman, "Unknotting Spheres", Ann. Math. 72 (1960), 350-361.

Consider now the set $\tilde{T}(x;\ Q_1,\ldots,Q_r;\ \mathbb{P}^n) = \bigcap_{i=1}^{r} (\{x\} \vee Q_i)$. If ℓ is any transversal through x, $Q_1,\ldots,Q_r$, then in particular ℓ meets x and Q_i so $\ell \subset (\{x\} \vee Q_i)$. Conversely, if p is any point in $\tilde{T}(x;\ Q_1,\ldots,Q_r;\ \mathbb{P}^n)$ other than x, then the line $(\{x\} \vee \{p\}) \subset (\{x\} \vee Q_i)\ \forall\ i$. Therefore $(\{x\} \vee \{p\})$ meets each plane Q_i and is a transversal. Thus we may call $\tilde{T}(x:\ Q_1,\ldots,Q_r;\ \mathbb{P}^n)$ the *transversal plane* of x, $Q_1,\ldots,Q_r$. Similarly, write $\tilde{T}_\pi(x;\ Q_1,\ldots,Q_r;\ \mathbb{P}^n) = \tilde{T}(x;\ Q_c : c \varepsilon \pi\ ;\ \mathbb{P}^n)$.

It is clear that $\tilde{T}_\pi(x;\ Q_1,\ldots,Q_r;\ \mathbb{P}^n) \supset T_\pi(x;\ Q_1,\ldots,Q_r;\ \mathbb{P}^n)$.

PROPOSITION 2.1: $\tilde{T}(x;\ Q_1,\ldots,Q_r;\ \mathbb{P}^n) = \cup\{T_\pi(x;\ Q_1,\ldots,Q_r;\ \mathbb{P}^n) : \pi \varepsilon \Pi_r\}$ provided $\tilde{T}(x;\ Q_1,\ldots,Q_r;\ \mathbb{P}^n) \neq \{x\}$.

PROPOSITION 2.2: Write $\pi \leq \pi'$, for π and π' partitions, if π refines π'. Then $\tilde{T}_\pi(x;\ Q_1,\ldots,Q_r;\ \mathbb{P}^n) \supset \tilde{T}_{\pi'}(x;\ Q_1,\ldots,Q_r;\ \mathbb{P}^n)$ if $\pi \leq \pi'$

THEOREM 2.3: If $T(x;Q_1,\ldots,Q_r;\ \mathbb{P}^n) \neq \emptyset$, then $\dim T(x;\ Q_1,\ldots,Q_r;\ \mathbb{P}^n) = \dim \tilde{T}(x;\ Q_1,\ldots,Q_r;\ \mathbb{P}^n)$.

PROOF: We will show that $T(x) = T(x;\ Q_1,\ldots,Q_r;\ \mathbb{P}^n)$ contains an open set in $\tilde{T}(x)$. ($T(x)$ will not necessarily be open in $\tilde{T}(x)$, because in general x will be a boundary point of $T(x)$.) Since $T(x) \neq \Phi$ by hypothesis, let $y \neq x\ \varepsilon\ T(x)$. Then the line $[\{x\} \vee \{y\}] = \ell$ meets $Q_i - \bigcup_{j \neq i} Q_j$, $1 \leq i \leq r$. If ℓ' is any other line through x in $(\{x\} \vee Q_i)$ making a sufficiently small angle, say less than θ_i, with ℓ, then $(\ell \cap Q_i) \subset Q_i - \bigcup_{j \neq i} Q_j$. Therefore, if ℓ' is any line lying in $\bigcap_1^r (\{x\} \vee Q_i) = \tilde{T}(x)$, passing through x and making an angle less than $\min \theta_i$ with ℓ, then $\ell' \subset T(x)$. Thus an entire open cone of lines around ℓ in $\tilde{T}(x)$ is contained in $T(x)$. It follows that a neighborhood of y in $\tilde{T}(x)$ is contained in $T(x)$. ||

Unlike $T_\pi(x; Q_1,\ldots,Q_r; \mathbb{P}^n)$, $\tilde{T}_\pi(x)$ is always a plane; in fact it is the plane $\bigcap(\{x\} \vee Q_c : c\epsilon\pi)$. By PROP. 1.2, $\dim \tilde{T}_\pi(x) \geq \sum_{c\epsilon\pi}(\dim Q_c + 1) - (|\pi| - 1)n$, where $|\pi|$ is the number of elements of π. For notational purposes, let $\sigma = \{Q_1,\ldots,Q_r\}$ and $t_\pi(\sigma) = \sum(\dim Q_c + 1) - (|\pi| - 1)n$. In particular, write $t(\sigma) = \sum_1^r q_i + r - (r-1)n$.

DEFINITION: Let $x_o \epsilon \mathbb{P}^n$. We shall say that x_o is in *general position with respect to* $\sigma = \{Q_1,\ldots,Q_r\}$ if $x_o \notin Q_1 \cup \ldots \cup Q_r$ and $\dim T_\pi(x_o; \sigma; \mathbb{P}^n) \leq t_\pi(\sigma)$ for every $\pi \epsilon \Pi_r$.

COROLLARY 2.4: If x_o is in general position with respect to σ and if $T_\pi(x_o; \sigma; \mathbb{P}^n) \neq \Phi$, then $\dim T_\pi(x_o; \sigma; \mathbb{P}^n) = \dim \tilde{T}_\pi(x_o; \sigma; \mathbb{P}^n) = t_\pi(\sigma)$.

PROOF: By THEOREM 2.3, $\dim T_\pi = \dim \tilde{T}_\pi$. By hypothesis $\dim T_\pi \leq t_\pi(\sigma)$. But $\dim \tilde{T}_\pi \geq t_\pi(\sigma)$. Therefore, $\dim T_\pi = \dim \tilde{T} = t_\pi(\sigma)$.

Remark: If $T_\pi(x_o; \sigma; \mathbb{P}^n) = \Phi$, and x_o is in general position with respect to σ, then $\dim \tilde{T}_\pi(x_o; \sigma; \mathbb{P}^n)$ may be greater than $t_\pi(\sigma)$. For example, let Q_1, Q_2, Q_3 be lines in $\mathbb{P}^3$ which will all lie in one plane Q and such that $Q_1 \cap Q_2 \cap Q_3 = \{p\}$. Let $x_o \epsilon \mathbb{P}^3 - Q$. Then it may be easily verified that x_o is in general position, since $T_\pi(x_o; \sigma; \mathbb{P}^3) = \Phi$ except for $\pi = \{\{1,2,3\}\}$, in which case $T_\pi(x_o; \sigma; \mathbb{P}^3)$ is the line $(\{x_o\} \vee \{p\} = \ell$. Now $\tilde{T}(x_o; \sigma; \mathbb{P}^3) = \ell$, so $\dim \tilde{T} = 1$. But $t(\sigma) = 1 + 1 + 3 - 2(3) = 0$.

THEOREM 2.5: Let $\sigma = \{Q_1,\ldots,Q_r\} \subset G\mathbb{P}^n$, and let $\mathcal{S}$ be the set of points in $\mathbb{P}^n$ in general position with respect to σ. Then $\mathcal{S}$ is open and dense in $\mathbb{P}^n$.

PROOF: By induction on n, beginning trivially for $n = 0$. We now assume the result for projective space of dimension less than n and prove the result for n by induction on r.

If $r = 1$, then clearly $\mathscr{G} = \mathbb{P}^n - Q_1$, which is open and dense. Thus we may assume the theorem is true for any $r' < r$.

Let $\mathscr{G}^*$ be the set of points in $\mathbb{P}^n$ such that for any $\pi \neq$ $\{\{1\}, \{2\}, \ldots, \{r\}\}$, $\dim T_\pi(x_o\ \sigma; \mathbb{P}^n) \leq t_\pi(\sigma)$. Recall that $T_\pi(x, \sigma; \mathbb{P}^n) = T(x, \sigma_\pi; \mathbb{P}^n)$, for $\sigma_\pi = \{Q_c, c \in \pi\}$ and $Q_c = \bigcap_{i \in c} Q_i$. Now since $|\pi| < r$ we have by induction that the set of points x satisfying $\dim T_\pi(x, \sigma; \mathbb{P}^n) \leq t_\pi(\sigma) = t(\sigma_\pi)$ is open and dense. Thus $\mathscr{G}^*$ is the finite intersection of open, dense sets, and hence $\mathscr{G}^*$ is open and dense. It suffices to show $\mathscr{G}$ is open and dense in $\mathscr{G}^*$. Observe that $x \in \mathscr{G}^*$ & $\dim T(x; \sigma; \mathbb{P}^n) \leq t(\sigma) \Leftrightarrow x \in \mathscr{G}$.

Denseness of $\mathscr{G}$: If for any $i \neq j$ $Q_i \vee Q_j \neq \mathbb{P}^n$, then for any $x \in \mathbb{P}^n - Q_i \vee Q_j$. $T(x; Q_1, \ldots, Q_r; \mathbb{P}^n) \subset T(x; Q_i, Q_j; \mathbb{P}^n) = \Phi$. Then $\mathscr{G} = \mathscr{G}^* \cap (\mathbb{P}^n - Q_i \vee Q_j)$ is dense in $\mathscr{G}^*$. So assume w.l.o.g. that $Q_i \vee Q_j = \mathbb{P}^n$ for $i \neq j$. Let $x_o \in \mathscr{G}^*$ and $\varepsilon > 0$. We must find $x \in \mathscr{G} \ni d(x, x_o) < \varepsilon$.

Let $S = (\{x_o\} \vee Q_r)$.

Since $Q_i \vee Q_r = \mathbb{P}^n$ and $\dim Q_i \cap Q_r \geq -1$, by PROP. 1.1 $q_i + q_r \geq n-1$. Therefore, again by PROP. 1.1, $\dim Q_i \cap S = q_i + q_r + 1 - n \geq 0$. Thus $Q_i \cap S \neq \Phi$. From this it follows that $(\{x_o\} \vee Q_i) \cap S = (\{x_o\} \vee (Q_i \cap S))$. Therefore $\tilde{T}(x_o; Q_1, \ldots, Q_r; \mathbb{P}^n) = \tilde{T}(x_o; Q_1', \ldots, Q_{r-1}'; S)$ for $Q_i' = Q_i \cap S$. By induction on r if $S = \mathbb{P}^n$ or on n if $S \neq \mathbb{P}^n$ (observing that S is a projective space) we may assume $\exists\, x' \in S \ni$ *(1)* $d(x_o, x^1) < \varepsilon$, *(2)* $x' \in \mathscr{G}^*$, and *(3)* $\dim T(x'; Q_1', \ldots, Q_{r-1}'; S)$

$$\leq \sum_1^{r-1} \dim Q_i' + (r-1) - (r-2) \dim S$$

$$= \sum_1^{r-1} (q_i + q_r + 1 - n) + (r-1) - (r-2)(q_r + 1)$$

$$= \sum_1^{r} (q_i + 1) + (r-1)(q_r - n) + (r-1) - (r-1)(q_r + 1)$$

$$= \sum_1^{r} (q_i + 1) - (r-1)n = t(\sigma).$$

If $T(x'; Q_1,\ldots,Q_r; \mathbb{P}^n) = \phi$, then $x' \in \mathcal{G}$. If not, let ℓ be a proper transversal of $Q_1,\ldots,Q_r$. Since $S = (\{x'\} \vee Q_r)$, $\ell < S$. Therefore $\ell \cap Q_i \subset Q_i \cap S = Q_i'$ and ℓ is a proper transversal of $Q_1',\ldots,Q_{r-1}'$ in S. Since $T(x'; Q_1',\ldots,Q_{r-1}'; S) \neq \phi$, $\dim \tilde{T}(x'; Q_1',\ldots,Q_{r-1}'; S) = t(\sigma)$. By the same argument as before, $\tilde{T}(x'; Q_1',\ldots,Q_{r-1}'; S) = \tilde{T}(x'; Q_1,\ldots,Q_r; \mathbb{P}^n)$, so $\dim T(x'; Q_1,\ldots,Q_r; \mathbb{P}^n) = \dim \tilde{T}(x'; Q_1,\ldots,Q_r; \mathbb{P}^n) = t(\sigma)$. Therefore $x' \in \mathcal{G}$. This proves $\mathcal{G}$ is dense in $\mathcal{G}^*$.

Openness of $\mathcal{G}$: Let $x_o \in \mathcal{G}$. Suppose first that $T(x_o; Q_1,\ldots,Q_r; \mathbb{P}^n) \neq \phi$. Then $\dim T(x_o) = \dim \tilde{T}(x_o) = t(\sigma)$. This means that the planes $(\{x_o\} \vee Q_1),\ldots,(\{x_o\} \vee Q_r)$ have a minimal intersection. By COROLLARY 1.6, any small perturbation of these planes will preserve the dimension of intersection. If x is close to x_o, $(\{x\} \vee Q_i)$ is close to $(\{x_o\} \vee Q_i)$. Therefore, if x is close to x_o, $\dim \tilde{T}(x; \sigma; \mathbb{P}^n) = t(\sigma)$ and $x \in \mathcal{G}$.

Assume instead that $T(x_o; \sigma; \mathbb{P}^n) = \phi$. We may assume dimension of $\tilde{T}(x_o; \sigma; \mathbb{P}^n) \geq 1$, for otherwise the argument of the last paragraph applies. Since every transversal through x_o and σ is improper, PROP. 2.2 implies $\tilde{T}(x_o; \sigma; \mathbb{P}^n) = \cup\{\tilde{T}_\pi(x_o; \sigma; \mathbb{P}^n) : |\pi| \neq r\}$. Since this is a union of planes, $\exists\ \pi \ni \tilde{T}(x_o; \sigma; \mathbb{P}^n) = \tilde{T}_\pi(x_o; \sigma; \mathbb{P}^n)$. Choose a maximal such π. Then $T_\pi(x ; \sigma; \mathbb{P}^n) \neq \Phi$; for if it were we could repeat the identical argument and find $\pi' > \pi$ with $\tilde{T}_{\pi'} = \tilde{T}_\pi$. Therefore $\dim \tilde{T}(x_o; \sigma; \mathbb{P}^n) = \dim \tilde{T}_\pi(x_o; \sigma; \mathbb{P}^n) = \dim \tilde{T}_\pi(x_o; \sigma; \mathbb{P}^n) = t_\pi(\sigma)$. For sufficiently small ε, if $d(x_o, x) < \varepsilon$ $\dim \tilde{T}_\pi(x; \sigma; \mathbb{P}^n) = t_\pi(\sigma)$ (by applying the previous paragraph). We may also quarantee, by THEOREM 1.4, that $\dim T_\pi(x) \leq t_\pi(\sigma)$. But $\tilde{T}_\pi(x) \subset \tilde{T}(x)$, so $\dim \tilde{T}(x; \sigma; \mathbb{P}^n) = t_\pi(\sigma)$. It follows that $\tilde{T}_\pi(x; \sigma; \mathbb{P}^n) = \tilde{T}(x; \sigma; \mathbb{P}^n)$. But this says every transversal is improper, so $T(x; \sigma; \mathbb{P}^n) = \Phi$. This proves $x \in \mathcal{B}$, and $\mathcal{B}$ is open.

ADDENDUM 2.6: Let x_o be in general position with respect to σ. Then $\exists\ \varepsilon > 0$ such that if $d(x, x_o) < \varepsilon$ then $\dim \tilde{T}_\pi(x; \sigma; \mathbb{P}^n) = \dim \tilde{T}_\pi(x_o; \sigma; \mathbb{P}^n)$ and $\dim T\ (x; \sigma; \mathbb{P}^n) = \dim T_\pi(x_o; \sigma; \mathbb{P}^n)$ for all partitions $\pi \in \Pi_r$.

PROOF: Suppose $T(x_o; \sigma; \mathbb{P}^n) = \Phi$. Then the proof of THEOREM 2.5 shows that for x near x_o $T(x; \sigma; \mathbb{P}^n) = \Phi$ and $\tilde{T}(x_o; \sigma; \mathbb{P}^n)$ has the same dimension, $t_\pi(\sigma)$, as $\tilde{T}(x; \sigma; \mathbb{P}^n)$ for some $\pi \in \Pi_r$. Now suppose $\exists$ a proper transversal ℓ of x_o and σ. Then $\ell \cap Q_i \subset Q_i - \bigcap_{j\neq i} Q_j$ for $1 \leq i \leq r$.

If ℓ' is any line sufficiently close to ℓ, then $\ell' \cap Q_i \subset Q_i - \bigcap_{j\neq i} Q_j$ (the intersection may be empty.) By the proof of THEOREM 2.5, $\dim \tilde{T}(x_o; \sigma; \mathbb{P}^n) = t(\sigma)$. By part *(2)* of PROP. 1.3, if x is chosen close enough to x_o that $\dim \tilde{T}(x; \sigma; \mathbb{P}^n) = t(\sigma)$ then the plane $\tilde{T}(x; \sigma; \mathbb{P}^n)$ is close to $\tilde{T}(x_o; \sigma; \mathbb{P}^n)$. We may therefore find a line ℓ' through x and $Q_1, \ldots, Q_r$ which is close to ℓ. $T(x; \sigma; \mathbb{P}^n) \neq \Phi \Rightarrow \dim T(x; \sigma; \mathbb{P}^n) = t(\sigma) = \dim T(x_o; \sigma; \mathbb{P}^n)$.

Finally, $T_\pi(x; \sigma; \mathbb{P}^n) = T(x; \sigma'; \mathbb{P}^n)$, so the above arguments may be repeated for T_π in place of T. ||

Remark: General position of x with respect to $Q_1,\ldots,Q_r$ is not an open condition on $Q_i,\ldots,Q_r$. Let Q_1, Q_2, Q_3 be 2-planes in $\mathbb{P}^4$ such that $Q_1 \cap Q_2$ = a line and $Q_1 \cap Q_2 \cap Q_3 = \Phi$. Suppose x is in general position with respect to Q_1, Q_2, Q_3. Then $\dim T(x;\ Q_1 \cap Q_2,\ Q_3;\ \mathbb{P}^4) \leq$ $\dim(Q_1 \cap Q_2) + \dim Q_3 + 2 - 4 = 1$, while $\dim \tilde{T}(x;\ Q_1 \cap Q_2,\ Q_3;\ \mathbb{P}^4) \geq 1$. But since $Q_1 \cap Q_2 \cap Q_3 = \Phi$, we have $T(x;\ Q_1 \cap Q_2, Q_3\colon \mathbb{P}^4)$ = a line ℓ. Let $\ell \cap Q_1 \cap Q_2 = \{y\}, \ell \cap Q_3 = \{z\}$. Perturb Q_1 to Q_1' in $\mathbb{P}^4$ so that $Q_1' \cap Q_2 = \{y\}$. Then $\ell \subset T(x;\ Q_1' \cap Q_2,\ Q_3;\ \mathbb{P}^4)$. But general position dictates that $\dim T(x;\ Q_1' \cap Q_2,\ Q_3;\ \mathbb{P}^4) \leq 0 + 2 + 2 - 4 = 0$. Thus x is not in general position with respect to Q_1', Q_2, Q_3.

It may be verified that if the planes $Q_1,\ldots,Q_r$ interect minimally this phenomenon does not occur.

DEFINITION: Let $x_1,\ldots,x_r$ be points in $\mathbb{P}^n$. Let $\mathcal{Q}$ be the subset of $G\mathbb{P}^n$ consisting of all planes spanned by subsets of $x_1,\ldots,x_r$. A point x is in *general position with respect to* $x_1,\ldots,x_r$ if for every subset σ of $\mathcal{Q}$ x is in general position with respect to σ. A set of points $x_1,\ldots,x_r$ is in *(Projective) general position* if each point is in general position with respect to the others.

THEOREM 2.7: The set $\mathcal{S}_r = \{(x_1,\ldots,x_r)\} \in (\mathbb{P}^n)^r \mid x_1,\ldots,x_r$ are in general position$\}$ is open and dense in $(\mathbb{P}^n)^r$.

LEMMA 2.8: Let $(x_1,\ldots,x\ ) \in \mathring{\mathcal{S}}_r$. Let $\varepsilon \geq 0$ be given such that if $d(x_i', x_i) < \varepsilon, 1 \leq i \leq r$, then $(x_1',\ldots,x_r') \in \mathcal{S}_r$. For any $\alpha \in 2^{\{1,\ldots,r\}}$ such that $|\alpha| < n$ let Q_α, Q_α' be the planes spanned by $\{x_i : i \in \alpha\}$, $\{x_i' : i \in \alpha\}$ respectively. Then $\dim(Q_{\alpha_1} \cap \ldots \cap Q_{\alpha_s}) = \dim(Q_{\alpha_1}' \cap \ldots \cap Q_{\alpha_s}')$ and for small ε $Q_{\alpha_1}' \cap \ldots \cap Q_{\alpha_s}'$ is close to $Q_{\alpha_1} \cap \ldots \cap Q_{\alpha_s}$.

PROOF: Assume w.l.o.g. that $x_i = x_i'$ for $i > 1$; that is, iterate the process of moving one point to change successively from $(x_1,\ldots,x_r)$ to $(x_1',\ldots,x_r)$ to $(x_1', x_2',\ldots,x_r)$, etc. For notational convenience let $Q_i^{(1)} = Q_{\alpha_i}^{(1)}$. Also assume w.l.o.g. that $1 \in \alpha_1,\ldots,\alpha_t$ but $1 \notin \alpha_{t+1},\ldots,\alpha_s$.

If $t = 0$ the lemma is trivial. Suppose $t = s$. Let $Q_i{}^*$ be the plane spanned by $\{x_j : j \in \alpha_i - \{1\}\}$. Then we have $Q_1 \cap \ldots \cap Q_s =$ $(\{x_1\} \vee Q_1{}^*) \cap \ldots \cap (\{x_1\} \vee Q_s{}^*) = \tilde{T}(x_1;\ Q_1{}^*,\ldots,Q_s{}^*;\ \mathbb{P}^n)$ and $Q_1' \cap \ldots \cap Q_s' =$ $\tilde{T}(x_1';\ Q_1{}^*,\ldots,Q_r{}^*;\ \mathbb{P}^n)$. Then ADDENDUM 2.6 followed by part *(2)* of PROP. 1.3 yields the theorem in this case.

Suppose instead that $1 \leq t < s$. Let $Q_i{}^*$ be as before for $1 \leq i \leq t$. Then one easily verifies that

$\{x_1\} \vee (Q_1 \cap \ldots \cap Q_s) = (\{x_1\} \vee Q_1{}^*) \cap \ldots \cap (\{x_1\} \vee Q_t{}^*) \cap (\{x_1\} \vee (Q_{t+1} \cap \ldots \cap Q_s))$. $= \tilde{T}(x_1:\ Q_1{}^*,\ldots,Q_t{}^*,\ Q_{t+1} \cap \ldots \cap Q_s;\ \mathbb{P}^n) = \tilde{T}_\pi(x_1;\ Q_1{}^*,\ldots,Q_t{}^*, Q_{t+1},\ldots,Q_s; \mathbb{P}^n)$ For $\pi = \{\{1\},\ldots,\{t-1\}, \{t\}, \{t+1,\ldots,s\}\}$. Similarly $\{x_1'\} \vee (Q_1' \cap \ldots \cap Q_s') =$ $\tilde{T}_\pi(x_1';\ Q_1{}^*,\ldots,Q_t{}^*,\ Q_{t+1},\ldots,Q_s;\ \mathbb{P}^n)$. Again by ADDENDUM 2.6 $\dim \tilde{T}\,(x_1') =$ $\dim \tilde{T}_\pi(x_1) = \dim\,(Q_1 \cap \ldots \cap Q_s) + 1 = \dim\,(Q_1' \cap \ldots \cap Q_s') + 1$ and the theorem now follows from part *(2)* of PROP. 1.3.

LEMMA 2.9: Let $x \in \mathbb{P}^n$ and $Q_1,\ldots,Q_r \in G\,\mathbb{P}^n$. For every $\alpha \in 2^{\{1,\ldots,r\}}$ let $Q_\alpha = \bigcap_{i\in\alpha} Q_i$. Suppose x is in general position with respect to $Q_1,\ldots,Q_r$. Then if x' is sufficiently close to x and Q_i' is sufficiently close to Q_i such that $\dim Q_\alpha' = \dim Q_\alpha\ \forall \alpha \in 2^{\{1,\ldots,r\}}$ then x' is in general position with respect to $Q_1',\ldots,Q_r'$.

PROOF: By choosing Q_i' subject to the hypothesis we may quarantee by 1.3 that also Q_α' is close to $Q_\alpha\ \forall\alpha$. For every partition $\pi \in \Pi_r$ we must verify that $\dim T_\pi(x';\sigma';\ \mathbb{P}^n)\quad t_\pi(\sigma') = t_\pi(\sigma)$, where $\sigma' = \{Q_1',\ldots,Q_r'\}$.

Suppose $T_\pi(x;\sigma;\mathbb{P}^n) = \emptyset$. Then, as in the proof of THEOREM 2.5, $\exists$ a maximal $\pi' > \pi \ni \tilde{T}'_\pi(x;\sigma;\mathbb{P}^n) = \tilde{T}_\pi(x;\sigma;\mathbb{P}^n)$ and $\dim \tilde{T}'_\pi(x;\sigma;\mathbb{P}^n) = t'_\pi(\sigma)$. $\tilde{T}'_\pi(x;\sigma;\mathbb{P}^n) = \bigcap_{\alpha\in\pi'}(\{x\}\vee\hat{Q}_\alpha)$ and similarly $\tilde{T}'_\pi(x';\sigma';\mathbb{P}^n) = \bigcap_{\alpha\in\pi'}(\{x\}\vee Q'_\alpha)$. If x is close to x' and Q_α is close to Q'_α we may guarantee by COROLLARY 1.6 that $\dim T'_\pi(x';\sigma';\mathbb{P}^n) = t'_\pi(\sigma)$. We may also guarantee by THEOREM 1.4 that $\dim \tilde{T}_\pi(x';\sigma';\mathbb{R}^n) \le \dim \tilde{T}_\pi(x;\sigma;\mathbb{P}^n) = t'_\pi(\sigma)$. $\tilde{T}_\pi(x';\sigma';\mathbb{P}^n) = \tilde{T}'_\pi(x';\sigma';\mathbb{P}^n)$ and $T_\pi(x';\sigma';\mathbb{P}^n) = \emptyset$.

Suppose $T_\pi(x;\sigma;\mathbb{P}^n) \ne \emptyset$. Then $\dim \tilde{T}_\pi(x;\sigma;\mathbb{P}^n) = t_\pi(\sigma)$ and by the same reasoning as above we can arrange that $\dim \tilde{T}_\pi(x';\sigma';\mathbb{P}^n) = t_\pi(\sigma)$. ||

COROLLARY 2.10: Assume that general position of fewer than r points in $\mathbb{P}^n$ is an open and dense condition. Then the condition "$x_2, \ldots, x_r$ are in general position and x_1 is in general position with respect to $x_2, \ldots, x_r$" is open and dense in $(\mathbb{P}^n)^r$.

PROOF: Given $x_1, \ldots, x_r$ and $\epsilon > 0$ there exists $x_2', \ldots, x_r'$ such that $d(x_i, x_i') < \epsilon$ and $(x_2', \ldots, x_r') \in \mathcal{G}_{r-1}$. By THEOREM 2.5 we may find x_1' such that $d(x_1', x_1) < \epsilon$ and x_1' is in general position with respect to $x_2', \ldots, x_r'$ (we need only find x_1' in general position with respect to every family of planes spanned by $x_2', \ldots, x_r'$.) Thus the condition is dense.

Suppose $x_1, \ldots, x_r$ satisfy the condition. Then let $\mathcal{Q}$ be the set of planes spanned by $x_2, \ldots, x_r$. If $Q_1, \ldots, Q_s$ are planes in $\mathcal{Q}$, then by LEMMA 2.8 if $x_2', \ldots, x_r'$ are near $x_2, \ldots, x_r$ then $\dim Q_1 \cap \ldots \cap Q_s = \dim Q_1' \cap \ldots \cap Q_s'$. Therefore by LEMMA 2.9 if x_1' is close to x_1 and $x_2', \ldots, x_r'$ close to $x_2, \ldots, x_r$ then x_1^1 is in general position with respect to $x_2', \ldots, x_r'$. Thus the condition is open.

PROOF of THEOREM 2.7: By induction on r. Assume the theorem for $r-1$ points. Let $\mathcal{G}_r^{(i)}$ be the set of r-tuples $(x_1, \ldots, x_r) \in (\mathbb{P}^n)^r$ such that $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_r$ are in general position and x is in general position with respect to $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_r$. Then by COROLLARY 2.10 $\mathcal{G}_r^{(i)}$ is open and dense in $(\mathbb{P}^n)^r$. Hence $\mathcal{G}_r = \bigcap_{i=1}^{r} \mathcal{G}_r^{(i)}$ is open and dense.

THEOREM 2.11: Let $x_1, \ldots, x_r$ be points in general position in $\mathbb{P}^n$. Let $Q_{c_1}, \ldots, Q_{c_k}$ be planes in $\mathbb{P}^n$ such that $c_i \in 2^{\{1, \ldots, r\}}$ and Q_{c_i} is the plane spanned by $\{x_j : j \in c_i\}$. Then $\exists\, c_i' \subset c_i,\ 1 \le i \le k \ni Q_{c_1} \cap \ldots \cap Q_{c_k} = Q_{c_1'} \cap \ldots \cap Q_{c_k'}$ and the distinct planes among $Q_{c_1'}, \ldots, Q_{c_k'}$ intersect in the minimal dimension predicted by PROPOSITION 1.2.

PROOF : To prove this theorem we need only show that if $\dim (Q_{c_1} \cap \ldots \cap Q_{c_k}) > \dim Q_{c_1} + \ldots + \dim Q_{c_k} - (k-1)n$, then we may find $c_i' \subset c_i \ni Q_{c_1} \cap \ldots \cap Q_{c_k} = Q_{c_1'} \cap \ldots \cap Q_{c_k'}$ and $c_i \neq c_i'$ for some i. Then iterating this procedure until the intersection is minimal yields the result.

General position requires that $x_i \in Q_{c_j}$ iff $i \in c_j$. By renumbering, we may assume w.l.o.g. that $x_1 \in Q_{c_1}, \ldots, Q_{c_\ell}$, but $x_1 \notin Q_{c_{\ell+1}}, \ldots, Q_{c_k}$, and that $\ell < k$. Let $c_j^* = c_j - \{1\}$ for $1 \le j \le \ell$. Then $Q_{c_j} = \{x_1\} \vee Q_{c_j^*}$. Therefore $Q_{c_1} \cap \cdots \cap Q_{c_k}$
$= (\{x_1\} \vee Q_{c_1^*}) \cap \ldots \cap (\{x_1\} \vee Q_{c_\ell^*}) \cap Q_{c_{\ell+1}} \cap \cdots \cap Q_{c_k}$.
and $x_1 \vee (Q_{c_1} \cap \cdots \cap Q_{c_k}) = \tilde{T}_\pi (x_1; Q_{c_1^*}, \ldots, Q_{c_\ell^*}, Q_{c_{\ell+1}}, \ldots, Q_{c_k}; \mathbb{P}^n)$ for $\pi = \{\{1\}, \ldots, \{\ell\}, \{\ell+1, \ldots, k\}\}$.

We may assume Q_{c+1} intersect minimally; otherwise we may reduce these planes by indication on the number of planes in THEOREM 2.11.

Then $\dim(Q_{c_{\ell+1}} \cap \cdots \cap Q_{c_k}) = \dim Q_{c_{\ell+1}} + \cdots + \dim Q_{c_k} - (k-\ell-1)n. \Rightarrow$
$t_\pi(Q_{c_1^*}, \ldots, Q_{c_\ell^*}, \ldots, Q_{c_k}) = \dim Q_{c_1^*} + \cdots + \dim Q_{c_\ell^*}$
$+ \dim Q_{c_{\ell+1}} + \cdots + \dim Q_{c_k} - (k-\ell-1)n + \ell + 1 - \ell n = \sum_{i=1}^{k} \dim Q_{c_i} - (k-1)n+1.$

By hypothesis $\dim(\{x_i\} \vee (Q_{c_1} \cap \cdots \cap Q_{c_k})) > \sum_1^k \dim Q_{c_i} - (k-1)n+1$; that is, $\dim \widetilde{T}_\pi(x_1) > t_\pi$. Since x_1 is in general position with respect to $Q_{c_1^*}, \ldots, Q_{c_\ell^*}, Q_{c_{\ell+1}}, \ldots, Q_{c_k}$, we must have

$$T_\pi(x_1; Q_{c_1^*}, \ldots, Q_{c_\ell^*}, Q_{c_{\ell+1}}, \ldots, Q_{c_k}) = \emptyset \text{ and}$$

$\widetilde{T}_\pi(x_1) = \widetilde{T}_{\pi'}(x_1)$ for maximal $\pi' \in \Pi_k$.

For $\alpha \in \pi'$, let $R_\alpha = \bigcap \{Q_{c_j^*} : j \in \alpha\}$ if $\{\ell+1, \ldots, k\} \not\subset \alpha$ and if $\{\ell+1, \ldots, k\} \subset \bar\alpha$, $R_{\bar\alpha} = \bigcap \{Q^*_{c_j} : j \le \ell, j \in \alpha\} \cap Q_{\ell+1} \cap \cdots \cap Q_k$. $\widetilde{T}_\pi(x_1) = \widetilde{T}(x_1; R_{\alpha_1}, \ldots, R_{\alpha_s}; \mathbb{P}^n)$, and $\widetilde{T}_\pi(x_1) = \sum_1^s \dim R_{\alpha_i} + s - (s-1)n$. Suppose each R_α has minimal dimension, namely $\sum_{j \in \alpha} \dim Q_{c_j^*} - (|\alpha| - 1)n$, and $\dim R_{\bar\alpha} = \sum_{\substack{j \in \bar\alpha \\ j \le \ell}} \dim Q_{c_j^*} + \sum_{\ell+1}^{k} \dim Q_{c_j} - (|\bar\alpha|-1)n.$ Then $\dim \widetilde{T}_\pi(x_1)$

$$= \sum_{i=1}^{s-1} \Big(\sum_{j \in \alpha_i} \dim Q_{c_j^*} - (|\alpha_i|-1)n \Big) + \sum_{\substack{j \in \bar\alpha \\ j \le \ell}} \dim Q_{c_j^*} + \sum_{\ell+1}^{k} \dim Q_{c_j} - (|\bar\alpha|-1)n$$
$$+ s - (s-1)n$$

$$= \sum_{j=1}^{\ell} \dim Q_{c_j^*} + \sum_{\ell+1}^{k} \dim Q_{c_j} - kn + sn + s - (s-1)n$$

$$= \sum_{j=1}^{k} \dim Q_{c_j} - (k-1)n + s - \ell. \text{ Since } \pi' > \pi,\ s = |\pi'| < |\pi| = \ell+1.$$

Therefore $\dim \widetilde{T}_\pi(x_1) < \sum \dim Q_{c_j} - (k-1)n+1$, which is a contradiction. Thus some R_α does not have minimal dimension.

Suppose $\alpha \not\supset \{\ell+1,\ldots,k\}$. By induction we may find

$c_{j'} \subset c_j^*$, $j \in \alpha$, with $\bigcap_{j\in\alpha} Q_{c_j^*} = \bigcap_{j\in\alpha} Q_{c_j'}$.

Then $\{x_1\} \vee (Q_{c_1} \cap \ldots \cap Q_{c_k}) = \tilde{T}(x_1; R_{\alpha_1},\ldots,R_{\alpha_s}; \mathbb{P}^n)$

$= \tilde{T}_{\pi'}(x_1; \{Q_{c_j^*} : j \le \ell, j \notin \alpha\}, \{Q_{c_{j'}} : j \in \alpha\}, Q_{\ell+1},\ldots,Q_k; \mathbb{P}^n)$

$\subset \{x_1\} \vee [(\cap \{Q_{c_j} : j \notin \alpha\}) \cap (\cap Q_{c_{j'} \cup \{1\}} : j \in \alpha)]$

$\{x_1\} \vee (Q_{c_1} \cap \ldots \cap Q_{c_k})$

$\Rightarrow$ the inclusions are equalities and the planes Q_{c_i} have been reduced.

The case $\alpha \supset \{\ell+1,\ldots,k\}$ is handled similarly. $\|$

DEFINITION: Let $\{x_1,\ldots,x_r\}$ be points in $\mathbb{P}^n$, spanning a plane Q. If Q' is spanned by a subset of Q, say Q' is a *face* of Q; if Q' is spanned by a proper subset, say Q' is a *proper face*. The *interior* of Q is the complement in Q of all proper faces; denote this by $\mathring{Q}$. Then THEOREM 2.11 may be restated:

THEOREM 2.11': If $\{x_1,\ldots,x_r\}$ are in general position and $Q_1,\ldots,Q_k$ are planes spanned by subsets of these points, then

$\dim(\mathring{Q}_1 \cap \ldots \cap \mathring{Q}_k) \le \dim Q_1 + \ldots + \dim Q_k - (k-1)n.$ $\|$

DEFINITION: An r-tuple $(x_1,\ldots,x_r) \in (\mathbb{E}^n)^r$ is in *general position* if it is in projective general position when $(\mathbb{E}^n)^r$ is viewed as a subset of $(\mathbb{P}^n)^r$. Since $(\mathbb{E}^n)^r$ is open and dense in $(\mathbb{P}^n)^r$, it follows that general position is open and dense in $(\mathbb{E}^n)^r$.

One may transfer all definitions from projective space to Euclidean space. For example, a (proper) transversal of $x, Q_1,\ldots,Q_r$ in $\mathbb{E}^n$ is a line ℓ through x such that $\tilde{\ell}$ is a transversal (proper) of

$x, \tilde{Q}_1,\ldots,\tilde{Q}_r$ in $\mathbb{E}^n$. x is in general position with respect to $Q_1,\ldots,Q_r$ if $\dim T_\pi(x; \tilde{Q}_1,\ldots,\tilde{Q}_r; \mathbb{E}^n) \leq t_\pi$ for all partitions π of $\{1,\ldots,r\}$.

EXAMPLE: Let Q_1 be a point in $\mathbb{R}^2$, Q_2 and Q_3 parallel lines in $\mathbb{R}^2$ with $Q_1 \cap (Q_2 \cup Q_3) = \emptyset$. Let $x_1 \in \mathbb{R}^2$ be chosen on the line through Q_1 parallel to Q_2. Then no line through x_1 in $\mathbb{R}^2$ passes through Q_1, Q_2, and Q_3. Nevertheless, x_1 is not in general position with respect to Q_1, Q_2, Q_3, because $T(x_1; \tilde{Q}_1, \tilde{Q}_2 \cap \tilde{Q}_3; \mathbb{E}^n) \neq \emptyset$ and it should be.

III GENERAL POSITION OF LINEAR MAPS

In this section K will denote a finite simplicial complex, with dim K = $k < n$. $L(K, \mathbb{R}^n)$ is the space of linear maps $f: K \to \mathbb{R}^n$ with the uniform topology. Since every such map is characterized by its values on the vertices of K, one sees easily that $L(K, \mathbb{R}^n)$ is naturally homeomorphic to $\mathbb{R}^{nq}$, for q the number of vertices of K.

A map $f \in L(K, \mathbb{R}^n)$ is in *general position* if, given simplexes $\sigma_1, \ldots, \sigma_r$ of K of dimensions $a_1, \ldots, a_r$ respectively, then $\dim f\mathring{\sigma}_1 \cap \ldots \cap f\mathring{\sigma}_r \leq a_1 + \ldots + a_r - (r-1)n$. It is obvious that such a map is nondegenerate; that is, $f|\sigma$ is 1 - 1 for any simplex σ. $L_{GP}(K, \mathbb{R}^n) \subset L(K, \mathbb{R}^n)$ is the subspace of maps in general position.

THEOREM 3.1: $L_{GP}(K, \mathbb{R}^n)$ is dense in $L(K, \mathbb{R}^n)$.

PROOF: Let $\mathbb{R}^n$ be viewed as a subspace of $\mathbb{P}^n$, as in section I. For σ a simplex of K, let $[f(\sigma)]$ be the plane in $\mathbb{P}^n$ spanned by the vertices of $f(\sigma)$. Then $f(\mathring{\sigma}) \subset \mathring{[f(\sigma)]}$, the interior of the plane $[f(\sigma)]$, providing $F|\sigma$ is non-degenerate. Suppose f maps the vertices of K into general position in $\mathbb{R}^n \subset \mathbb{P}^n$. Then by THEOREM 2.11', for simplexes $\sigma_1, \ldots, \sigma_r$ of K, we have $\dim (f\mathring{\sigma}_1 \cap \ldots \cap f\mathring{\sigma}_r) \leq \dim ([f(\mathring{\sigma}_1)] \cap \ldots \cap [f(\mathring{\sigma}_r)]) = a_1 + \ldots + a_r - (r-1)n$. Therefore, if f maps the vertices of K into general position, then $f \in L_{GP}(K, \mathbb{R}^n)$; so by THEOREM 2.7 $L_{GP}(K, \mathbb{R}^n)$ is dense. ||

THEOREM 3.2: $L_{GP}(K, \mathbb{R}^n)$ is open in $L(K, \mathbb{R}^n)$.

LEMMA 3.3: Let A, B, C be convex linear cells, with B a face of A, in $\mathbb{R}^n$. Let $[X]$ denote the plane spanned by X. If $\mathring{B} \cap \mathring{C} \neq \emptyset$ (where $\mathring{B}$ is the interior of B in $[B]$, i.e., the manifold interior) and $[B]\vee[C]=\mathbb{R}^n$, then $\mathring{A}\cap\mathring{C}\neq\emptyset$ and $\mathring{B}\cap\mathring{C}\subset\overline{\mathring{A}\cap\mathring{C}}$.

PROOF: Let $a\in\mathring{A}$. Then $([B]\vee\{a\})\cap A$ is a convex cell A', $\mathring{A}'\subset\mathring{A}$. It suffices to show $\mathring{A}'\cap\mathring{C}\neq\emptyset$. Thus we may assume w.l.o.g. $\dim A=\dim B+1$. Since $[B]\vee[C]=\mathbb{R}^n$., $\dim([B]\cap[C])=\dim B+\dim C-n\geq 0$, and $\dim([A]\cap[C])=\dim([B]\cap[C])+1$. Let $x\in\mathring{B}\cap\mathring{C}$. Let ℓ be a line in $[A]\cap[C]-[B]\cap[C]$ through x (that is, $\ell-\{x\}\subset[A]\cap[C]-[B]\cap[C]$).

Since $x\in\mathring{C}$ and $\ell\subset[C]$, $\exists\,\epsilon>0\ni$ if $x'\in\ell$ and $d(x,x')<\epsilon$, $x'\in\mathring{C}$. Since $x\in\mathring{B}$ and B is a principal face of A, A is a neighborhood of x in cl(component of A in $[A]-[B]$) $=A^+$. Then for ϵ small, if $x'\in\ell\cap A^+$ and $d(x',x)<\epsilon$, $x'\in\mathring{A}\cap\mathring{C}$. Hence $\mathring{A}\cap\mathring{C}\neq\emptyset$ and $\mathring{B}\cap\mathring{C}\subset\overline{\mathring{A}\cap\mathring{C}}$. ||

PROPOSITION 3.4: If $f\in L_{GP}(K\mathbb{R}^n)$ and $f\mathring{\sigma}_1\cap\ldots\cap f\mathring{\sigma}_r\neq\emptyset$, then $\dim(f\mathring{\sigma}_1\cap\ldots\cap f\mathring{\sigma}_r)=\dim\sigma_1+\ldots+\dim\sigma_r-(r-1)n$.

LEMMA 3.5: Let $f\in L_{GP}(K\mathbb{R}^n)$ and let $\sigma_1,\ldots,\sigma_r$ be simplexes of K such that $f\mathring{\sigma}_1\cap\ldots\cap f\mathring{\sigma}_r=\emptyset$ but the intersection becomes nonempty if any term is deleted. If $\tau=\sigma_1\cap\ldots\cap\sigma_r$, then $f\tau=f\sigma_1\cap\ldots\cap f\sigma_r$.

PROOF: Let $\tau_1,\ldots,\tau_r$ be faces of $\sigma_1,\ldots,\sigma_r$ such that $f\mathring{\tau}_1\cap\ldots\cap f\mathring{\tau}_r\neq\emptyset$. (If none exist, then both sides of the above equation are empty.) We must show $\tau_1=\ldots=\tau_r$. Suppose $\tau_1=\tau_2=\ldots=\tau_s$, but $\tau_1\neq\tau_j$ for $j>s<r$. By hypothesis, $f\mathring{\sigma}_1\cap\ldots\cap f\mathring{\sigma}_s\neq\emptyset$. Thus $f\sigma_1\cap\ldots\cap f\sigma_s$ is a convex linear cell A with $f\tau_1=B$ a face. By PROPOSITION 3.4, the hypotheses of LEMMA 3.3 are satisfied for A, B, and $C=f\tau_{s+1}\cap\ldots\cap f\tau_r$ (namely $[B]\vee[C]=\mathbb{R}^n$). By LEMMA 3.3, $\mathring{A}\cap\mathring{C}=f\mathring{\sigma}_1\cap\ldots\cap f\mathring{\sigma}_s\cap f\mathring{\tau}_{s+1}\cap\ldots\cap f\mathring{\tau}_r\neq\emptyset$. Iterating this argument we must either arrive at all simplexes being equal or $f\mathring{\sigma}_1\cap\ldots\cap f\mathring{\sigma}_r\neq\emptyset$. Therefore, eventually we find $\tau_i\subset\sigma_i$ with $\tau_1=\ldots=\tau_r$ and $f\tau_i=f\sigma_1\cap\ldots\cap f\sigma_r$. ||

PROOF of THEOREM 3.2: Let $f \in L_{GP}(K, E^n)$. Let $\sigma_1, \ldots, \sigma_r$ be simplexes of K. If $f\mathring{\sigma}_1 \cap \ldots \cap f\mathring{\sigma}_r \neq \emptyset$, then $\dim (f\mathring{\sigma}_1 \cap \ldots \cap f\mathring{\sigma}_r) = ([f\sigma_1] \cap \ldots \cap [f\sigma_r]) = \dim \sigma_1 + \ldots + \dim \sigma_r - (r-1)n$. By COROLLARY 1.6, if $f'(\sigma_i)$ is close to $f(\sigma_i)$, the planes $[f'\sigma_1], \ldots, [f'\sigma_r]$ intersect minimally and therefore $\dim (f'\sigma_1) \cap \ldots \cap (f'\sigma_r) = \sum_1^r \dim \sigma_i - (r-1)n$. In fact, by PROP. 1.3 we may guarantee that $f'\mathring{\sigma}_1 \cap \ldots \cap f'\mathring{\sigma}_r \neq \emptyset$, so that $\dim (f'\mathring{\sigma}_1 \cap \ldots \cap f'\mathring{\sigma}_r) = \dim \sigma_1 + \ldots + \dim \sigma_r - (r-1)n$.

Suppose $f\mathring{\sigma}_1 \cap \ldots \cap f\mathring{\sigma}_r = \emptyset$. We show that if f' is close to f, $f'\mathring{\sigma}_1 \cap \ldots \cap f'\mathring{\sigma}_r = \emptyset$. Thus it suffices to prove this for $\sigma_1, \ldots, \sigma_r$ satisfying the hypothesis of LEMMA 3.5. Then $f\sigma_1 \cap \ldots \cap f\sigma_r = f\tau$ (τ possibly empty). If $f\sigma_1 \cap \ldots \cap f\sigma_r = \emptyset$, then by compactness if f' is close to f, $f'\sigma_1 \cap \ldots \cap f'\sigma_r = \emptyset$. If $\tau \neq \emptyset$ then $f'\sigma_1 \cap \ldots \cap f'\sigma_r \supset f'\tau$. By assumption $f\mathring{\sigma}_2 \cap \ldots \cap f\mathring{\sigma}_r \neq \emptyset$, and f' close to f implies that $f'\mathring{\sigma}_2 \cap \ldots \cap f'\mathring{\sigma}_r \neq \emptyset$ and is close to $f\mathring{\sigma}_2 \cap \ldots \cap f\mathring{\sigma}_r$. Since $f\mathring{\sigma}_1 \cap (f\mathring{\sigma}_2 \cap \ldots \cap f\mathring{\sigma}_r) = \emptyset$, and since $f\sigma_1 \cap (f\sigma_2 \cap \ldots \cap f\sigma_r) = f\tau$, there is a minimal angle Θ between line segments $\ell \subset f\sigma_1$ and $\ell' \subset f\sigma_2 \cap \ldots \cap f\sigma_r$ such that ℓ and ℓ' each intersects $f\tau$ perpendicularly at the same point. But then if f' is close to f, the minimal angle will remain positive, so $f'\sigma_1 \cap f'\sigma_2 \cap \ldots \cap f'\sigma_r = f'\tau$.

Thus if f' is close to f, the condition of general position is preserved. ||

IV STRUCTURAL STABILITY

Let X,Y be topological spaces and Map (X,Y) a collection of continuous maps from X to Y. Two maps $f, g \in$ Map (X,Y) are called *structurally equivalent*[1] if there are homeomorphisms $j: X \to X$, $k: Y \to Y$ and a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "j"'] & Y \arrow[d, "k"] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

The maps are *strongly structurally equivalent* if k can be chosen so that $j = 1_X$, the identity map on X.

Suppose now that Map (X,Y) has a topology. A map $f \in$ Map (X,Y) is *structurally stable* if there is a neighborhood N of f in Map (X,Y) and a commutative diagram

$$\begin{tikzcd} N \times X \arrow[r, "1_N \times f"] \arrow[d, "j"'] & N \times Y \arrow[d, "k"] \\ N \times X \arrow[r, "E"'] & N \times Y \end{tikzcd}$$

such that 1. $E(f^1, x) = (f^1, f^1(x))$

2. For every $f^1 \in N$ there exist homeomorphisms $J_{f^1}: X \to X$ and $K_{f^1}: Y \to Y$ such that $J_f = 1_X$, $K_f = 1_Y$, and $J(f^1, x) = (f^1, J_{f^1}(x))$ and $K(f^1, y) = (f^1, K_{f^1}(y))$.

A map F is *strongly structurally stable* if J can be chosen to be the identity on N x X.

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[1] See Levine, "Singularities of Differentiable Mappings", p. 41 for the differentiable analogue.

The following are simple consequences of these definitions:

4.1 If f and g are strongly structurally equivalent they are structurally equivalent. If f is strongly structurally stable, it is structurally stable.

4.2 If f is (strongly) structurally stable, there is a neighborhood of f in Map (X,Y) such that if $f^1 \varepsilon N$, f and f^1 are (strongly) structurally equivalent.

PROOF: J_{f^1}, K_{f^1} satisfy the definition. ||

4.3 If N is a "structurally stable neighborhood" of f, i.e., if it satisfies the definition, and if $f^1 \varepsilon \overset{o}{N}$, then N is a structurally stable neighborhood of f^1.

PROOF: Define $J^1: N \times X \to N \times X$ by $J^1\ (g,x) = (g, J_{f^1}^{-1}\ (x))$ and $K^1: N \times Y \to N \times Y$ by $K^1(g,y) = (g, K_{f^1}^{-1}(y))$.

The diagram ; yields

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "J_{f^1}"'] & Y \arrow[d, "K_{f^1}"] \\ X \arrow[r, "f^1"] & Y \end{tikzcd}$$

$f \circ J_{f^1}^{-1} = K_{1}^{-1} \circ f^1 : X \to Y.$

Therefore, $K^1 \circ (1_N \times f^1)\ (g,x) = K^1(g f^1 x) = (g, K_{f^1}^{-1} f^{-1}(x))$

$= (g, f J_{f^1}^{-1}(x)) = 1_N \times f\ (g, J_{f^1}^{-1}(x))$

$= 1_N \times f \circ J^1\ (g,x\ ).$

We now have a commutative diagram

$$\begin{tikzcd} N \times X \arrow[r] \arrow[d, "J^1"] \arrow[dd, bend right=60, "J \circ J^1"'] & N \times Y \arrow[d, "K^1"] \arrow[dd, bend left=60, "K \circ K^1"] \\ N \times X \arrow[d, "J"] & N \times Y \arrow[d, "K"] \\ N \times X \arrow[r, "E"'] & N \times Y \end{tikzcd}$$

the outer rectangle of which establishes the proposition if N is a neighborhood of f^1; that is, if $f^1 \varepsilon \mathring{N}$. ||

COROLLARY 4.4: Structural stability is an open condition on Map (X,Y).

If the spaces X,Y have structure, we may require in the definitions that homeomorphisms $J_g : X \to X$ and $K_g : Y \to Y$ are structure-preserving. In our applications, these maps will be required to be piecewise linear.

PROPOSITION 4.5: Let $\nu : I \to \text{Map}\ (X,Y)$ be a path with $\nu\ (t)$ structurally stable $\forall\ t \varepsilon I$. Then there is a diagram of level preserving homeomorphisms

$$\begin{array}{ccc} I \times X & \xrightarrow{(1 \times y\ (o))} & I \times Y \\ J \downarrow & & \downarrow K \\ I \times K & \xrightarrow[E]{} & I \times Y \end{array}$$

where $E(t,x) = (t, \{\nu\ (t)\}\ (x))$.

PROOF: For each $t \varepsilon I$, there exists a structurally stable neighborhood N_t of $\nu\ (t)$ in Map (X,Y) and maps J_t, K_t. For each t, there exists ε_τ such that $\nu\ (t - \varepsilon_t, t + \varepsilon_t) \subset \mathring{N}_t$. We may therefore, by compactness, choose $0 = t_o < t_1 < \ldots < t_n$ and $N_1, \ldots, N_n$ such that $\nu[\ t_i, t_{i+1}\ ] \subset \mathring{N}_{i+1}$ and N_i is a structurally stable neighborhood of $\nu\ (t_i)$. Thus there are diagrams

$$\begin{array}{ccc} N_i \times X & \xrightarrow{1 \times \nu\ (t_i)} & N_i \times Y \\ J_i \downarrow & & \downarrow K_i \\ N_i \times X & \xrightarrow[E]{} & N_i \times Y \end{array}$$

which yield

$$\begin{array}{ccc} [\ t_i - 1, t_i]\ \times X & \xrightarrow{1 \times \nu\ (t_i - 1)} & [\ t_i - 1, t_i]\ \times Y \\ J_i{}^1 = J_i \circ\ (\nu \times 1) \downarrow & & \downarrow K_i \circ\ (\nu \times 1) = K_i{}^1 \\ [\ t_i - 1, t_i]\ \times X & \xrightarrow{E\ \circ\ (\nu \times 1)} & [t_i - 1, t_i]\ \times Y. \end{array}$$

For $t \in [t_i - 1, t_i]$ define $J_i^{\,1}(t) : X \to X$ and $K_i^{\,1}(t) : Y \to Y$ by $J_i^{\,1}(t)(x) = J_i^{\,1}(t,x)$ and $K_i^{\,1}(t)(x) = K_i^{\,1}(t,x)$.

Now define $J: I \times X \to I \times X$ by

$$J(t,x) = (t, K_j^{\,1}(t) \circ K_{j-1}^{\,1}(t_{i-1}) \ldots \circ K^1(t_1)(X))$$

for $t \in [t_{j-1}, t_j]$ and similarly $K: I \times Y \to I \times Y$. These are well defined since $J_i^{\,1}(t_{i-1}) = 1_X$ and $K_i^{\,1}(t_{i-1}) = 1_Y$.

The diagram

$$\begin{array}{ccc} I \times X & \xrightarrow{1 \times \nu(t_o)} & I \times Y \\ J\downarrow & & \downarrow K \\ I \times X & \xrightarrow{E} & I \times Y \end{array}$$

is the desired one. ||

COROLLARY 4.6: Let $\nu : I \to \text{Map}(X,Y)$ be a path with $\nu(t)$ strongly structurally stable $\forall\, t \in I$. Then the diagram of 1.5 can have $J = 1_{I \times X}$.

COROLLARY 4.7: Suppose an isotopy E of embeddings of X into Y can be realized by a path $\nu: I \to \text{Map}(X,Y)$ with each embedding strongly structurally stable. Then there is an isotopy of homeomorphisms H of Y realizing this isotopy; i.e., there is a commutative diagram

$$\begin{tikzcd} I \times X \arrow[r, "1 \times f"] \arrow[dr, "E"'] & I \times Y \arrow[d, "H"] \\ & I \times Y \end{tikzcd}$$

This is the *isotopy extension theorem*.

DEFINITION: A map $F: N \times A \to N \times B$ is a *Fiberwise map* if $pr_1 \circ F(g,a) = g$. It is a *Fiberwise embedding* if also $pr_2 \circ F|\{g\} \times A$ is an embedding for each $g \in N$. It is a *Fiberwise homeomorphism* if $pr_2 \circ F|\{g\} \times A$ is a homeomorphism.

THEOREM 4.8: If X is locally compact, locally connected, and Hausdorff, then a *Fiberwise homeomorphism* $J : N \times X \to N \times X$ is a homeomorphism.

PROOF: Let $\mathcal{C}(X, X)$ be the set of continuous maps from X to X, with the compact - open topology. Let A (X) be the subspace of homeomorphisms of X. Since the map $pr_2 \circ J: N \times X \to X$ is continuous, the associated map $J^*: N \to \mathcal{C}(X, X)$ given by $[J^*(F)](x) = pr_2 \circ J(F, x)$ is continuous. By hypothesis $J^*(N) \subset A(x)$.

[*THEOREM (Arens)*: If X is locally compact, locally connected and T_2, inversion is continuous in A(X).]

The map $J^1: N \to A(X)$ given by $J^1(g) = J^*(g)^{-1}$ is continuous. Since X is compact and T_2, the associated map $K: N \times X \to X$ given by $K(g,x) = [J^1(g)](x)$ is continuous. But $K = pr_2 \circ J^{-1}$, and $pr_1 \circ J^{-1}$ is clearly continuous. $\therefore J^{-1}$ is continuous. ||

V STRUCTURAL STABILITY in $L(K, \mathbb{R}^n)$

The objective of this section is to prove the following

THEOREM 5.1: If $f \in L(K, \mathbb{R}^n)$ maps the vertices of K into general position, then f is structurally stable. That is, there is a neighborhood N of f in $L(K, \mathbb{R}^n)$ and a commutative diagram

$$\begin{array}{ccccc} N \times K & \xrightarrow{E} & F(N \times K) & \to & N \times \mathbb{R}^n \\ \downarrow \lambda' & & \downarrow \lambda & & \downarrow \lambda \\ N \times K & \xrightarrow{1 \times f} & N \times f(K) & \to & N \times \mathbb{R}^n \end{array}$$

such that λ, λ' are Fiberwise homeomorphisms, and the maps $\lambda'_g: K \to K$ and $\lambda_g: \mathbb{R}^n \to \mathbb{R}^n$ given by $\lambda'(g, x) = (g, \lambda'_g(x))$ and $\lambda(g, y) = (g, \lambda_g(y))$ are PL homeomorphisms.

COROLLARY 5.2: The structurally stable linear maps of K into $\mathbb{R}^n$ are dense and open in $L(K, \mathbb{R}^n)$.

PROOF: The set of maps in $L(K, \mathbb{R}^n)$ sending the vertices of K into general position is dense in $L(K, \mathbb{R}^n)$. The corollary now follows from THEOREM 5.1 and COROLLARY 4.4. ||

Recall that $L(K, \mathbb{R}^n)$ is naturally homeomorphic to $\mathbb{R}^{ns}$ for s the number of vertices of K. Given f as in THEOREM 5.1, chose $\varepsilon > 0$ such that if $d(g, f) < \varepsilon$, g maps the vertices of K into general position in $\mathbb{R}^n$. Let $N = N(f, \epsilon)$. We will prove the theorem for this choice of N, for small ε.

For the duration of this section, we will adopt the following notation:

$\{v_1, \ldots, v_s\}$ are the vertices of K. We may assume $s > n$, since otherwise the result is trivial.

$\mathcal{S}$ = the set of subsets of $\{v_1, \ldots, v_s\}$ of order $\leq n$.

$\mathcal{S}'$ = the set of subsets of $\{v_1, \ldots, v_s\}$ of order n.

View K as a subset of $\mathbb{R}^s$ by extending linearly the embedding $V_i \to (0, \ldots, 1, \ldots, 0)$, 1 in the ith place.

α will be a generic name for an element of $\mathcal{S}$, and ν for a subset of $\mathcal{S}$.

σ_α = simplex spanned by $\{V_i : V_i \epsilon \alpha\}$ for $\alpha \epsilon \mathcal{S}$. Note that dim. n-1 and that σ_α need not lie in K.

P_g^α = plane spanned by the points $\{g(V_i) : V_i \epsilon \alpha\}$ in $\{g\} \times \mathbb{R}^n$ $\subset N \times \mathbb{R}^n$. If $\sigma_\alpha \subset K$, this is the plane spanned by $F(\{g\} \times \sigma_\alpha)$.

$P_N^\alpha = \bigcup_{g \epsilon N} P_g^\alpha \subset N \times \mathbb{R}^n$.

View $\mathbb{R}^n \subset \mathbb{P}^n$. Every p - plane in $\mathbb{R}^n$ extends uniquely to a p-plane in $\mathbb{P}^n$.

$\overline{P}_g^\alpha$ = plane in $\{g\} \times \mathbb{P}^n \subset N \times \mathbb{P}^n$ determined by $P_g^\alpha, \alpha \epsilon \mathcal{S}$.

$\overline{P}_{\nu,g} = \bigcap_{\alpha \epsilon \nu} \overline{P}_g^\alpha \subset \{g\} \times \mathbb{P}^n$, where $\nu \subset \mathcal{S}$.

$\overline{P}_{\nu,N} = \bigcup_{g \epsilon n} \overline{P}_{\nu,g} \subset N \times \mathbb{P}^n$.

PROPOSITION 5.3: For any $\alpha \epsilon \mathcal{S}$, $\overline{P}_g^\alpha = \bigcap_{\substack{\alpha' \supset \alpha \\ \alpha' \epsilon \mathcal{S}^1}} \overline{P}_g^{\alpha'}$.

PROOF: By decreasing induction on $|\alpha|$, the cardinality of α. If $|\alpha| = n$, then the equality is trivial. If $|\alpha| < n$, then since $n < s$, there exist at least two points V_i, V_j not in α. Then by general position of $\{g(V) : V \epsilon \alpha \cup \{V_i, V_j\}\}$ in $\mathbb{R}^n$, $\overline{P}_g^{\alpha \cup \{V_i\}} \cap \overline{P}_g^{\alpha \cup \{V_j\}} = \overline{P}_g^\alpha$. Applying induction to the left side of this equation, the result follows easily.

PROPOSITION 5.4: For any $\nu \subset \mathcal{S}$, there exists $\nu^1 \subset \mathcal{S}^1$ such that $\overline{P}_{\nu,g} = \overline{P}_{\nu^1,g}$ for all $g \epsilon N$.

PROOF: This follows directly from 5.3.

LEMMA 5.5: $\dim \overline{P}_{\nu,g} = \dim \overline{P}_{\nu,f}$ for $\forall \nu \subset \mathcal{S}$, $g \in N$.

PROOF: Let $\nu = \{\alpha_1, \ldots, \alpha_c\}$. Then $\overline{P}_{\nu,f} = \overline{P}_f^{\alpha_1} \cap \ldots \cap \overline{P}_f^{\alpha_c}$.
By THEOREM 2.11, there exists $\alpha_i' \subset \alpha_i$ such that $\overline{P}_f^{\alpha_1} \cap \ldots \cap \overline{P}_f^{\alpha_c} = \overline{P}_f^{\alpha_1'} \ldots \overline{P}_f^{\alpha_c'}$ and $\dim \overline{P}_f^{\alpha_1'} \cap \ldots \cap \overline{P}_f^{\alpha_c'} = \dim \overline{P}_f^{\alpha_1'} \ldots \dim \overline{P}_f^{\alpha_c'} - (c-1)n$
$= \sum_1^c (|\alpha_i'| - 1) - (c-1)n.$
By PROPOSITION 1.2, $\dim \overline{P}_g^{\alpha_1'} \cap \ldots \cap \overline{P}_g^{\alpha_c'} \geq \sum_1^c (|\alpha_i'| - 1) - (c-1)n$ for all $g \in N$. Since $\overline{P}_g^{\alpha_i} \supset \overline{P}_g^{\alpha_i'}$ for $1 \leq i \leq c$, it follows that $\dim \overline{P}_{\nu,g} \geq \dim \overline{P}_g^{\alpha_1'} \cap \ldots \cap \overline{P}_g^{\alpha_c'} \geq \sum_1^c (|\alpha_i'| - 1) - (c-1)n = \dim \overline{P}_{\nu,f}$; now by symmetry between f and g, $\dim \overline{P}_{\nu,f} \geq \dim \overline{P}_{\nu,g}$. ||

PROPOSITION 5.6: Let $\nu, \nu' \subset \mathcal{S}'$ such that $\overline{P}_{\nu,g} \subsetneq \overline{P}_{\nu',g}$. Then there exists $\nu'' \subset \mathcal{S}'$ such that $\overline{P}_{\nu,g} \subset \overline{P}_{\nu'',g} \subset \overline{P}_{\nu',g}$ and $\dim \overline{P}_{\nu',g} = \dim \overline{P}_{\nu'',g} + 1$.

PROOF: Since $\overline{P}_{\nu\cup\nu',g} = \overline{P}_{\nu,g} \cap \overline{P}_{\nu',g} = \overline{P}_{\nu,g}$, we may assume w.l.o.g. that $\nu \supset \nu'$. Hence there is some $\alpha \in \nu - \nu'$ such that $\overline{P}_{\nu,g} \subset \overline{P}_{\nu' \cup \{\alpha\},g} \subsetneq \overline{P}_{\nu',g}$. $\overline{P}_{\nu' \cup \{\alpha\},g} = \overline{P}_{\nu',g} \cap \overline{P}_g^{\alpha}$. By PROPOSITION 1.2: $\dim \overline{P}_{\nu' \cup \{\alpha\},g} \geq \dim \overline{P}_{\nu',g} + \dim \overline{P}_g^{\alpha} - n =$
$\dim \overline{P}_{\nu',g} + (n-1) - n = \dim \overline{P}_{\nu',g} - 1$. Since $\overline{P}_{\nu' \cup \{\alpha\},g} \neq \overline{P}_{\nu',g}$, $\dim \overline{P}_{\nu' \cup \{\alpha\},g} = \dim \overline{P}_{\nu',g} - 1$. Set $\nu'' = \nu' \cup \{\alpha\}$. ||

COROLLARY 5.7: Given $\nu \subset \mathcal{S}'$, let $\nu_1, \ldots, \nu_r$ be all the subsets of $\mathcal{S}'$ such that $\overline{P}_{\nu_i,g} \subsetneq \overline{P}_{\nu,g}$. Then $\overline{P}_{\nu,g} - \bigcup_i \overline{P}_{\nu_i,g}$ is a disjoint union of open convex cells in $\overline{P}_{\nu,g}$.

PROOF: By 5.6, there are $\nu_1', \ldots \nu_{r'}'$ such that $\bigcup_1^r \overline{P}_{\nu_i,g} = \bigcup_1^{r'} \overline{P}_{\nu_i',g}$ and $\dim \overline{P}_{\nu_i',g} = \dim \overline{P}_{\nu,g} - 1$. But the complement of a family of hyperplanes (planes of codimension 1) in $\mathbb{R}^n$ is a disjoint union of open convex cells. ||

THEOREM 5.8: Let $P_1, \ldots, P_r$ be hyperplanes (codimension 1) in $\mathbb{P}^n$. Let $F\colon N \times (\overset{r}{\underset{1}{\cup}} P_i) \to N \times \mathbb{P}^n$ be a *Fiberwise embedding* with N a disc, such that

1. $\forall x \in N$, $1 \le i \le r$, $F(\{x\} \times P_i)$ is a hyperplane in $\{x\} \times \mathbb{P}^n$

and

2. $F \mid \{x_0\} \times \overset{r}{\underset{1}{\cup}} P_i$ is the inclusion map into $\{x_0\} \times \mathbb{P}^n$.

Then F can be extended to a *Fiberwise homeomorphism* $H\colon N \times \mathbb{P}^n \to N \times \mathbb{P}^n$ such that $H \mid \{x_0\} \times \mathbb{P}^n$ is the identity map.

PROOF: Let $\pi\colon S^n \to \mathbb{P}^n$ be the double covering of projective space by the sphere. Let $S_i^{n-1} = \pi^{-1}(P_i)$. We have a diagram

$$\begin{tikzcd}
\{x_0\} \times \overset{r}{\underset{1}{\cup}} S_i^{n-1} \arrow[r, hook] \arrow[d, hook] & \{x_0\} \times S^n \arrow[d] \\
N \times \overset{r}{\underset{1}{\cup}} S_i^{n-1} \arrow[r, dashed, "F^*"] \arrow[d, "1\times\pi"'] & N \times S^n \arrow[d, "1\times\pi"] \\
N \times \overset{r}{\underset{1}{\cup}} P_i \arrow[r, "F"] & N \times \mathbb{P}^n
\end{tikzcd}$$

Since $\{x_0\} \times \overset{r}{\underset{1}{\cup}} S_i^{n-1}$ is a deformation retract of $N \times \overset{r}{\underset{1}{\cup}} S_i^{n-1}$ we can lift F to F^* with $F^* \mid \{x_0\} \times \overset{r}{\underset{1}{\cup}} S_i^{n-1}$ the inclusion map. It suffices to extend F^* to a *Fiberwise homeomorphism* H^* of $N \times S^n$ which sends antipodal points to antipodal points.

The complement of $\overset{r}{\underset{1}{\cup}} S_i^{n-1}$ in S^n is a disjoint union of convex linear cells C_+^j, C_-^j, $1 \le j \le q$, where C_+^j and C_-^j are antipodal. Given a point $x \in C_+^j$, every point of $\mathring{C}_+^j$ can be joined to x by a unique geodesic in $\mathrm{cl}(C_+^j)$. The corresponding statement is false in $\mathbb{P}^n$.

Suppose first that $r = 1$. Let C_+^j be one of the cells in $S^n - S_1^{n-1}$. Let v be the center point of C_+^1 (the unique point farthest from S_1^{n-1}. Then we may define $F^1 : N \to S^n$ to be the unique continuous map such that $F^1(x_o) = v$ and $F^1(x)$ is the center point of a complementary domain of $F(\{x\} \times S_1^{n-1})$. (One way to see that F^1 exists is to define $G : N \to P^n$ and lift to S^n). Define $H^* | \{x\} \times cl(C_+^1)$ as the join of $F | \{x\} \times S_1^{n-1}$ and the map $v \to F^1(x)$. Define $H^* | \{x\} \times (C_-^1)$ to be the antipodal map to $H^* | \{x\} \times cl(C_+^1)$.

Now assume $r > 1$. For any cell C_+^j, let $a_j, b_j, \in \dot{C}_+^j$ which lie on different spheres S_{j1}^{n-1}, S_{j2}^{n-1}. Let v_j be the midpoint of the geodesic joining a_j and b_j (Since a_j and b_j cannot be antipodal, there is a unique such point and $v_j \in C_+^j$) Define $F_j^1 : N \to S^n$ by $F_j^1(x)$ = the midpoint of the geodesic joining $F^*(x, a_j)$ and $F^*(x, b_j)$ in $\{x\} \times S^n$. Since $F^* | N \times \{a_j\}$ and $F^* | N \times \{b_j\}$ are continuous, so is F_j^1. Define $H^* | \{x\} \times cl(C_+^j)$ as the join of $F^* | \{x\} \times \dot{C}_+^j$ and the map $v_j \to F_j(x)$. Define $H^* | \{x\} \times cl(C_-^j)$ to be the corresponding map antipodal to this.

Note that since S^n and S_j^{n-1} are orientable, we may consistently distinguish the regions of $\{x\} \times S^n - F(\{x\} \times S_j^{n-1})$, calling one + and the other − . In this way each open convex cell in $S^n - \bigcup S_j^{n-1}$ is identified by a collection of signs. For any v_j chosen above, the construction guarantees that $F_j(x)$ has the same signs assigned to it as v_j, since it is easily verified that $F_j(x)$ can never lie on $F(\{x\} \times S_i^{n-1})$. Therefore, the map H^* defined above gives a well-defined homeomorphism extending F^*, $H^* : N \times S^n \to N \times S^n$. //

PROOF of THEOREM 5.1: If $P \in G_k \mathbb{P}^n$, then $\hat{P} \in G_{k+1} \mathbb{R}^{n+1}$, Let $Q \in G_{n-k-1} \mathbb{P}^n$ be the plane such that $\hat{Q} \in G_{n-k} \mathbb{R}^{n+1}$ is the orthogonal complement to $\hat{P}$. Call $Q = P^{\perp}$ the *orthogonal complement* of P. If $x \in \mathbb{R}^n - Q$, there is a unique point $\pi(x) \in P$ closest to x; thus there is a natural projection map $\pi_P : \mathbb{R}^n - Q \to P$. If $P^1 \in G_k \mathbb{R}^n$ and $P^1 \subset \mathbb{R}^n - Q$, then $\pi_P | P^1 : P^1 \to P$ is easily seen to be a homeomorphism.

Recall that F maps the vertices of K into general position in $\mathbb{R}^n \subset \mathbb{R}^n$. Choose $\epsilon > 0$ such that if $d(g, F) < \epsilon$, then

1. g maps the vertices of K into general position, and
2. if $\nu \in \mathcal{S}$, then $Pr_2(\overline{P}_{\nu,g}) \subset \mathbb{R}^n - pr_2(\overline{P}_{\nu,f})^{\perp}$.

That 2. can be satisfied follows from PROPOSITION 1.3, together with LEMMA 5.5.

Let $N = N(f, \epsilon)$. Observe that N is homeomorphic to a disc of dimension ns.

For any $\nu \in \mathcal{S}$, define a *Fiberwise homeomorphism* $\Psi_\nu : \overline{P}_{\nu,N} \to N \times \overline{P}_{\nu,f}$ by $\Psi_\nu(g, x) = (g, \pi_{\overline{P}_{\nu,f}}(x))$. The family of maps Ψ_ν, $\nu \in \mathcal{S}$ are not compatible; that is, if $\overline{P}_{\nu,f} \subset \overline{P}_{\nu',f'}$ then $\Psi_{\nu'} | \overline{P}_{\nu,N} \neq \Psi_\nu$. We will modify these maps inductively so that they are compatible.

By PROPOSITION 5.4, it suffices to consider $\nu \in \mathcal{S}'$. If $\dim \overline{P}_{\nu,f} = 0$, let $\varphi_\nu = \Psi_\nu : \overline{P}_{\nu,N} \to N \times \overline{P}_{\nu,f}$. Suppose now that φ_ν is defined for all $\nu \in \mathcal{S}'$ such that $\dim \overline{P}_{\nu,f} < r$, such that if $\nu_1 \subset \nu_2$, then $\varphi_{\nu_1} | \overline{P}_{\nu_2,N} = \varphi_{\nu_2}$. We will construct φ_ν for $\nu \in \mathcal{S}'$ with $\dim \overline{P}_{\nu,f} = r$.

Let $\sigma = \{\nu' : \nu < \nu' \subset \mathcal{S}' \text{ and } \dim \overline{P}_{\nu',f} < r\}$. Let $\sigma^1 = \{\nu' \in \sigma \mid \dim \overline{P}_{\nu',f} = r - 1\}$. By PROPOSITION 5.6, $\bigcup_{\nu \in \sigma'} \overline{P}_{\nu',g} = \bigcup_{\nu' \in \sigma} \overline{P}_{\nu',g}$. For each $\nu' \in \sigma$, there is a commutative diagram

$$\begin{array}{ccc} \overline{P}_{\nu',N} & \hookrightarrow & \overline{P}_{\nu,N} \\ \swarrow \varphi_{\nu'} & & \downarrow \Psi_\nu \\ N \times \overline{P}_{\nu',f} & \xrightarrow{\ \theta_\nu^{\nu'}\ } & N \times \overline{P}_{\nu,f} \end{array}$$

where $\Theta_{\nu}^{\nu'} = \psi_{\nu} \circ \varphi_{\nu'}^{-1}$ is a level preserving embedding. The statement that the maps $\varphi_{\nu'}$ are compatible for $\nu' \in \sigma$ translates into the statement that the above diagrams yield a commutative square

$$\begin{array}{ccc} \bigcup_{\nu' \in \sigma} \overline{P}_{\nu',N} & \hookrightarrow & \overline{P}_{\nu,N} \\ \downarrow \cup \varphi_{\nu'} & & \downarrow \psi_{\nu} \\ N \times \bigcup_{\nu' \in \sigma} \overline{P}_{\nu',f} & \xrightarrow[\Theta_{\nu}]{} & N \times \overline{P}_{\nu,f} \end{array}$$

Namely, if $\nu_1 \supset \nu_2 \in \sigma$, $\Theta_{\nu}^{\nu_1} = \psi_{\nu} \circ \varphi_{\nu_1}^{-1} = \psi_{\nu} \circ (\varphi_{\nu_2} \mid \overline{P}_{\nu_1,N})^{-1}$

$$= \Theta_{\nu}^{\nu_2} \mid N \times \overline{P}_{\nu_1,f} .$$

Θ_{ν} is a map satisfying the hypotheses of THEOREM 5.8, so there is an extension of Θ_{ν} to a *Fiberwise homeomorphism* of $N \times \overline{P}_{\nu,f}$.

Define $\varphi_{\nu}: P_{\nu,N} \longrightarrow N \times \overline{P}_{\nu,f}$ by $\varphi_{\nu} = \Theta_{\nu}^{-1} \circ \psi_{\nu}$.

Then for any $\nu' \supset \nu$,

$$\varphi_{\nu} \mid \overline{P}_{\nu',N} = \Theta_{\nu}^{-1} \psi_{\nu} \mid \overline{P}_{\nu',N} = (\Theta_{\nu}^{\nu'})^{-1} \psi_{\nu} \mid \overline{P}_{\nu',N} = \varphi_{\nu'} .$$

By induction, we have made the maps φ_{ν} compatible. Now we repeat the process one last time:

Combining the maps $\varphi_{\nu}^{-1}: N \times \overline{P}_{\nu,f} \longrightarrow \overline{P}_{\nu,N} \subset N \times \mathbb{P}^n$ to a *Fiberwise embedding* φ^{-1}: $N \times \bigcup_{\nu \in \mathcal{S}} \overline{P}_{\nu,f} \longrightarrow N \times \mathbb{P}^n$, and reapplying THEOREM 5.8, we get a *Fiberwise homeomorphism* φ^{-1}: $N \times \mathbb{P}^n \rightarrow N \times \mathbb{P}^n$, such that $(\varphi^{-1})^{-1} = \varphi$ is compatible with the φ_{ν}'s. $\varphi / \overline{P}_{\nu,N} = \varphi_{\nu} : \overline{P}_{\nu,N} \longrightarrow N \times \overline{P}_{\nu,f}$.

The homeomorphism φ was not constructed using the linear structure of $\mathbb{R}^n$, but rather that of $\mathbb{P}^n$, so it must be modified to prove the theorem.

Let C be an open convex cell in $\{g\} \times \overline{P}_{\nu,f} - \{g\} \times \bigcup_{\nu' \supset \nu} \overline{P}_{\nu',f}$. Then $\Theta_{\nu} \mid \overline{C}$ is the join of $\Theta_{\nu} \mid \dot{C}$ and a map of a point to a point. If $\overline{C} \subset \{g\} \times \mathbb{R}^n$ and $\varphi_{\nu^{-1}}(\overline{C}) \subset \{g\} \times \mathbb{R}^n$ and $\varphi_{\nu^{-1}} \mid \dot{C}$ is PL, we may replace $\varphi_{\nu-1}|\overline{C}$ by the PL map which is the join of $\varphi_{\nu}^{-1}|\mathring{C}$ and the map $\varphi_{\nu}^{-1}|\{v\} = \varphi_{\nu}^{-1} \circ \theta_{\nu}|\{v\}$, using the *linear* structure of $\mathbb{R}^n$ to define the join.

Let Q_g denote the convex linear cell in $\{g\} \times \mathbb{R}^n$ spanned by the points $g(v_1), \ldots, g(v_s)$. By the construction of φ, it is clear that $\varphi^{-1}(\{g\} \times Q_f) = Q_g$. Using the preceding argument inductively on the skeleta of Q_f we see that the map $\varphi^{-1}/N \times Q_f : N \times Q_f \longrightarrow \bigcup_{g \in N} Q_g$ can be taken to be a *Fiberwise PL homeomorphism*. (Note: The construction of φ really was a skeletonwise construction of the cell complex $\{g\} \times \mathbb{R}^n$, using the cellular subdivision displayed by the planes $\overline{P}_{\nu, g}$.)

Let $\nu: N \to \bigcup Q_g$ be a *Fiberwise embedding* such that $\nu(g) \in \mathring{Q}_g$. There is a homeomorphism $\varphi'^{-1}: \{g\} \times \mathbb{R}^n \to \{g\} \times \mathbb{R}^n$ defined as the join of the map $\varphi^{-1}|\{g\} \times \mathring{Q}_f : \{g\} \times \mathring{Q}_f \to \mathring{Q}_g$ and the map $\{g\} \times \{\nu(f)\} \longrightarrow \{\nu(g)\}$. This is well defined, since Q_g is convex and is hence starlike from any point in its interior. This gives two *Fiberwise homeomorphisms* which are PL:

$$\varphi^{-1}/N \times Q_f : N \times Q_f \longrightarrow \bigcup Q_g \text{, and}$$
$$\varphi'^{-1}/N \times \overline{\mathbb{R}^n - Q_f} : N \times \overline{\mathbb{R}^n - Q_f} \longrightarrow \bigcup (\overline{\mathbb{R}^n - Q_g})$$

which agree on $N \times \dot{Q}_f$. Combining them, we get a PL homeomorphism *(fiberwise)* $\lambda^{-1} : N \times \mathbb{R}^n \longrightarrow N \times \mathbb{R}^n$.

λ^{-1}, like φ^{-1}, has the property that $\lambda^{-1}(N \times f(k)) = F(N \times K)$. We complete the theorem by defining $\lambda' | N \times K$. Let τ be a simplex of k. Define $\lambda': \{g\} \times \tau = (1 \times F|\tau^{-1}) \circ \lambda \circ F$ Then the commutative diagram of THEOREM 5.1 is completed.

The *PROOF* of THEOREM 5.1 shows that the definition of general position is really a projective space definition. For what has been actually proved is a structural stability theorem for points in projective space:

THEOREM 5.9: Let $(x_1, \ldots, x_r) \in \mathcal{B}_r \subset (\mathbb{P}^n)^r$. Then there exists neighborhood N of $(x_1, \ldots, x_r)$ in $\mathbb{P}^{nr}$ and a *Fiberwise homeomorphism* $\lambda: N \times \mathbb{P}^n \to N \times \mathbb{P}^n$ such that

1. $\lambda(n, x_i) = (n, n_i)$, where $n = (n_1, \ldots, n_r)$; and
2. λ takes the plane spanned by $x_{i_1}, \ldots, x_{i_s}$ to the plane spanned by $n_{i_1}, \ldots, n_{i_s}$.

VI THE C^1 TOPOLOGY

In this section $\mathbb{R}^n$ will be viewed as a normed linear space, K a finite simplicial complex embedded in $\mathbb{R}^s$, s = the number of vertices of K, as in section V, and inheriting the metric from $\mathbb{R}^s$. Thus K is endowed with a metric which is "linear" on each simplex of K. We will assume $\dim K = k < n$.

Let σ be an r - simplex, and let $f : \sigma \to \mathbb{R}^n$ be a linear map. For any $b \in \sigma$, define $df_b : \sigma \to \mathbb{R}^n$ by $df_b(x) = f(x) - f(b)$. Similarly, if $f: K \to \mathbb{R}^n$ is a linear map, then for $b \in \mathring{\sigma} \in K$, we may define $df_b : St(\sigma, k) \to \mathbb{R}^n$ by $df_b(x) = f(x) - f(b)$, where $St(\sigma, k) = \cup \{\tau \in K \mid \sigma \text{ is a face of the (closed) simplex}\}$.

DEFINITION: Let $f, g : J \to \mathbb{R}^n$ be PL maps, where J is a subcomplex of some Euclidean space $\mathbb{R}^s$. For $\delta > 0$, say that g is a (uniform) δ - approximation[1] to f if

1. $\|f(b) - g(b)\| \leq \delta \;\; \forall b \in J$.
2. If J^1 is a subdivision of J such that f and g are linear with respect to J^1, then $\|df_b(x) - dg_b(x)\| \leq \delta\|x - b\|$ for all $b \in J$ and all x in $St(b, J^1) = St(\sigma, J^1)$ for $b \in \mathring{\sigma}$.

Note that $\| \; \|$ represents the appropriate Euclidean space norm. The definition is easily seen to be independent of the choice of subdivision J^1. The definition of δ- approximation depends on the "PL metric" on J in the following sense: If $| \; |_1$ and $| \; |_2$ are two linear metrics on a t - simplex σ, that is, metrics induced by linear embeddings of σ into $\mathbb{R}^t$, then there exists $\alpha, \beta > 0$ such that

[1] Munkres, p. 83.

$|x - b|_1 \leq \alpha |x - b|_2$ and $|x - b|_2 \leq \beta |x - b|_1$ for all $x, b \in \sigma$. Hence, if $g : \sigma \to \mathbb{R}^n$ is a δ - approximation to a PL map $F : \sigma \to \mathbb{R}^n$ with respect to $|\ |_1$, then it is a max $(\delta, \delta\alpha)$ - approximation with respect to $|\ |_2$. Thus for any finite complex K, the embedding of K into large - dimensional Euclidean space is irrelevant. Similarly, it is irrelevant which linear norm $\|\ \|$ is placed on $\mathbb{R}^n$.

DEFINITION: <u>PL $(K, \mathbb{R}^n)$</u> is the space of piecewise - linear maps of K into $\mathbb{R}^n$ with the topology induced by taking for a neighborhood basis at f $N(f, \epsilon) = \{ g \in PL(K, \mathbb{R}^n) \mid g$ is an ϵ - approximation to $f \}$, $\epsilon > 0$.

Given $f, g \in PL(K, \mathbb{R}^n)$, $\alpha, \beta \in \mathbb{R}$, define $\alpha f + \beta g : K \to \mathbb{R}^n$ by $(\alpha f + \beta g)(x) = \alpha \cdot f(x) + \beta \cdot g(x)$. If f and g are linear with respect to K^1, a subdivision of K, then $\alpha f + \beta g$ is linear w.r.t. K^1. Say $\|f\| = \delta$ f $\|f(x)\| \leq \delta \ \forall x \in K$ and $\|f(x) - f(b)\| \leq \delta \|x - b\| \ \forall b \in K$ and $x \in St(b, K^1)$, where f is linear with respect to K^1, but the inequality is violated for any $\delta' < \delta$.

With the above definitions, one may verify that PL $(K, \mathbb{R}^n)$ is a real normed linear space. Observe that f and g are δ- approximations to each other if f $\|f - g\| \leq \delta$.

The remainder of this section is concerned with some of the properties of δ- approximations of PL maps.

PROPOSITION 6.1: Let $f : \mathbb{R}^k \to \mathbb{R}^n$ be a PL map, $k \leq n$. If g is a δ - approximation to f, then $\|(f(x) - f(b)) - (g(x) - g(b))\| \leq \delta \|x - b\|$ for all $b, x \in \mathbb{R}^k$.

PROOF: Let J be a triangulation of $\mathbb{R}^k$ such that both f and g are linear with respect to J. Let $[b, x]$ be the line segment in

$\mathbb{R}^k$ joining b to x. Then $[b, x]$ is broken up into line segments $[b, b_1]$, $[b_1 b_2]$, . . ., $[b_{n-1}, x]$, each of which lies in a single simplex. Then we have

$$\|(f(x) - f(b)) - (g(x) - g(b))\|$$
$$= \|(f(x) - f(b_{n-1}) + f(b_{n-1}) - f(b_{n-2}) +, \ldots f(b_1) - f(b))$$
$$- (g(x) - g(b_{n-1}) + \ldots + g(b_1) - g(b))\|$$
$$\leq \|dfb_{n-1}(x) - dgb_{n-1}(x)\| + \ldots + \|df_b(b_1) - dgb(b_1)\|$$
$$\leq \delta(\|x - b_{n-1}\| + \|b_{n-1} - b_{n-2}\| + \ldots + \|b_1 - b\|$$
$$= \delta\|x - b\| . \qquad \|$$

THEOREM 6.2: Let σ be an r - simplex, $[\sigma]$ the r - plane spanned by σ in some Euclidean space. Let $f : \sigma \to \mathbb{R}^n$ be a non - degenerate linear map, $r < n$, and let $\bar{f}$ be the linear extension of f to $[\sigma]$.

Then for every $\epsilon > 0$ there exists $\delta > 0$ such that if $g : \sigma \to \mathbb{R}^n$ is a PL δ- approximation to f, then g extends to a PL ϵ- approximation $\bar{g} : [\sigma] \to \mathbb{R}^n$ to $\bar{f}$.

LEMMA : Let m, n be vectors in Euclidean space, and let $\pi - \varphi_1$ be the angle between m and n. Then $|m + n| \geq \sin\left(\frac{\varphi_1}{2}\right)(|m| + |n|)$. If $|m| = |n|$, this becomes an equality.

PROOF: We have $|m + n|^2 = |m|^2 + |n|^2 - 2|m||n|\cos\varphi_1$.
$= (|m| + |n|)^2 - 2|m||n|(1 + \cos\varphi_1)$

Then $|m + n|^2 / \sin^2(\varphi_1/2) - (|m| + |n|)^2$

$$= (1/\sin^2(\varphi_1/2) - 1)(|m| + |n|)^2 - 2\frac{(1 + \cos\varphi_1)}{\sin^2\varphi_1/2} . |m||n|$$

if $0 < \varphi_1 \leq \pi$.

Thus $(|m + n|^2 / \sin^2(\varphi_1/2) - (|m| + |n|)^2$

$$= \left(\frac{2}{1 - \cos\varphi_1} - 1\right)(|m| + |n|)^2 - \frac{4(1 + \cos\varphi_1)}{(1 - \cos\varphi_1)}|m||n|$$

$$= \frac{(1 + \cos \varphi_1)}{1 - \cos \varphi_1} \left[(|m| + |n|)^2 - 4|m||n| \right.$$

$$= (1 + \cos \varphi_1 / 1 - \cos \varphi_1) \left[|m| - |n| \right]^2 \geq 0$$

Equality occurs if $|m| = |n|$.

We conclude that $|m + n|^2 \geq (|m| + |n|)^2 \sin^2 (\varphi_1 /2)$

or $|m + n| \geq (|m| + |n|) \sin (\varphi_1 /2)$.

If $\varphi_1 = 0$, the lemma is of course trivial. ||

COROLLARY 6.3: If $\measuredangle(m , n) = \pi - \varphi_1$ and $\measuredangle(m + n, p) = \pi - \varphi_2$, then $|m + n + p| \geq \sin (\varphi_1 /2) \sin (\varphi_2/2) |m + n|$.

PROOF: $|m + n + p| \geq \sin (\varphi_2/2) (|m + n| + |p|)$

$\geq \sin (\varphi_2 /2) (|m + n|) \geq \sin \varphi_1/2 \sin \varphi_2/2 (|m| + |n|)$.

PROOF of THEOREM 6.2: Triangulate $[\sigma]$ by J such that

1. σ is a simplex of K.
2. If α is a face of σ, then $[\alpha]$ is a subcomplex of J.
3. If $\alpha\tau$ is a simplex of J with α a face of σ and τ disjoint from σ, then $\|a - t\| \geq 1$ for $\forall\, a \in \alpha$, $t \in \tau$.

For $a_1, a_2 \in \alpha$ and $t \in \tau$, let $\theta_1(a_1, a_2, t)$ be the smaller angle between the line $[a_1, a_2]$ and the line $[a_1, t]$. This angle is positive. If a_1 and t are kept fixed and a_2 varies, then θ_1 varies continuously with a_2. Since the values of θ_1 are all determined by the values on a small sphere about a_1 in α, $\min_{a_2 \neq a_1} \theta_1 > 0$ by compactness. By two more applications of compactness, we see that θ_1 achieves a minimum value for $a_1, a_2 \in \alpha$ and $t \in \tau$. Therefore, define

$\theta_1(\alpha\tau) = \min \theta_1(a_1, a_2, t) > 0$.

Similarly, let $\theta_2(\alpha\tau)$ be the minimum over all $a_1, a_2 \in \alpha$, $t_1, t_2 \in \tau$, of the angle between the plane $[a_1, a_2, t_2]$ and the line

$[t_1, t_2]$. $\Theta_2(\alpha\tau) > 0$. Let Θ_i = minimum over all choices of α, τ of $\Theta_i(\alpha\tau)$. Then $\Theta_i > 0$.

Let $\delta = \epsilon \sin \Theta_1/2 \sin \Theta_2/2$.

Let $g: \sigma \to \mathbb{R}^n$ be a PL δ- approximation to f, linear with respect to the triangulation L^1 of σ. Let J^1 be a subdivision of J which extends L^1 and adds no vertices to $J - \sigma$. Extend g to a linear map $\bar{g}: J^1 \to \mathbb{R}^n$ by

$$\bar{g}(v) = \begin{cases} g(v) & \text{if v is a vertex of } L^1 \\ \bar{f}(v) & \text{if v is a vertex of } K^1 - L^1. \end{cases}$$

We will show that $\bar{g}$ is an ϵ- approximation to $\bar{f}$.

Let $\alpha'\tau$ be a simplex of J^1, $\alpha' \subset L^1$, $\tau \subset J^1 - L^1$. Let $b, x \in \alpha'\tau$. We may write $\underline{b = \beta a_1^1 + (1-\beta) t_1}$ and $x = \chi a_2^1 + (1-\chi) t_2$, for $a_1^1, a_2^1 \in \alpha'$, $t_1, t_2 \in \tau$, and $0 \le \chi, \beta \le 1$. Let $\underline{b^1 = \beta a_1^1 + (1-\beta) t_2}$, $\underline{x^1 = \chi a_1^1 + (1-\chi) t_2}$.

Since $\bar{f}$ and $\bar{g}$ are linear on , we have for instance

$\bar{F}(b) = \bar{f}(a_1^1) + (1-\beta)\bar{f}(t_1)$.

Now $\|\bar{g}(b) - \bar{f}(b)\| = \|\beta g(a_1^1) + (1-\beta)\bar{f}(t_1) - \beta f(a_1^1) - (1-\beta)\bar{f}(t_1)\| = \beta\|g(a_1^1) - f(a_1^1)\| \le \beta\delta \le \delta \le \epsilon$.

Next, $d\bar{g}_b(x) = \bar{g}(x) - \bar{g}(b) =$

$$[\bar{g}(b^1) - \bar{g}(b)] + [\bar{g}(x^1) - \bar{g}(b^1)] + [\bar{g}(x) - \bar{g}(x^1)]$$
$$= (1-\beta)[\bar{f}(t_2) - \bar{f}(t_1)] + (\chi-\beta)[g(a_1^1) - \bar{f}(t_2)] + \chi[g(a_2^1) - g(a_1^1)].$$

Similarly, $d\bar{f}_b(x) = (1-\beta)[\bar{f}(t_2) - \bar{f}(t_1)] + (\chi-\beta)[f(a_1^1) - \bar{f}(t_2)] + \chi[f(a_2^1) - f(a_1^1)]$.

Hence $d\bar{g}_b(x) - d\bar{f}_b(x) = (\chi-\beta)[g(a_1^1) - f(a_1^1)] + \chi[dg_{a_1^1}(a_2^1) - df_{a_1^1}(a_2^1)]$.

So we have $\|d\,\bar{g}_b(x) - d\bar{f}_b(x)\| \le |\chi - \beta|\delta + \chi\|a_2^1 - a_1^1\|\delta$.

On the other hand, $x - b = (b^1 - b) + (x^1 - b^1) + (x - x^1)$

$$= (1-\beta)(t_2 - t_1) + (\chi - \beta)(a_1^1 - t_2) + \chi(a_2^1 - a_1^1).$$

By COROLLARY 6.3,

$$\begin{aligned}\|x - b\| &= \|[(1-\beta)(t_2 - t_1)] + [(\chi-\beta)(a_1^1 - t_2)] \\ &\quad + [\chi(a_2^1 - a_1^1)]\| \\ &\ge (\|(\chi-\beta)(a_1^1 - t_2)\| + \|\chi(a_2^1 - a_1^1)\|)\sin\varphi_1/2\,\sin\varphi_2/2 \\ &\ge (|\chi-\beta| + \chi\|a_2^1 - a_1^1\|)\sin\varphi_1/2\,\sin\varphi_2/2.\end{aligned}$$

Here $\pi - \varphi_1$ is the angle between the vectors $(a_2^1 - a_1^1)$ and $(a_1^1 - t_2)$ while $\pi - \varphi_2$ is the angle between the vectors $(t_2 - t_1)$ and $(\chi-\beta)(a_1^1 - t_2) + \chi(a_2^1 - a_1^1)$. It follows that $\varphi_1 \ge \theta_1$ and $\varphi_2 \ge \theta_2$.

We conclude that

$$\begin{aligned}\|d\,\bar{g}_b(x) - d\bar{f}_b(x)\| &\le (|\chi-\beta| + \chi\|a_2^1 - a_1^1\|)\delta \\ &= (|\chi-\beta| + \chi\|a_2^1 - a_1^1\|)\epsilon\sin\theta_1/2\,\sin\theta_2/2 \\ &\le (|\chi-\beta| + \chi\|a_2^1 - a_1^1\|)\epsilon\sin\varphi_1/2\,\sin\varphi_2/2) \\ &\le \epsilon\|x - b\|.\end{aligned}$$

Thus g is an ϵ- approximation to f.

THEOREM 6.4: Let $i: \mathbb{R}^k \hookrightarrow \mathbb{R}^n$, $k \le n$ and let $g: \mathbb{R}^k \to \mathbb{R}^n$ be a δ- approximation to i. If $\delta < 1$, then g is a closed embedding. Furthermore, if $\pi: \mathbb{R}^n \to \mathbb{R}^k$ is the orthogonal projection map, then $\pi|g\mathbb{R}^k : g\mathbb{R}^k \to \mathbb{R}^k$ is a PL homeomorphism.

PROOF: By PROPOSITION 6.1, $\|g(x) - g(b) - x + b\| \le \delta\|x - b\|$ For any $b, x \in \mathbb{R}^k$. Then if $g(x) = g(b)$, $\|x - b\| \le \delta\|x - b\|$ $\Rightarrow x = b$. Hence g is $1-1$. Since $\|g(x) - x\| < 1\ \forall x \in \mathbb{R}^k$, g is a closed embedding.

By the sine law, $\dfrac{\|(g(x) - g(b)) - (x - b)\|}{\|x - b\|}$

$$= \frac{\sin \measuredangle (g(x) - g(b), x - b)}{\sin \measuredangle (g(x) - g(b) - x + b, g(x) - g(b))}$$

Therefore, $\sin \measuredangle (g(x) - g(b), x - b) \leq \delta$

and $\measuredangle (g(x) - g(b), x - b) \leq \sin^{-1} \delta < \pi/2$.

Hence $\pi | g \mathbb{R}^k$ is 1 - 1, and since $\|\pi g(x) - x\| \leq \delta \ \forall x \in \mathbb{R}^k$, πg is onto, so $\pi | g \mathbb{R}^k$ is onto. Thus we have shown $\pi | g \mathbb{R}^k$ is a PL homeomorphism.

COROLLARY 6.5: If ϵ is sufficiently small, then in THEOREM 6.2 $\overline{g}: [\sigma] \to \mathbb{R}^n$ is a closed embedding.

PROOF: There exists M such that $\|x - b\| \leq M \|\overline{F}(x) - \overline{f}(b)\| \ \forall b, x \in [\sigma]$, since $\overline{f}$ is linear and non singular. Then if $\epsilon < 1/M$,
$\|\overline{g}(x) - \overline{g}(b) - \overline{f}(x) + \overline{f}(b)\| \leq \epsilon \|x - b\| \leq M \|\overline{f}(x) - \overline{f}(b)\| =$
$\delta \|\overline{f}(x) - \overline{f}(b)\|$ for $\delta = \epsilon M < 1$.

THEOREM 6.6: Let $f, g: \mathbb{R}^n \to \mathbb{R}^n$ be PL maps such that f is a δ_1 - approximation to $1_{\mathbb{R}^n}$ and g is a δ_2 - approximation to $1_{\mathbb{R}^n}$. Then

1. gf is a $(\delta_1 + \delta_2 + \delta_1 \delta_2)$ - approximation to $1_{\mathbb{R}^n}$,
2. If $\delta_1 < 1$, then $f^{-1}: \mathbb{R}^n \to \mathbb{R}^n$ is a $\delta_1/1 - \delta_1$ - approximation to $1_{\mathbb{R}^n}$.
3. If h is a δ_3 - approximation to F, then h is a $\delta_1 + \delta_3$ - approximation to $1_{\mathbb{R}^n}$.

PROOF: 1. For $x \in \mathbb{R}^n$, $\|g f(x) - x\| \leq \|g f(x) - f(x)\| + \|f(x) - x\| \leq \delta_2 + \delta_1 \leq \delta_1 + \delta_2 + \delta_1 \delta_2$. For $b \in \mathbb{R}^n$, x close to b, $\|(gf(x) - gf(b)) - (x - b)\|$
$\leq \|(gf(x) - gf(b)) - (f(x) - f(b))\| + \|(f(x) - f(b)) - (x - b)\| \leq \delta_2 \|f(x) - f(b)\| + \delta_1 \|x - b\|$
$\leq \delta_2 \|f(x) - f(b) - x + b\| + \delta_2 \|x - b\| + \delta_1 \|x-b\|$
$\leq (\delta_1 \delta_2 + \delta_2 + \delta_1) \|x - b\|$.

2. $\|f^{-1}(x) - x\| = \|f^{-1}(x) - ff^{-1}(x)\| \leq \delta_1 \leq \delta_1 / 1 - \delta_1$.

(Note that f^{-1} is a homeomorphism by 6.4)

$$\|f(x) - f(b)\| \geq \|x - b\| - \|f(x) - f(b) - x + b\|$$

$$\geq (1 - \delta) \|x - b\|$$

So $\|x^1 - b^1\| \geq (1 - \delta_1) \| f^{-1}x^{-1} - f^{-1}b^1 \|$

$\| df_b{}^{-1}(x^1) - d1_{b1}(x') \| =$

$\|(f^{-1}(x^1) - f^{-1}(b^1)) - (ff^{-1}(x^1) - ff^{-1}(b^1))\|$

$\leq \delta_1 \|f^{-1} x^1 - f^{-1} b^1\|$.

$\leq \delta_1 / (1 - \delta_1) \| x^1 - b^1 \|$.

3. $\|hx - x\| \leq \|hx - fx\| + \|fx - x\| \leq \delta_3 + \delta_1$.

$\|(hx - hb) - (x - b)\| \leq \|(h(x) - h(b)) - (f(x) - f(b))\|$

$+ \|(f(x) - f(b)) - (x - b)\|$

$\leq \delta_3\|x - b\| + \delta_1\|x - b\|$

$= (\delta_1 + \delta_3) \|x - b\|$.

Q.E.D.

VII STRUCTURAL STABILITY FOR PL $(K, \mathbb{R}^n)$

Since this section is concerned with *Fiberwise maps*, it will be convenient to review some facts about $\mathcal{C}(X,Y)$, ths space of continuous functions from a space X to a space Y with the compact - open topology.

If X is locally compact and Hausdorff, and if A is any space, then a function $\varphi: A \to \mathcal{C}(X, Y)$ is continuous iff the corresponding map $\Phi: A \times X \to Y$ defined by $\Phi(a, x) = \varphi(a)(x)$ is continuous. If also X is locally connected, then $\mathcal{H}(X) \subset \mathcal{C}(X,X)$, the group of homeomorphisms of X, is a topological group.

If $f; J \to \mathbb{R}^n$ is a PL map and $\delta > 0$, then $N(f,\delta)$ is the set of all PL δ- approximations to f. Denote by $N^*(f,\delta)$ the set $N(f,\delta)$ endowed with the compact-open topology. $N(f,\delta)$ will have the C^1 topology. The "identity" map $N(f,\delta) \to N^*(f,\delta)$ is automatically continuous.

THEOREM 7.1: Let P, Q be planes in $\mathbb{R}^n$ such that $\dim(P \cap Q) = \dim P + \dim Q - n \geq 0$. Let $i: P \hookrightarrow \mathbb{R}^n$, $j: Q \hookrightarrow \mathbb{R}^n$, $k: P \cap Q \hookrightarrow \mathbb{R}^n$ be the inclusion maps. Let $\epsilon > 0$ be given. Then there exists $\delta > 0$ anda continuous map $\Theta: N^*(i,\delta) \times N^*(j,\delta) \to N^*(k,\epsilon)$ such that $\Theta(f, g)(P \cap Q) = fP \cap gQ$.

PROOF: Let $\delta < 1$ and let $f \in N(i,\delta)$. By THEOREM 6.4, the orthogonal projection $\pi: \mathbb{R}^n \to P$ gives a homeomorphism $\pi|fP: fP \to P$. Let $P(f) = (\pi|fP)^{-1}: P \to fP$.

Define a map G: $N^*(i,\delta)$ $(\mathbb{R}^n)$ as follows:
for $f \in N^*(i,\delta)$, define $G(f): \mathbb{R}^n \to \mathbb{R}^n$ by $G(f)(x) = x - \pi(x) + f\pi(x)$.

$G(f)$ is PL and $G(f)\,|\,P = f$.

For $x \in \mathbb{R}^n$, $\|G(f)(x) - x\| = \|f\pi(x) - \pi(x)\| \leq \delta$. For any $b, x \in \mathbb{R}^n$, $\|dG(f)_b(x) - d1_b(x)\|$

$= \|(x - \pi(x) + f\pi(x)) - (b - \pi(b) + f\pi(b)) - x + b\|$

$= f\pi(x) - f\pi(b) - \pi(x) + \pi(b)\| \leq \delta\,\|\pi x \quad \pi b\| \leq \delta\,\|x - b\|$.

Thus $G(f) \in N^*(1_{\mathbb{R}^n}, \delta)$, and since $\delta < 1$ $G(f)$ must be a homeomorphism.

Suppose $G(f) \in N^*(1, \delta)$ and let $\beta > 0$ and let C be a compact subset of $\mathbb{R}^n$.

Let $f^1 \in N^*(i, \delta)$ such that $\|f(x) - f^1(x)\| < \beta \quad \forall x \in \pi(C) \subset P$. Then for any $y \in C$, $\|G(f)(y) - G(f^1)(y)\|$

$= \|y - \pi y + f\pi y - y + \pi y - f^1 \pi y\|$

$= \|f^1 \pi y - f\pi y\| < \beta$. This proves G is continuous. Let $H(f) = G(f)^{-1}$. Then THEOREM 6.6 implies that $H(f) \quad N^*(1_{\mathbb{R}^n}, \delta/1 - \delta)$. Therefore, if $\delta/1 - \delta < 1$, there is a continuous map $H: N^*(i, \delta) \longrightarrow N^*(1_{\mathbb{R}^n}, \delta/1 - \delta)$ such that for any $f \quad N^*(i, \delta)$, $H(f)\,|\,fP = f^{-1} : fP \longrightarrow P$.

If $g \in N(j, \delta)$, then by THEOREM 6.6, for any $f \in N(i, \delta)$, $H(f) \circ g : Q \longrightarrow \mathbb{R}^n$ is a $(\delta + \delta/1 - \delta + \delta^2/1 - \delta) = 2\delta/1 - \delta$-approximation to the inclusion $j: Q \hookrightarrow \mathbb{R}^n$. Since composition of maps is continuous in the compact open topology, the map

$$\begin{tikzcd} N^*(i,\delta) \times N^*(j,\delta) \arrow[r, "H \times 1"] \arrow[dr, "\chi"'] & N^*(1_{\mathbb{R}^n}, \delta/1-\delta) \times N^*(j,\delta) \arrow[d] \\ & N^*(j, 2\delta/1-\delta) \end{tikzcd}$$

given by $X(f, g) = H(f) \circ g$ is continuous.

Suppose we can find a continuous function $\psi: N^*(j, 2\delta/1 - \delta) \longrightarrow N^*(k, \epsilon')$, for ϵ' arbitrary and δ sufficiently small, such that if $g: Q \to \mathbb{R}^n$ is a $2\delta/1 - \delta$- approximation to $j: Q \hookrightarrow \mathbb{R}^n$

then $\Psi(g)$ is a homeomorphism of $P\cap Q$ onto $P\cap gQ$. Then we get a map

$$\Psi\circ X : N^*(i,\delta)\times N^*(j,\delta)\to N^*(k,\epsilon')$$

such that $[\Psi\circ X(f,g)]\,(P\cap Q) = P\cap H(f)\circ g\,Q$.

The map $\Phi : N^*(i,\delta)\times N^*(j,\delta)\to N^*(1_{\mathbb{R}^n},\delta)$ defined by $\Phi(f,g)$ is continuous. Therefore, we may define θ by the diagram

$$\begin{tikzcd} N^*(i,\delta)\times N^*(j,\delta) \arrow[r, "(\Phi,\Psi\circ X)"] \arrow[dr, "\theta"'] & N^*(1,\delta)\times N^*(k,\epsilon') \arrow[d, "\circ"] \\ & N^*(k,\epsilon'+\delta+\epsilon'\delta) \end{tikzcd}$$

That is, $(f,g) = [G(f)]\circ[\Psi(H(f)\circ g)]$. We have

$$[G(f)]\circ[\Psi(H(f)\circ g)]\,(P\cap Q) = [G(f)]\,(P\cap H(f)\circ g\,Q)$$

$$= G(f)(P)\cap G(f)\,H(f)\,gQ = P\cap gQ.$$

Therefore, Let $\epsilon' = \min(1,\epsilon/3)$. Choose $\delta\le\min(1,\epsilon/3)$ such that Ψ exists (this is yet to be proved). Then $\epsilon'+\delta+\epsilon'\delta\le\epsilon$ and θ satisfies the theorem.

Thus we are reduced to the following situation: Given $g:Q\to\mathbb{R}^n$ a δ'- approximation to $j: Q\hookrightarrow\mathbb{R}^n$, δ' sufficiently small, we must find an embedding $\Psi(g) : P\cap Q\hookrightarrow\mathbb{R}^n$ which is an ϵ'-approximation to $k: P\cap Q\hookrightarrow\mathbb{R}^n$ such that $\Psi(g)\,(P\cap Q) = P\cap gQ$.

CLAIM: We may assume w.l.o.g. that Q is orthagonal to P.

PROOF: We may certainly assume w.l.o.g. that $0\in Q\cap P$, so that Q and P are actually linear subspaces of $\mathbb{R}^n$. Let Q^* be the plane of the same dimension as Q passing through $P\cap Q$ and orthogonal to P. Let $L: \mathbb{R}^n\to\mathbb{R}^n$ be a linear automorphism of $\mathbb{R}^n$ which keeps P pointwise fixed and maps Q^* to Q.

Let $m = \|L\|$ be the norm of L, and likewise let $n = \|L^{-1}\|$.

Suppose $g \in N(j, \delta')$. Then letting $\delta'' = \max(n\delta, mn\delta')$, $L^{-1} g (L | Q^*) : Q^* \longrightarrow \mathbb{R}^n$ is a δ''-approximation to $j^* : Q^* \hookrightarrow \mathbb{R}^n$.

For given $x \in Q^*$,

$$\|L^{-1} g L(x) - x\| = \|L^{-1}(gL(x) - L(x))\|$$
$$\leq n \|gL(x) - L(x)\| \leq n\delta'.$$

If $b, x \in Q^*$, then $\|L^{-1} gL(x) - L^{-1} gL(b) - x + b\|$

$$= \|L^{-1}(gL(x) - gL(b) - L(X) + L(b))\|$$
$$\leq n \| gL(x) - gL(b) - L(x) + L(b)\|$$
$$\leq n\delta' \|L(x) - L(b)\| \leq mn\delta' \|x - b\|.$$

Secondly, $L^{-1} gL(Q^*) \cap P = L^{-1} gQ \cap P$

$$= L^{-1} gQ \cap L^{-1} P = L^{-1}(gQ \cap P) = gQ \cap P.$$

Finally, if $g, g^1 \in N^*(j, \delta')$ and $\|g(x) - g^1(x)\| < \beta \ \forall x \in C$, then $\|L^{-1} gL(y) - L^{-1} g^1L(y)\| \leq n\|gL(y) - g^1L(y)\| \leq n\beta$ $\forall y \ L^{-1}C$. Thus the map $\alpha : N^*(j, \delta') \longrightarrow N^*(j^*, \delta'')$ given by $g \longrightarrow L^{-1} gL$ is continuous. If we can find $\Psi : N^*(j^*, \delta'') \longrightarrow N^*(k, \epsilon')$, then $\Psi \circ \alpha$ is the desired map.

Q.E.D. CLAIM

Now we prove the theorem assuming Q is orthogonal to P. We are given $\epsilon' > 0$ and must construct $\Psi : N^*(j, \delta') \longrightarrow N^*(k, \epsilon')$ for δ' sufficiently small, such that $\Psi(g) (P \cap Q) = P \cap gQ$.

Let $\pi : \mathbb{R}^n \longrightarrow Q$ be orthogonal projection. If $\delta' < 1$ and $g \in N(j, \delta')$, then $\pi | gQ : gQ \longrightarrow Q$ is a Pl homeomorphism. Let $\rho(g) = (\pi | gQ)^{-1}$. Define $\Psi(g) = \rho(g) \mid P \cap Q$. Since $\pi(P) = P \cap Q$, $\pi | gQ \cap P$ maps $gQ \cap P$ homeomorphically to $Q \cap P$. (Here is where we need the condition dim $(P \cap Q)$ = dim P + dim Q − n, in order that $\pi(x) \in P \cap Q$, if *and only if* $x \in P$). Therefore,

$\Psi(g) : P \cap Q \xrightarrow{\cong} P \cap gQ$.

We will show that $\Psi(g) \in N^*(k, \delta'/\sqrt{1-\delta'^2})$ for $g \in N^*(j, \delta')$. As in THEOREM 6.4, if $x, b \in Q$ then $\angle(g(x) - g(b), x - b) \leq \sin^{-1}\delta'$. for any $x \in Q$, let x^1 be the point such that $\pi g(x^1) = x$. Then $g(x^1) = \rho(x)$. Since the vector $p(x) - x$ is perpendicular to the plane Q, $\|\rho(x) - x\| \leq \|\rho(x) - x^1\| = \|g(x^1) - x^1\| \leq \delta'$.

Since the vector $x - b$ is the orthogonal projection of $\rho(x) - \rho(b)$ onto the plane Q, $\angle(\rho(x) - \rho(b), x - b) \leq \angle(\rho(x) - \rho(b), v)$ for any vector v in Q. Let $b^1 = g^{-1}\rho(g)\, b$ and $x^1 = g^{-1}\rho(g)\, x$. Then

$$\angle(\rho(x) - \rho(b), x - b) \leq (\rho(x) - \rho(b), x^1 - b^1)$$
$$= (g(x^1) - g(b^1), x^1 - b^1) \leq \sin^{-1}\delta'.$$

Therefore, since the vector $(\rho(x) - \rho(b)) - (x - b)$ is perpendicular to the vector $x - b$, we have

$$\|(\rho(x) - \rho(b)) - (x - b)\| = \|x - b\| \tan \angle(\rho(x) - \rho(b), x - b)$$
$$\leq \|x - b\| \tan(\sin^{-1}\delta')$$
$$= \delta'/\sqrt{1-\delta'^2}\, \|x - b\|.$$

This shows that $P = \rho(g)$ is a $\delta'/\sqrt{1-\delta'^2}$ - approximation to $j : Q \hookrightarrow \mathbb{R}^n$. Therefore, by restriction choosing $\delta'/\sqrt{1-\delta'^2} \leq \epsilon'$, we have $\Psi(g) \in N^*(k, \epsilon')$.

Finally, we must show $\Psi : N^*(j, \delta')$ $N^*(k, \epsilon')$ is continuous. Let g $N^*(j, \delta')$ and $h = \Psi(g)$. Let $\alpha > 0$ and C compact $\subset P \cap Q$ be given. CLAIM: If $\|g^1(x) - g(x)\| \leq \alpha/2 + \epsilon'$ $\forall x$ $g^{-1}P(g)(C)$, $g^1 \in N^*(j, \delta)$, then $\|\Psi(g^1)(y) - h(y)\| \leq \alpha$ $\forall y$ C.

PROOF: Let $y \in C$. Since $h(y) \in g(Q)$, there exists $Z \in Q \ni g(z) = h(y)$. We have

$$\|\Psi(g^1)(y) - h(y)\| \leq \|\Psi(g^1)(y) - g^1(z)\| + \|g^1(z) - h(y)\|$$
$$= \|\Psi(g^1)(y) - g^1(z)\| + \|g^1(z) - g(z)\| .$$

By hypothesis, $z = g^{-1}h(y) = g^{-1}(\rho(g)\pi)h(y) = g^{-1}\rho(g)\cdot y \in g^{-1}\rho(g)C.$

Therefore, $\|g^1(z) - g(z)\| \leq \alpha/_{2+\epsilon'}$. Since $\pi: \mathbb{R}^n \longrightarrow Q$ is orthogonal projection, $\|\pi g^1(z) - \pi g(z)\| \leq \alpha/2 + \epsilon'$. Since $\pi g(z) = \pi h(y) = y$, we have $\|\pi g^1(z) - y\| \leq \alpha/2 + \epsilon'$. We have already shown that $\rho(g^1): Q \rightarrow g^1 Q$ is an ϵ'- approximation to $j: Q \hookrightarrow \mathbb{R}^n$. This implies that

$$\|\rho(g^1)[\pi g^1(z)] - \rho(g^1)[y] - \pi g^1(z) + y\| \leq \epsilon' \|\pi g^1(z) - y\|$$
$$\Rightarrow \|\rho(g^1)\pi g^1(z) - \rho(g^1)(y)\| \leq (1+\epsilon')\|\pi g^1(z) - y\| \leq (1+\epsilon')\alpha/(2+\epsilon')$$

Now we get that $\|\Psi(g^1)(y) - h(y)\|$

$$\leq \|\Psi(g^1)(y) - g^1(z)\| + \|g^1(z) - h(y)\|$$
$$= \|\rho(g^1)(y) - \rho(g^1)\pi g^1(z)\| + \|g^1(z) - g(z)\|$$
$$\leq (1+\epsilon')\alpha/(2+\epsilon') + \alpha/2 + \epsilon' = \alpha .$$

This proves Ψ is continuous.

COROLLARY 7.2: Let $P_1, \ldots, P_s$ be planes in $\mathbb{R}^n$ such that $\dim(P_1 \cap \ldots \cap P_s) = \dim P_1 + \ldots + \dim P_s - (s-1)n \geq 0$. Then, if $i_j: P_j \hookrightarrow \mathbb{R}^n$ are the inclusive maps and $k: P_1 \cap \ldots \cap P_s \hookrightarrow \mathbb{R}^n$ is the inclusion, and $\epsilon > 0$, there exists $\delta > 0$ such that if $f_j \in N(i_j, \delta)$, there is a map $\Theta(f_1, \ldots f_s): P_1 \cap \ldots \cap P_s \hookrightarrow \mathbb{R}^n$ such that

1. $\Theta(f_1, \ldots, f_s) \in N(k, \epsilon)$, and
2. $\Theta(f_1, \ldots, f_s)(P_1 \cap \ldots \cap P_s) = f_1 P_1 \ldots f_s P_s$. Furthermore, $\Theta: N^*(i, \delta) \times \ldots \times N^*(i_s, \delta) \quad N^*(k, \epsilon)$ is continuous.

PROOF: By induction and 7.1. Since $\dim((P_1 \cap \ldots \cap P_{s-1}) \cap P_s)$ $\geq \dim(P_1 \cap \ldots \cap P_{s-1}) + \dim P_s - n \geq \dim P_1 + \ldots + \dim P_{s-1} - (s-2)n$ $+ \dim P_s - n$

(since $P_1 \cap \ldots \cap P_s \neq \emptyset$), it follows that $\dim(P_1 \cap \ldots \cap P_{s-1})$ $= \dim P_1 + \ldots + \dim P_{s-1} - (s-2)n$. By induction, we may define $\Theta': N^*(i_1, \delta') \times \ldots \times N^*(i_{s-1}, \delta')^* \to N^*(k^*, \delta)$ for any δ, if δ' is small, where $k^*: P_1 \cap \ldots \cap P_{s-1} \hookrightarrow \mathbb{R}^n$. By THEOREM 7.1 if δ is small we may find

$\Theta'': N^*(k^*, \delta) \times N^*(i_s, \delta) \to N^*(k, \epsilon)$. This gives a map $N^*(i_1, \delta') \times \ldots \times N^*(i_s, \delta') \to N^*(k, \epsilon)$ defined by

$$\Theta(f_1, \ldots, f_s) = \Theta''(\Theta'(f_1, \ldots, f_{s-1}), f_s).$$

Furthermore, $\Theta(f_1, \ldots, f_s)\ (P_1 \cap \ldots \cap P_s)$

$$= [\Theta'(f_1, \ldots, f_{s-1})\ (P_1 \cap \ldots \cap P_{s-1})] \cap f_s P_s$$

$$= f_1 P_1 \ldots f_{s-1} P_{s-1} \cap f_s P_s.$$

DEFINITION : Let C be a compact set in $\mathbb{R}^n$, $J \subset \mathbb{R}^n$, $j: J \to \mathbb{R}^n$ the inclusion map. Then $N(j, \delta, C)$

$$= \{ g \in N(j, \delta) \mid cl \{ x \mid g(x) \neq j(x) \} \subset \text{int } C \}.$$

ADDENDUM 7.3: In COROLLARY 7.2, $\Theta(N^*(i_1, \delta, C) \times \ldots \times N^*(i_s, \delta, C))$ $\subset N^*(k, \epsilon, C)$.

PROOF: This follows directly from the construction of Ψ in 7.1.

LEMMA 7.4: Let $P_1, \ldots, P_r$ be $(n-1)$ - planes in $\mathbb{R}^n$, and $i: \bigcup_1^r P_i$ $= J \hookrightarrow \mathbb{R}^n$ the inclusion map. Let C be a convex linear n - cell in $\mathbb{R}^n$. Given $1 > \epsilon > 0$, there exists $\delta > 0$ and a map $H: N^*(i, \delta, C) \to N^*(1_{\mathbb{R}^n}, \epsilon, C)$ with $H(i) = 1_{\mathbb{R}^n}$ and $H(f) \mid J = f$.

PROOF: $\mathbb{R}^n - J$ is a disjoint union of open convex cells $D_1, \ldots, D_s$. Let $E_i = D_i \cap C$. Then $\mathring{E}_i$ is either empty or a convex linear n - cell (open). Assume $E_1, \ldots, E_{s^1}$ are nonempty. Let $x_i \in \mathring{E}_i$, $1 \leq i \leq s^1$. Then $\overline{E}_i$ is a closed convex linear cell and $\overline{E}_i = x_i * \dot{E}_i$, the join of x_i and the boundary of E_i. Let $q_i = \min\{\|x_i - b\| \mid b \in \dot{E}_i\}$. Then $q_i > 0$. Let θ_i = the minimum angle between line segments $[x_i, b]$ and $[b, c]$, where $b \in \dot{E}_i$ and $[b, c]$ is a line segment contained in $\overline{E}_i$. Then $\theta_i > 0$. Let $\delta_i = \sin(\theta_i/2) \min(\epsilon, \epsilon q_i)$. Let $\delta = \min \delta_i$.

If $\overline{D}_i \subset \mathring{C}$, then $\mathring{E}_i = \mathring{D}_i \subset J$. Define $H(f) \mid \overline{D}_i$ to be the join of the map $f: \dot{D}_i \to \mathbb{R}^n$ and the map $x_i \to x_i$. If $\overline{D}_i \cap C \neq \emptyset$, define $H(f) \mid \dot{E}_i \cap \dot{C}$ to be the identity (which we may do since $f \mid \dot{D}_i \cap \dot{C}$ is the identity map by hypothesis). Now define $H(f) \mid \overline{E}_i$ to be the join of $H(f): \dot{E}_i \to \mathbb{R}^n$ and the map $x_i \to x_i$. Define $H(f) \mid \mathbb{R}^n - C$ to be the identity.

Now $H(f)$ has been defined on $\mathbb{R}^n$. It is clear that $H(f) \mid J = f$ and that $H(f)$ is continuous. It is also clear that $H: N^*(i, \delta, C) \longrightarrow \mathcal{C}(\mathbb{R}^n, \mathbb{R}^n)$ is continuous. So we need only verify that $H(f) \in N(1_{\mathbb{R}^n}, \epsilon, C)$.

Let $a, b \in \overline{E}_i$. Then $a = \alpha x_i + (1 - \alpha) a^1$, and $b = \beta x_i + (1 - \beta) b^1$, where $a^1, b^1 \in \dot{E}_i$. Let $a^* = \beta x_i + (1 - \beta) a^1$.

$$\|H(f)(a) - a\| = \|\alpha x_i + (1 - \alpha) f(a^1) - \alpha x_i - (1 - \alpha) a^1\|$$
$$= (1 - \alpha) \|f(a^1) - a^1\| \leq (1 - \alpha)\delta \leq \delta_i \leq \epsilon.$$

$$H(f)(a) - H(f)(b) = H(f)(a) - H(f)(a^*) + H(f)(a^*) - H(f)(b^*)$$
$$= (\alpha x_i + (1 - \alpha) f(a^1)) - (\beta x_i + (1 - \beta) f(a^1))$$
$$+ (\beta x_i + (1 - \beta) f(a^1)) - (\beta x_i + (1 - \beta) f(b^1))$$
$$= (\alpha - \beta)(x_i - f(a^1)) + (1 - \beta)(f(a^1) - f(b^1)).$$

Thus $\|H(f)(a) - H(f)(b) - a + b\|$

$$= \| (\alpha-\beta)(x_i - f(a^1)) + (1-\beta)(f(a^1) - f(b^1)) - (\alpha-\beta)(x_i - a^1) - (1-\beta)(a^1 - b^1) \|$$

$$\leq |\alpha-\beta| \, \|a^1 - f(a^1)\| + |1-\beta| \, \|f(a^1) - f(b^1) - a^1 + b^1\|$$

$$\leq |\alpha-\beta| \delta_i + |1-\beta| \delta_i \, \| a^1 - b^1 \| .$$

Consider the vectors $a - a^*$, $a^* - b$. Since $a - a^* = (\alpha-\beta)(x_i - a^1)$ and $a^* - b \leq (1-\beta)(a^1 - b^1)$, the angle between $a - a^*$ and $a^* - b$ is the same as that between $(x_i - a^1)$ and $(a^1 - b^1)$, and therefore it is greater than θ_i. By 6.2,

$$\epsilon \|a - b\| = \epsilon \|(a - a^*) + (a^* - b)\| \geq \epsilon \sin \theta_i/2 (\|a - a^*\| + \|a^* - b\|)$$

$$= \epsilon \sin \theta_i/2 \, \|a - a^*\| + \epsilon \sin \theta_i/2 \, \|a^* - b\|$$

$$\geq \epsilon \sin \theta_i/2 \, |\alpha-\beta| q_i + \epsilon \sin \theta_i/2 \, |1-\beta| \, \|a^1 - b^1\|$$

$$\geq |\alpha-\beta| \delta_i + |1-\beta| \delta_i \, \|a^1 - b^1\|$$

$$\geq \|H(f)(a) - H(f)(b) - a + b\| .$$

This proves that $H(f)$ is an ϵ- approximation to $1_{\mathbb{R}^n}$. Since by construction $H(f)(x) = x \; \forall x \; \mathbb{R}^n - C$, $H(f) \; N^*(1_{\mathbb{R}^n}, \epsilon, C)$.

By an identical argument to the preceding one, we have the following approximation lemma.

LEMMA 7.5: Let $f: \sigma \to \mathbb{R}^n$ be a linear map of a k- simplex.

Let b be the barycenter of σ; let $q = \min \{ |x - b| \mid x \in \dot{\sigma} \}$; let θ_i = minimum $\measuredangle$ between lines $[b, x]$ and $[x, y]$ for any $x, y \in \tau$, τ a face of σ. Given $\epsilon > 0$ with $\epsilon < 1$, let $\delta = \sin \theta/2 \min(\epsilon, \epsilon q)$. Then if $g: \dot{\sigma} \to \mathbb{R}^n$ is an δ- approximation to $f \mid \dot{\sigma}$, the join of g with $f \mid \{b\}$ is an ϵ - approximation to f.

COROLLARY 7.6: Let $L \subset K$ (finite) be a subcomplex, and let $f: K \to \mathbb{R}^n$ be a linear map. Then given $0 < \epsilon < 1$, there exists $\delta > 0$ such that there is a map $\Theta: N^*(f \mid L, \delta) \to N^*(f, \epsilon)$ such that $\Theta(g) \mid L = g$ for all $g \in N^*(f \mid L, \delta)$.

PROOF: Given any map $g \in N^*(f/L, \delta)$ we may canonically extend g to K skeletonwise on the simplexes of $K - L$ using LEMMA 7.5, where δ is small enough to satisfy the Lemma. The extension $\bar{g}$ clearly varies continuously with g. //

We are now ready to prove the main theorem of this section.

THEOREM: Let K be a finite simplicial complex of dimension $k < n$. Let $f\colon K \to \mathbb{R}^n$ be a linear map which sends the vertices of K into general position. Then there is a $\delta > 0$ and a diagram of *Fiberwise maps*

$$\begin{array}{ccc} N(f,\delta) \times k & \xrightarrow{\quad E \quad} & N(f,\delta) \times \mathbb{R}^n \\ \lambda' \downarrow & & \downarrow \lambda \\ N(f,\delta) \times k & 1 \times f & N(f,\delta) \times \mathbb{R}^n \end{array}$$

where $E(g, x) = (g, g(x))$ and λ, λ' are *fiberwise homeomorphisms.* Furthermore, the map $\Lambda\colon N(f,\delta) \longrightarrow \mathcal{H}(\mathbb{R}^n)$ cOrresponding to λ (i.e., $\lambda(g, y) = (g, \Lambda(g)(y))$) has its image contained in $N^*(1_{\mathbb{R}^n}, \epsilon, C)$. Here ϵ may be made small by choosing δ small, and C is a convex linear n- cell containing $f(K)$ in its interior.

COROLLARY: The set of structurally stable maps is dense and open in $PL(K, \mathbb{R}^n)$.

PROOF of COROLLARY: Let $f \in PL(K, \mathbb{R}^n)$ and $\epsilon > 0$ be arbitrary. Let f be linear with respect to a subdivision K^1. Let g be an ϵ- approximation to f sending the vertices of K^1 into general position and linear with respect to K^1. Then g is structurally stable. This proves denseness, and openness is an automatic property of structural stability. //

PROOF of THEOREM: We will adopt the same notation as that used in V; this will be made more explicit shortly.

First we assert that the theorem will follow if, given $0<\epsilon<1$, we can find a map $\Lambda^* : N^*(f,\delta) \rightarrow N^*(1_{\mathbb{R}^n}, \epsilon, C)$, for δ and C appropriately chosen, such that for any simplex σ of K and $g \in N^*(f,\delta)$, $\Lambda^*(g)(g(\sigma)) = f(\sigma)$. Then the associated map $\Lambda : N(f,\delta) \rightarrow N^*(1_{\mathbb{R}^n}, \epsilon, C)$ given by $\Lambda(g) = \Lambda^*(g)$ is automatically continuous. Let $\lambda : N(f,\delta) \times \mathbb{R}^n \rightarrow N(f,\delta) \times \mathbb{R}^n$ be the associated map to Λ. Define $\lambda' : N(f,\delta) \times K \rightarrow N(f,\delta) \times K$ as follows: Given σ a simplex of K, $f^{-1} \mid (f)(\sigma) \rightarrow \sigma$ is defined since f is non - degenerate. Define $\lambda' \mid N(f,\delta) \times \sigma$ to be the map $(1 \times f^{-1}) \circ \lambda \circ F$. Then $\lambda' \mid N(f,\delta) \times \sigma$ is a *Fiberwise homeomorphism* and hence, so is λ'. Also $(1 \times f) \circ \lambda' = \lambda E$ by construction.

Next we observe that we may assume K to be the (n - 1) - skeleton of an (s - 1) - simplex. Assume $\{v_1, \ldots, v_s\}$ are the vertices of K and that K is embedded linearly in $\mathbb{R}^s$, as in V. Let $\hat{K}$ be the (n - 1) - skeleton of the (s - 1) - simplex spanned by $v_1, \ldots, v_s$ in $\mathbb{R}^s$. Suppose there is a map $\Lambda^* : N^*(\bar{f},\delta) \rightarrow N^*(1_{\mathbb{R}^n}, \epsilon, C)$, where $\bar{f}$ is the linear map on $\hat{K}$ which extends f. By COROLLARY 7.6, there exists δ' and a map $N^*(f,\delta') \rightarrow N^*(\bar{f},\delta)$. The composition of these two maps is the desired map $\Lambda^* : N^*(f,\delta) \rightarrow N^*(1_{\mathbb{R}^n}, \epsilon, C)$.

NOTATION: $\mathcal{S}$ is the set of subsets of $\{v_1, \ldots, v_s\}$ of order $\leq$ n.

$\mathcal{S}'$ is the set of subsets of $\{v_1, \ldots, v_s\}$ of order n.

σ_α is the simplex spanned by $\{v_i : i \in \alpha\}$ for $\alpha \in \mathcal{S}$.

P^α is the plane in $\mathbb{R}^s$ spanned by $\{v_i : i \in \alpha\}$.

The map f extends by linearity to $J = \bigcup_{\alpha \in \mathcal{S}} P^\alpha$ in a unique way.

Triangulate J such that

1. Each plane P^{α} is a subcomplex of J.

2. Each plane P^{α} is triangulated as in the *PROOF of* THEOREM 6.2

Then given any $g \in N^*(f, \delta)$, δ sufficiently small, g can be extended to a PL δ' approximation to $f: J \to \mathbb{R}^n$, and the map $N^*(f \mid K, \delta)$ $N^*(f \mid J, \delta')$ is continuous. Furthermore, if C is a convex linear n - cell in $\mathbb{R}^n$ such that the simplicial neighborhood of $f(\sigma_\alpha)$ in $f(P^{\alpha})$ is contained in $\mathring{C}$, then each such extension $g: J \to \mathbb{R}^n$ has the property $g(x) = f(x)$ $\forall x \ni f(x) \in \mathbb{R}^n - C$.

Now we define $P_g^{\alpha} = g(P^{\alpha}), \alpha \in \mathcal{S}, g \in N(f, \delta)$. $P_{\nu, g}$ $= \bigcap \{P_g^{\alpha}, \alpha \in \nu\}$ for $\nu \subset \mathcal{S}$.

The remainder of the argument requires δ to be chosen small enough so that a certain finite number of theorems hold. A function ϵ of δ will be written generically as $\epsilon = \mathcal{Q}(\delta)$ if $\lim_{\delta \to 0} \epsilon = 0$. The composition of two functions of δ each of which is $\mathcal{Q}(\delta)$ is again $\mathcal{Q}(\delta)$. Then we need only construct $\Lambda^*: N^*(f, \delta) \longrightarrow N^*(1_{\mathbb{R}^n}, \mathcal{Q}(\delta), C)$.

Applying 2.11 followed by 7.2, we may construct maps $\Theta_\nu: N^*(f, \delta) \to N^*(i_\nu, \mathcal{Q}(\delta))$ for $i_\nu; P_{\nu, f} \hookrightarrow \mathbb{R}^n$ such that $\Theta_\nu(g)\ (P_{\nu, f}) = P_{\nu, g}$ and $\Theta_\nu(g) \mid P_{, f} - C = 1$. However, these maps are not compatible, in that if $\nu' \subset \nu$, and thus $P_{\nu, f} \subset P_{\nu', f}$, $\Theta_{\nu'}(g) \mid P_{\nu, f} \neq \Theta_\nu(g)$.

Define $\varphi_\nu: N^*(f, \delta) \longrightarrow N^*(i_\nu, \mathcal{Q}(\delta))$ inductively as follows: Assume w.l.o.g. that $\nu \subset \mathcal{S}'$. If $\dim P_{\nu, f} = 0$, define $\varphi_\nu = \Theta_\nu$. Now suppose φ_ν is defined for all $\nu \subset \mathcal{S}$ such that $\dim P_{\nu, f} < r$, such that if $\nu_1 \subset \nu_2$, then $\varphi_{\nu_1}(g) \mid P_{\nu_2, f} = \varphi_{\nu_2}(g)$.

Let $\nu_1, \ldots, \nu_s$ be the subsets of $\mathcal{S}'$ containing ν such that $\dim P_{\nu_i, f} = r - 1$. Then by induction, we may define a map $\varphi^* : N^*(f, \delta) \to N^*(i, \mathcal{S}(\delta))$, where $i: P_{\nu_1, f} \cup \ldots \cup P_{\nu_s, f} \hookrightarrow \mathbb{R}^n$ by $\varphi^*(g) \mid P_{\nu_i, f} = \varphi_{\nu_i}(g)$. Since $\Theta_\nu(g) : P_{\nu, f} \to \mathbb{R}^n$ is an $\mathcal{S}(\delta)$ - approximation to the inclusion, $\Theta_\nu(g)^{-1} : P_{\nu, g} \to P_{\nu, f} \subset \mathbb{R}^n$ is an $\mathcal{S}(\delta)$ - approximation to the inclusion, and $\Theta_\nu(g)^{-1} \circ \varphi^*(g)$ is an $\mathcal{S}(\delta)$ - approximation to the inclusion $P_{\nu_1, f} \cup \ldots \cup P_{\nu_s, f} \hookrightarrow P_{\nu, f}$. By LEMMA 7.4, extend $\Theta_\nu(g)^{-1} \circ \varphi^*(g)$ to a homeomorphism of $P_{\nu, f}$ which is an $\mathcal{S}(\delta)$ - approximation to $1_{P_{\nu, f}}$. This gives a map $\psi: N^*(f, \delta) \to N^*(1_{P_{\nu, f}}, \mathcal{S}(\delta))$. Now follow this by Θ_ν and get a map $\varphi_\nu: N^*(f, \delta_\nu) \to N^*(i_\nu, \epsilon)$ with $\varphi_\nu(g) = \Theta_\nu(g) \circ \psi_\nu(g)$. Then $\varphi_\nu(g)(P_{\nu, f}) = P_{\nu, g}$ and $\varphi_\nu(g) \mid P_{\nu_i, f} = \Theta_\nu(g) \circ \Theta_\nu(g)^{-1} \varphi_{\nu_i}(g) = \varphi_{\nu_i}(g)$. Lastly, observe that all of these maps leave points in $\mathbb{R}^n - C$ fixed. This completes the induction.

Amalgamating the maps $\varphi_\nu, \nu \subset \mathcal{S}$, we get a map $\varphi: N^*(f, \delta) \to N^*(i, \mathcal{S}(\delta), C)$ where $i: \bigcup_{\nu \subset \mathcal{S}'} P_{\nu, f} \hookrightarrow \mathbb{R}^n$. Applying LEMMA 7.4 once more now extends this map to

$\Lambda^*: N^*(f, \delta) \to N^*(1_{\mathbb{R}^n}, \mathcal{S}(\delta), C)$.

THEOREM : Let $f : K \to \mathbb{R}^n$ be an embedding. Then f is strongly structurally stable. That is, there exists $\delta > 0$ and a diagram of *Fiberwise maps*

$$\begin{tikzcd} N(f,\delta) \times K \arrow[r, "F"] \arrow[dr, "1 \times f"'] & N(f,\delta) \times \mathbb{R}^n \arrow[d, "\lambda"] \\ & N(f,\delta) \times \mathbb{R}^n \end{tikzcd}$$

where $F(g, x) = (g, g(x))$ and λ is a homeomorphism.

COROLLARY: If $H: K \times I \;\; \mathbb{R}^n \times I$ is a C^1 - isotopy; that is H is an isotopy and the associated function $h: I \to PL\ (K, \mathbb{R}^n)$ is continuous, then H may be extended to an ambient isotopy of $\mathbb{R}^n \times I$. (See 4.7 for *proof*).

PROOF of THEOREM: Subdivide K to K^1 such that $f: K^1 \to \mathbb{R}^n$ is linear. Extend K^1 to a triangulation J of $\mathbb{R}^n$ such that $K^* = N\ (K^1, J)$ the simplicial neighborhood of K^1 in J_1 is a regular neighborhood. By COROLLARY 7.6, if $\epsilon < 1$ is given, there exists $\delta > 0$ and a continuous function $\theta: N^*\ (f, \delta) \longrightarrow N^*\ (i, \epsilon)$ for $i: K^* \to \mathbb{R}^n$ the inclusion. By construction, $\theta(g) \mid \partial(K^*)$ is the identity. Therefore, extend $\theta(g)$ by the identity on $\mathbb{R}^n - K^*$ to an ϵ- approximation to $1_{\mathbb{R}^n}$, which by THEOREM 6.4 is therefore a homeomorphism. This gives $\Lambda: N^*\ (f, \delta) \longrightarrow N^*\ (1_{\mathbb{R}^n}, \epsilon)$. ||

VIII RELATIVE GENERAL POSITION, RELATIVE STRUCTURAL STABILITY and TRANSVERSALITY

With some small modifications, the results of sections II and VII can be generalized to relative versions, in which maps in PL $(K, \mathbb{R}^n)$ which all agree on a subcomplex L can be compared. A speedy corollary of this will be a transversality theorem.

DEFINITION: Let $x_1, \ldots, x_r, y_1, \ldots, y_s$ be points in $\mathbb{R}^n$. The points $x_1, \ldots, x_r$ are in *relative general position with respect to* $y_1, \ldots, y_s$ if each point x_i is in general position with respect to the points $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_r, y_1, \ldots, y_s$. Note that if $s = 0$ this is the usual definition of general position of $x_1, \ldots, x_r$. The points $y_1, \ldots, y_s$ should be thought of as fixed and the points $x_1, \ldots, x_r$ as variable.

NOTATION: If α is any subset of $1, \ldots, r$ and β any subset of $\{1, \ldots, s\}$ with $|\alpha| + |\beta| < n$, then $Q_{\alpha,\beta}$ is the plane spanned by $\{x_i : i \in \alpha\} \cup \{y_j : j \in \beta\}$.

LEMMA 8.1: Let $x_1, \ldots, x_r$ be in relative general position (R.G.P.) with respect to $y_1, \ldots, y_s$, and assume that if $d(x_i, x_i') < \epsilon$, $i = 1, \ldots, r$, $x_1', \ldots, x_r'$ are also in R.G.P. w.r.t. $y_1, \ldots, y_s$. Let $Q_{\alpha_1, \beta_1}, \ldots, Q_{\alpha_q, \beta_q}$ be any planes spanned by $x_1, \ldots, x_r, y_1, \ldots, y_s$ and $Q'_{\alpha_1, \beta_1}, \ldots, Q'_{\alpha_q, \beta_q}$ the corresponding planes for $x_1', \ldots, x_r', y_1, \ldots, y_s$. Then $\dim (Q_{\alpha_1,\beta_1} \cap \ldots \cap Q_{\alpha_q,\beta_q}) = \dim (Q'_{\alpha_1,\beta_1} \cap \ldots \cap Q'_{\alpha_q,\beta_q})$ and the two planes are close together for small.

PROOF: This is identical to LEMMA 2.8, and the proof carries over without modification, since we need only know that each point x_i is in general position with respect to $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_r, y_1 \ldots y_s$.

LEMMA 8.2: Assume relative general position of fewer than r points in $\mathbb{P}^n$ with respect to $y_1, \ldots, y_s$ is an open and dense condition. Then the condition, " $x_2, \ldots, x_r$ are in R.G.P. with respect to $y_1, \ldots, y_s$ and x_1 is in general position with respect to $x_2, \ldots, x_r, y_1, \ldots y_s$" is open and dense in $(\mathbb{P}^n)^r$.

PROOF: This follows from LEMMA 2.9 exactly as the proof of COROLLARY 2.10.

THEOREM 8.3: The set of points $\{(x_1, \ldots, x_r) \in (\mathbb{P}^n)^r \mid x_1, \ldots, x_r$ are in relative general position w.r.t. $y_1, \ldots, y_s$ is open and dense in $(\mathbb{P}^n)^r$.

PROOF: For $r = 1$, this follows from THEOREM 2.5. The inductive case follows from LEMMAS 8.1 and 8.2 exactly as in the *PROOF of* THEOREM 2.7.

It is necessary now to have an analogue of THEOREM 2.11, which is in a sense the key lemma for structural stability. Let $Q_{\alpha,\beta}$ be a plane as before and consider the face $Q_{\emptyset,\beta}$ of $Q_{\alpha,\beta}$. If ν is any subset of $2^{\{1,\ldots,s\}}$, let $Q_{\emptyset,\nu}$ denote $\bigcap\{Q_{\emptyset,\beta} : \beta \in \nu\}$. Let $Q_{\alpha,\nu} = Q_{\alpha,\emptyset} \vee Q_{\emptyset,\nu}$. We must consider a plane $Q_{\alpha,\nu}$ to be a face (as in 2.11) of $Q_{\alpha',\nu'}$ whenever $\alpha \subset \alpha'$, and $Q_{\emptyset,\nu} \subset Q_{\emptyset,\nu'}$. Note that if $\nu = \{\beta\}$, $\nu' = \{\beta'\}$ and $\beta \subset \beta'$, this is the old definition. Call any face which is not of the old type a *pseudoface*.

THEOREM 8.4: (Compare with 2.11) If $x_1, \ldots, x_r$ are in R.G.P. with respect to $\{y_1, \ldots, y_s\}$ and $Q_{\alpha_1,\beta_1}, \ldots, Q_{\alpha_t,\beta_t}$ are planes spanned by these points, then there exist planes $Q_1', \ldots Q_t'$ such that Q_i' is a face or pseudoface of Q_{α_i,β_i}, $Q_{\alpha_1,\beta_1} \cap \ldots \cap Q_{\alpha_t,\beta_t} = Q_1' \cap \ldots \cap Q_t'$, and the distinct planes among the Q_i' have minimal intersection.

PROOF: This differs very little from the proof of 2.11, but we will repeat the proof in order to display the full detail.

In order to apply an inductive argument, we prove a slightly more general theorem. Let $Q_{\alpha_1,\nu_1}, \ldots, Q_{\alpha_t,\nu_t}$ be planes of the type described above, and suppose $\dim (Q_{\alpha_1,\nu_1} \cap \ldots \cap Q_{\alpha_t,\nu_t})$ $> \sum_1^t \dim Q_{\alpha_i,\nu_i} - (t-1)n$. We show that we may replace these planes by faces or pseudofaces such that the distinct ones meet minimally.

Now we use a reduction argument exactly parallel to that of 2.11, except that there are many more cases.

CASE 1: $\alpha_1 = \ldots = \alpha_t = \emptyset$. Replace $Q_{\emptyset,\nu_1} \ldots Q_{\emptyset,\nu_t}$ by $Q_{\emptyset,\nu_1 \cup \ldots \cup \nu_t}$.

CASE 2: $\alpha_1 = \ldots = \alpha_t \neq \emptyset$. Assume w.l.o.g. that $1 \in \alpha_1$.
Let $\alpha' = \alpha_1 - \{1\}$ (possibly empty). Then $Q_{\alpha_1,\nu_1} \cap \ldots \cap Q_{\alpha_t,\nu_t}$
$= (x_1 \vee Q_{\alpha',\nu_1}) \cap \ldots \cap (x_1 \vee Q_{\alpha',\nu_t}) = \widetilde{T}(x_1; Q_{\alpha',\nu_1}, \ldots, Q_{\alpha',\nu_t}; \mathbb{P}^n)$.
Since $\dim T(x_1) \leq \sum_1^t (\dim (Q_{\alpha',+}) + 1) - (t-1)n < \widetilde{T}(x_1)$ by hypothesis, and since x_1 is in general position, we must have $T(x_1) = \emptyset$ and $\widetilde{T}(x_1) = \widetilde{T}_\pi(x_1)$. For $c \in \pi$, let $R_c = \bigcap_{i \in c} Q_{\alpha',\nu_i}$. As in THEOREM 2.11, we must have that R_c does not have the minimal dimension $\sum_{i \in c} \dim (Q_{\alpha',\nu_i}) - (|c| - 1)n$.

By induction, replace Q_{α',ν_i} by Q_{a_i,η_i} a face or pseudoface so that the distinct planes among Q_{a_i,η_i}, $i \in C$, meet minimally, and such that $\bigcap_{i\in c} Q_{\alpha_i',\nu_i} = \bigcap_{i\in c} Q_{\alpha_i,\eta_i}$.

Then

$$Q_{\alpha_1,\nu_1} \cap \cdots \cap Q_{\alpha_t,\nu_t} = \tilde{T}\,(x_1;\ \{R_c : c \in \pi\};\ \mathbb{R}^n)$$

$$= \bigcap_{\substack{d\in\pi \\ d\neq c}} \left[x_1 \vee \left(\bigcap_{i\in d} Q_{\alpha_i',\nu_i}\right)\right] \cap \left[x_1 \vee \left(\bigcap_{i\in c} Q_{\alpha_i,\eta_i}\right)\right]$$

$$\subset \bigcap\{Q_{\alpha_i,\nu_i} : i \notin c\} \cap \left[\bigcap\{Q_{\alpha_i\cup\{1\},\eta_i} : i \in c\}\right]$$

$\subset Q_{\alpha_1,\nu_1} \cdots Q_{\alpha_t,\nu_t}$. Thus the containments are equalities and the planes Q_{α_i,ν_i} have been reduced.

CASE 3: Other possibility. This case is identical to the one treated in 2.11, so the argument will not be repeated.

DEFINITION: Let K be a finite simplicial complex, L a subcomplex of K, and $f \in PL\ (K, \mathbb{R}^n)$. Let $h = f \mid L$. Denote by $PL\ (K, \mathbb{R}^n; h)$ the subspace $\{g \in PL\ (K, \mathbb{R}^n) \mid g \mid L = h\}$. Then we say f is *structurally stable relative to L* there is a neighborhood N of f in $PL\ (K, \mathbb{R}^n; h)$ and a commutative diagram of *Fiberwise maps*

$$\begin{tikzcd} N \times K \arrow[r, "E"] \arrow[d, "\lambda'"] & N \times \mathbb{R}^n \arrow[d, "\lambda"] \\ N \times K \arrow[r, "1 \times f"] & N \times \mathbb{R}^n \end{tikzcd}$$

with λ and λ' homeomorphisms

THEOREM 8.5: The set of maps in $PL\ (K, \mathbb{R}^n, h)$ structurally stable relative to L, for $h: L \to \mathbb{R}^n$ any nondegenerate map, is dense in $PL\ (K, \mathbb{R}^n; h)$.

PROOF: The peculiarity of 8.4 is the pseudoface. If this concept had not appeared, the proof would proceed as in section VII. However, here we have to go through a few preliminary contortions, whose purpose will soon be apparent.

Let $g: K \to \mathbb{R}^n$ be such that $g \mid L = h$. Suppose K' subdivides K and $g: K' \longrightarrow \mathbb{R}^n$ is linear. Let K'' be the first barycentric subdivision of K' mod L. Then $g: K'' \longrightarrow \mathbb{R}^n$ is linear. Perturb the vertices of $K'' - L$ slightly so that they are in relative general position with respect to the vertices of L. If we show that any such map f is structurally stable relative to L, this will prove the theorem.

Using COROLLARY 7.6, we may assume that K' is the $(n - 1)$ - skeleton of a simplex and that K'' is the first barycentric subdivision mod L. This makes K'' a homogeneous $(n - 1)$ - complex and we may assume $f: K'' \to \mathbb{R}^n$ maps all the vertices of $K'' - L$ into general position w.r.t. the vertices of L. Embed K'' linearly in some large dimensional Euclidean space $\mathbb{R}^{r+s}$ so that the vertices $v_1, \ldots, v_r$ of $K'' - L$ and $w_1, \ldots, w_s$ of L span an $r + s - 1$ simplex.

If σ is a simplex of K'', then f extends by linearity to the plane $[\sigma]$. Triangulate each such plane to satisfy THEOREM 6.2. Then we may consistently extend PL maps $g: K'' \longrightarrow \mathbb{R}^n$ to the planes $[\sigma]$ so that if g is a δ- approximation to f, then the extension of g is an $\mathcal{Q}(\delta)$ - approximation to the extension of f, which agrees with f outside some fixed compact neighborhood of K''.

The outline of the proof now follows section VII. Let $P_g^{\sigma} = g([\sigma])$. If $\sigma_1, \ldots, \sigma_\ell$ are simplexes of K'' and $\nu = \{\sigma_1, \ldots, \sigma_\ell\}$, let $P_{\nu, g} = \bigcap_{\sigma_i \in \nu} P_g^{\sigma_i}$. We construct maps from $P_{\nu, f}$ to $P_{\nu, g}$ in the same canonical way as in VII, modify them to make them consistent, and then extend the combined maps to all of $\mathbb{R}^n$. All of this works provided we know that as g varies in a neighborhood of f, $\dim P_{\nu, g}$

remains constant and THEOREM 7.1 can be applied.

If τ is a face of two simplexes σ_1 and σ_2 and g is any δ - approximation to f, then $(g \mid [\sigma_1]) \mid [\tau] = (g \mid [\sigma_2]) \mid [\tau]$. But if Q is a common pseudoface of $P_f^{\sigma_1}$ and $P_f^{\sigma_2}$ then in general $g \mid (f/[\sigma_1])^{-1}\ Q \neq g \mid (f \mid [\sigma_2])^{-1}\ Q$, since these two planes are distinct in the domain.

A simple example of this type of difficulty is the following: Let $K = [0, 1]$ and triangulate K with vertices at 0, 1/2, 1. Let $L = \{0, 1\}$. Define a linear map $f: K \to \mathbb{R}^2$ by $f(0) = f(1) = (0, 0)$, $f(1/2) = (1, 0)$. Clearly f maps the vertex of K - L into relative general position with respect to L. But this map, while being *linearly* structurally stable relative to L, is certainly not structurally stable relative to L. This difficulty is traced to the fact that the lines $(f(0) \vee f(1/2))$ and $(f(1) \vee f(1/2))$ meet in more than a minimal dimension, but they are identical when viewed as the pseudoface $(f(0) \cap f(1)) \vee f(1/2)$.

This is precisely the reason for taking a barycentric subdivision. Let $\sigma_1 = k_1 \ell_1$ and $\sigma_2 = k_2 \ell_2$ be simplexes of K'' with ℓ_1, ℓ_2 in L and k_1, k_2 in K - L. If v is a common vertex of k_1 and k_2, then of course v can be joined to both ℓ_1 and ℓ_2 in K''. But this implies by the nature of barycentric subdivision mod L, that ℓ_1 and ℓ_2 are common faces of a simplex ℓ of L, and therefore $f[\ell_1] \cap f[\ell_2] = f[\ell_1 \cap \ell_2]$. This remark shows that the reduction process of THEOREM 8.5 always yields faces and not pseudofaces in this situation, and that the technique of CHAPTER VII works with no further modification. Note that the above is false if h is degenerate.

A corollary of THEOREM 8.5 is a transversality result for PL maps into $\mathbb{R}^n$.

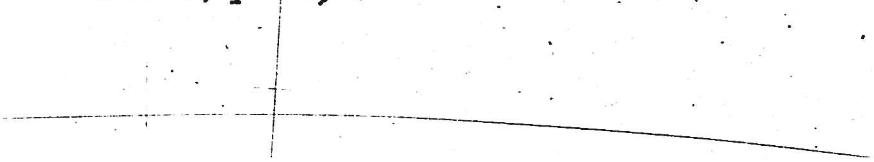
DEFINITION: Let M^m be a closed PL manifold, $m < n$. By embedding M in some Euclidean space we may assume M has a metric which is linear on each simplex of some triangulation K. Thus we may define $PL(M, \mathbb{R}^n) = PL(K, \mathbb{R}^n)$. Let $E(M, \mathbb{R}^n)$ be the subspace of embeddings of M into $\mathbb{R}^n$, and let $LF(M, \mathbb{R}^n)$ be the subspace of locally flat embeddings.

PROPOSITION 8.6: $E(M, \mathbb{R}^n)$ is open in $PL(M, \mathbb{R}^n)$, and $LF(M, \mathbb{R}^n)$ is open and closed in $E(M, \mathbb{R}^n)$.

PROOF: Since embeddings are strongly structurally stable, any embedding $F: M \to \mathbb{R}^n$ has a neighborhood N consisting of embeddings strongly structurally equivalent to F. This immediately gives the first assertion, while the second follows from the observation that strongly structurally equivalent embeddings are either both locally flat or both non-locally flat.

THEOREM 8.7: Let P be a complete submanifold of $\mathbb{R}^n$. Then almost all embeddings $f \in LF(M, \mathbb{R}^n)$ map M transverse to P. That is, the set of maps which fail to do so are nowhere dense.

REMARKS:
1. The condition that P be complete merely assures that M avoids hitting the "boundary" of P.
2. It is false to say that transversality is an open condition. For let $P = \{(x, y) \in \mathbb{R}^2 / x = 0, y \geq 0$ or $x \geq 0, y = 0\}$.



Let $M = [-1, 1]$ and define f by $f(t) = \begin{cases} (t, 0) & \text{if } t \leq 0 \\ (t, t) & \text{if } t \geq 0. \end{cases}$

Then f maps M transverse to P but a slight perturbation of f to f', where $f'(t) = f(t) + (\epsilon, 0)$, $\epsilon > 0$, destroys this. Observe that there are homeomorphisms of $\mathbb{R}^2$ which would change the picture of M and P to a structurally stable one.

PROOF: The key idea here is to use the t - shift of E.C. Zeeman to verify that a map is already transverse. Suppose $f \in LF(M, \mathbb{R}^n)$ such that the map $f \cup i: M \cup P \to \mathbb{R}^n$ is structurally stable relative to P, where $f \cup i(x) = \begin{cases} f(x) & \text{if } x \in M \\ x & \text{if } x \in P \end{cases}$.

(If P is noncompact, we need only consider the portion of P lying in a large n - cell containing f(M) well within itsinterior.) I claim: f must map M transverse to P. To see this, let K be a triangulation of M for which f is simplicial to a triangulation of $\mathbb{R}^n$ such that P appears as a subcomplex. Assume we have verified that f is transverse to P except on the t - skeleton, $0 \leq t \leq k$. Let $x \in f(\mathring{\sigma}) \cap P$, a t - simplex of K. Zeeman and Armstrong[1] show that an arbitrarily small t - shift performed relative to σ, using the linear structure of $\mathbb{R}^n$, will achieve transversality in a neighborhood of $\mathring{\sigma}$. But a t - shift is a C^1 - perturbation of f. Since f is structurally stable relative to P, it follows that $f \cup i$ followed by a small t - shift is structurally equivalent to $f \cup i$. Therefore, f must already be transverse to P at x.

THEOREM 8.8: Let X, Y be compact polyhedra, dim X, dim $Y \leq n - 3$. If $Y \subset \mathbb{R}^n$, then the set of PL maps $f: X \to \mathbb{R}^n$ mapping X transverse to Y

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[1] Armstrong and Zeeman, p. 453.

in the sense of Armstrong[1] is the complement of a nowhere dense set.

PROOF: The same as 8.7, referring this time to the paper by Armstrong.[2]

COROLLARY 8.9: Structural stability of X relative to Y is a generalization of transversality.

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[1]Armstrong, p. 175.

[2]Armstrong, p. 186.