FOCI OF PLANE CURVES[1]

By Mo Yeh

1. Introduction

A line whose slope is i or $-i$ will be called an isotropic line, or simply an isotropic [1]. Two isotropics of the same kind are two isotropics whose slopes, m_1 and m_2, have the same value, i or $-i$; otherwise, they are of different kinds. A tangent to a curve which is an isotropic is called an isotropic tangent to the curve. The finite intersections of the isotropic tangents to a curve are called the foci of the curve, and a chord of contact of any two isotropic tangents through a given focus is called a directrix of the curve corresponding to the given focus [2]. According to these definitions a focus or a directrix of a curve may be real or imaginary.

2. Some Special Types of Curves

Given a line D as the y-axis and a point F $(p,0)$, a conic is the locus of a point P (x,y) such that $FP=eMP$, where MP is the perpendicular distance from P to the line D, and e is a non-zero constant. Here F is the focus of the conic and D, the corresponding directrix.

Now we consider the algebraic curve defined by

(1) $$(FP^2)^m = eMP^n,$$

where m, n are positive integers such that $n \geqq 2$, $n>m$, and $(m, n)=1$, that is, m and n are relatively prime. Equation (1) may be written as

(2) $$\{(x-p)^2+y^2\}^m = ex^n.$$

If $r=\max(2m,n)$ then (2) represents an algebraic curve of order r.

Let us consider the isotropic

(3) $$y=i\,(x-p).$$

This line cuts $x=0$ at the point $(0, -ip)$. The translation

$$x=x', \quad y=y'-ip$$

carries (2) and (3) into

(4) $$\{x'^2+y'^2-2p\,(x'+iy')\}^m = ex'^n$$

and

(5) $$x'+iy'=0$$

respectively. Since $n \geqq 2$, $n>m$, the terms of lowest degree in (4) with respect to x', y' are given by $\{-2p\,(x'+iy')\}^m$. Hence (5) is a tangent to (4) at the new origin, and consequently (3) is a tangent to (2) at the point $(0, -ip)$. Similarly we can prove that the isotropic

(6) $$y=-i\,(x-p)$$

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is a tangent to (2) at the point $(0, ip)$. The point of intersection of (3) and (6) is $F(p, 0)$. The points of contact, $(0, -ip)$ and $(0, ip)$, lie on the line $x=0$. This proves:

THEOREM 1. *The point F and the line D are respectively a focus and a corresponding directrix of the algebraic curve defined by* (1).

A central conic is defined as the locus of a point $P(x,y)$, the sum or difference of whose distances from two given points, $F(c,0)$ and $F'(-c,0)$ is a constant $2a$. Hence the equation is $\pm F'P \pm FP = 2a$, and it is known that the points F and F' are the real foci of the central conic.

Now we consider the algebraic curve defined by

$$\pm\lambda F'P \pm \mu FP = a, \tag{7}$$

where λ, μ are two nonzero constants with absolute values both different from $a/2c$. We may write (7) in the form

$$\pm\lambda\sqrt{(x+c)^2+y^2} \pm \mu(\sqrt{x-c)^2+y^2} = a.$$

After the radicals in this equation are removed by rationalization we get

$$\{(\lambda^2-\mu^2)(x^2+y^2+c^2)+2(\lambda^2+\mu^2)cx-a^2\}^2 = 4a^2\mu^2\{(x-c)^2+y^2\}. \tag{8}$$

If $|\lambda| \neq |\mu|$, (8) represents a quartic curve.

Let us consider the isotropic

$$y = i(x-c) \tag{9}$$

A point of intersection of (8) and (9) is

$$x_1 = \frac{a^2}{4\lambda^2 c}, \quad y_1 = i\left(\frac{a^2}{4\lambda^2 c} - c\right)$$

and the translation $x = x'+x_1$, $y = y'+y_1$, carries (8) and (9) into

$$\{(\lambda^2-\mu^2)(x'^2+y'^2+2x_1x'+2y_1y')+2(\lambda^2+\mu^2)cx'\}^2 = 4a^2\mu^2(y'-ix')(y'+ix'+2y_1) \tag{10}$$

and

$$y' = ix' \tag{11}$$

respectively. Since $|\lambda| \neq a/2c$, we have $y_1 \neq 0$ and then the terms of lowest degree in (10) with respect to x', y' are given by $8a^2\mu^2y_1(y'-ix')$. Hence (11) is a tangent to (10) at the new origin and consequently (9) is a tangent to (8) at the point (x_1, y_1). Similarly we can prove that the isotropics

$$y = -i(x-c), \quad y = \pm i(x+c)$$

are tangents to (8). These four isotropic tangents intersect at four points $(c, 0)$, $(-c, 0)$, $(0, ic)$, $(0, -ic)$. This proves:

THEOREM 2. *The two points F and F′ are two foci of the algebraic curve defined by* (7).

The central conic has the remarkable property that the tangent to the curve at any point makes equal angles with the focal radii drawn to that point. It is easy to generalize this result as follows.

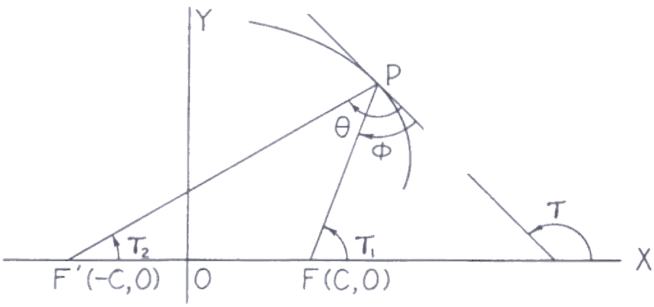
THEOREM 3. *If*

(12) $$\lambda F'P+\mu FP=a$$

is the real part of the algebraic curve defined by (7) *and if* θ *and* ϕ *are respectively the angles which the tangent line to this part at an arbitrary point* P *makes with the lines* PF' *and* PF, *then*

(13) $$\lambda \cos \theta+\mu \cos \phi=0.$$

Proof. Let the inclinations of PF, PF' and the tangent line at P be respectively τ_1, τ_2 and τ. Then



$$\tan \tau=\frac{dy}{dx}, \tan \tau_1=\frac{y}{x-c}, \tan \tau_2=\frac{y}{x+c},$$

and

(15) $$\phi=\tau-\tau_1, \quad \theta=\tau-\tau_2.$$

Equation (12) may be written as

$$\lambda\sqrt{(x+c)^2+y^2}+\mu\sqrt{(x-c)^2+y^2}=a.$$

Differentiating this equation we get

$$\lambda\frac{(x+c)+y\dfrac{dy}{dx}}{\sqrt{(x+c)^2+y^2}}+\mu\frac{(x-c)+y\dfrac{dy}{dx}}{\sqrt{(x-c)^2+y^2}}=0$$

From (14) and this equation we have

$$\lambda\frac{1+\tan \tau \tan \tau_2}{\sqrt{1+\tan^2 \tau_2}}+\mu\frac{1+\tan \tau \tan \tau_1}{\sqrt{1+\tan^2 \tau_1}}=0,$$

which can be reduced to

$$\lambda \cos (\tau-\tau_2)+\mu \cos (\tau-\tau_2)=0.$$

By (15), this equation becomes (13). The theorem is proved.

3. Conformal Transformation

Let

$$x' = P(x, y), \quad y' = Q(x, y) \tag{16}$$

be a conformal transformation, where P and Q are two real, single-valued, continuous functions with continuous partial derivatives of the first order in the whole plane except possibly at a finite number of real points and P, Q are not constants. We must have either

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \quad \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x} \tag{17}$$

or

$$\frac{\partial P}{\partial x} = -\frac{\partial Q}{\partial y}, \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad [3]. \tag{18}$$

The first set of conditions states that $P+Qi$ is an analytic function of $x+yi$. As for the second set, we can reduce it to the first by changing Q to $-Q$, that is, by a reflection about the x'-axis. Hereafter we shall restrict our discussion to transformations (16) which satisfy (17) only. Let us denote $x+iy$, $x'+iy'$, $x-iy$, $x'-iy'$ by z, z', $\bar{z}$, $\bar{z}'$ respectively. We can write (16) in the form $z'=f(z)$ where $f(z)=P+Qi$. By the conditions we impose on P and Q, we know that $f(z)$ is a single-valued analytic function in the whole plane except possibly at a finite number of singular points. We may call these singular points, the singular points of the transformation.

LEMMA 1. *The conformal transformation* (16) *carries a given isotropic into an isotropic of the same kind provided that the given isotropic does not pass through any singular point of the transformation.*

Proof. Let the given isotropic be

$$x+yi = a+bi, \tag{19}$$

where a, b are real. Since it does not pass through any singular point of the transformation, $a+bi$ is not a singular point of $f(z)$. Hence $f(a+bi)$ has a definite value, and (19) is carried into

$$x'+iy' = f(a+ib) \tag{20}$$

by (16). Now we consider the isotropic

$$x-iy = c-id \tag{21}$$

where c, d are real. By hypothesis (c,d) is not a singular point of the transformation; hence, $f(c+id)$ has a definite value. Let the conjugate of $f(z)$ be $F(\bar{z})$, that is, $\overline{f(z)}=F(\bar{z})$. Then $\bar{z}'=F(\bar{z})$ and $\overline{f(c+id)}=F(c-id)$. Therefore $F(c-id)$ also has a definite value and (21) is carried into $x'-iy'=F(c-id)$. The lemma is proved.

Example. The conformal transformation

$$x' = \frac{(x-1)^2-(y-1)^2}{[(x-1)^2+(y-1)^2]^2} + x^2 - y^2, \quad y' = 2xy - \frac{2(x-1)(y-1)}{[(x-1)^2+(y-1)^2]^2}$$

carries the isotropic $x-iy=2$ into the isotropic $x'-iy'=4-\frac{1}{2}i$.

LEMMA 2. *The conformal transformation* (16) *carries an isotropic tangent at* (x_1, y_1) *into an isotropic tangent to the conformal transform of the given curve at the point* $P(x_1, y_1)$, $Q(x_1, y_1)$, *provided that* $P(x,y)$, $Q(x,y)$ *and the partial derivatives* $P_1(x, y)$, $Q_1(x,y)$ *exist at* (x_1,y_1) *and*

$$P_1^2(x_1,y_1)+Q_1^2(x_1,y_1)\neq 0. \tag{22}$$

Proof. Let a curve C with equation $\phi(x,y)=0$ be carried by (16) into the curve C' with equation $\psi(x',y')=0$. Let (19) be an isotropic tangent to C at (x_1,y_1). Then $x_1+iy_1=a+bi$. If (a,b) is a singular point of the transformation, then $f(a+bi)$ is not defined. But

$$f(a+bi)=f(x_1+iy_1)=P(x_1,y_1)+iQ(x_1,y_1),$$

so there is a contradiction to our hypothesis. Therefore (a,b) cannot be a singular point of the transformation and (19) is carried into (20) by (16). Let $x_1'=P(x_1,y_1)$ $y_1'=Q(x_1,y_1)$. We need only prove that the curve C' has the slope i at (x_1',y_1'). The equation $\psi(x',y')=0$ may be obtained from $\phi(x,y)=0$ and (16) by eliminating x,y. Let us consider x as a parameter. We find

$$\frac{dy'}{dx'}=\frac{Q_1(x,y)+Q_2(x,y)\dfrac{dy}{dx}}{P_1(x,y)+P_2(x,y)\dfrac{dy}{dx}}.$$

Setting $x=x_1$, $y=y_1$, $\left(\dfrac{dy}{dx}\right)_1=i$ in this equation we have

$$\left(\frac{dy'}{dx'}\right)_1=\frac{Q_1(x_1,y_1)+Q_2(x_1,y_1)i}{P_1(x_1,y_1)+P_2(x_1,y_1)i}$$

By the aid of (17) and (22) this equation becomes

$$\left(\frac{dy'}{dx'}\right)_1=\frac{Q_1(x_1,y_1)+P_1(x_1,y_1)i}{P_1(x_1,y_1)-Q_1(x_1,y_1)i}=i.$$

Hence (20) is an isotropic tangent to C'. A similar proof holds for the case when the isotropic tangent to C is of the form (21). The lemma is proved.

We may note that the Jacobian of (16) is

$$J(x,y)=\begin{vmatrix}\dfrac{\partial P}{\partial x} & \dfrac{\partial P}{\partial y}\\[2ex] \dfrac{\partial Q}{\partial x} & \dfrac{\partial Q}{\partial y}\end{vmatrix}=\begin{vmatrix}\dfrac{\partial P}{\partial x} & -\dfrac{\partial Q}{\partial x}\\[2ex] \dfrac{\partial Q}{\partial x} & \dfrac{\partial P}{\partial x}\end{vmatrix}=\left(\frac{\partial P}{\partial x}\right)^2+\left(\frac{\partial Q}{\partial x}\right)^2.$$

Therefore the condition (22) is equivalent to $J(x_1,y_1)\neq 0$.

Since the point of intersection of two curves is preserved by a conformal transformation, from lemmas 1, 2 we immediately have the following important theorem.

THEOREM 4. *The conformal transform of a focus of a given curve is, in general, a focus of the conformal transform of the given curve.*

The conformal transformation (16) carries real points into real points. Hence under this conformal transformation, a real focus is carried, in general, into a real focus.

Example. Under the conformal transformation $x'=x^2-y^2$, $y'=2xy$, the parabola $y^2=2x$ is carried into the curve

$$x'=\tfrac{1}{4}t^4-t^2, \quad y'=t^3$$

and the focus of the parabola, $(\frac{1}{2},0)$, into a focus of the curve, $(\frac{1}{4},0)$.

We consider the inversion

$$x'=\frac{\gamma^2 x}{x^2+y^2}, \qquad y'=\frac{\gamma^2 y}{x^2+y^2}, \qquad \gamma>0, \tag{23}$$

where the origin is the center of inversion. From these two equations we have

$$x'-iy'=\frac{\gamma^2}{x+iy} \quad \text{or} \quad z'=\frac{\gamma^2}{z}$$

Hence an inversion is a conformal transformation which satisfies (18) and its singular point is the center of inversion. From theorem 4 we deduce:

COROLLARY. *The inverse of a focus of a given curve is, in general, a focus of the inverse of the given curve.*

In particular we can prove easily that a real focus is carried into a real focus by an inversion provided that the real focus is not at the center of inversion.

Example. The parabola $y^2=4ax$ with focus $(a,0)$ is carried into the cissoid $\gamma^2y^2=4ax(x^2+y^2)$ with focus $\left(\frac{\gamma^2}{a},0\right)$ by the inversion (23). Here $\left(\frac{\gamma^2}{a},0\right)$ is obviously the inverse of $(a,0)$.

The focus of a parabola has a remarkable property that the tangent to the curve at any point bisects the angle between the focal radius and the diameter through that point. By an inversion with the center of inversion at the vertex of the parabola we deduce the following interesting property of a cissoid.

THEOREM 5. *If a circle passing through the real focus and the cusp of a cissoid and another circle tangent to the axis of symmetry of the cissoid at its cusp intersect at another point on the cissoid, then the cissoid bisects the angle between the two circles at this point.*

This theorem suggests that any curve which is the conformal transform of a conic should itself possess interesting focal properties.

4. The Position of Foci

It is well known that for any nondegenerate conic any focus must lie on an axis of symmetry and the corresponding directrix is perpendicular to the axis [4]. Does this set of facts hold for any curve? First, there are curves which have axes of symmetry but have no foci. For example, the degenerate conic $x^2-y^2=0$ has axes of symmetry, the x- and y-axes, but its foci are not defined. Second, there are curves which have foci but have no axes of symmetry. For example, the curve $y=e^x$ has an infinite number of real foci which are given by $\left(-1, \frac{4k+1}{2}\pi\right)$, where k is an arbitrary integer, but it has no axis of symmetry. In order to establish a theorem for any curve, we must assume that the curve has an axis of symmetry and a focus. But even so the conditions are still not sufficient to prove a theorem such as the one we stated for the conics. We may illustrate this by an interesting example.

Consider the curve $xy^2=4a^3$. Its axis of symmetry is the x-axis. Its foci are the nine points given by

$$x=-\frac{3}{2}a(\omega^n+\omega^m), \qquad y=\frac{3}{2}ai(\omega^n-\omega^m),$$

where $n=0, 1, 2$; $m=0, 1, 2$; $\omega=\dfrac{-1+i\sqrt{3}}{2}$. We observe only three of the nine foci, namely: $(-3a,0)$, $(-3a\omega,0)$, $(-3a\omega^2,0)$, lie on the axis of symmetry of the curve. The other six foci can be put into three pairs, each pair consisting of two points symmetric to the axis of symmetry. The directrices of the foci $(-3a,0)$, $(-3a\omega,0)$, $(-3a\omega^2,0)$ are the lines $x=-a$, $x=-a\omega$, $x=-a\omega^2$, respectively, which are perpendicular to the axis of symmetry. However, we can prove the following theorem.

THEOREM 6. *If a curve has at least one axis of symmetry and at least one focus, then with respect to each axis there is at least one focus situated on it and one of the corresponding directrices is perpendicular to it; and any remaining foci are situated symmetrically with respect to this axis.*

Proof. Let us consider one of the axes of symmetry. We choose this axis as the x-axis. Let the equation of the curve be

$$f(x,y)=0. \tag{24}$$

Since the curve is symmetric with respect to the x-axis, we have

$$f(x,y)\equiv \pm f(x,-y). \tag{25}$$

If a focus $(a,0)$ of the curve lies on this axis, then two isotropic tangents, $y=i(x-a)$ and $y=-i(x-a)$, passing through this focus, will touch the curve at (x_0,y_0) and $(x_0,-y_0)$ respectively. Hence a directrix of the focus $(a,0)$ is the line $x=x_0$, which is perpendicular to the x-axis. By hypothesis the curve has at least one focus. If all the foci of the curve lie on the x-axis, the theorem holds obviously. Otherwise we may assume that the curve has a focus (x_1,y_1) with $y_1\neq 0$. It is the intersection of the isotropic tangents

$$y-y_1=i(x-x_1), \tag{26}$$
$$y-y_1=-i(x-x_1). \tag{27}$$

The reflection $x=x'$, $y=-y'$ carries (24), (26), (27) into

$$f(x',-y')=0, \tag{28}$$
$$y'+y_1=-i(x'-x_1), \tag{29}$$
$$y'+y_1=i(x'-x_1), \tag{30}$$

respectively. Since tangency is invariant under a reflection, (29), (30) are isotropic tangents to (28). By the aid of (25) we prove that

$$y+y_1=-i(x-x_1), \tag{31}$$
$$y+y_1=i(x-x_1) \tag{32}$$

are two isotropic tangents to the curve (24). The point of intersection of (31) and (32), that of (26) and (31), and that of (27) and (32) are $(x_1, -y_1)$, $(x_1+iy_1, 0)$, and $(x_1-iy_1, 0)$, respectively. The focus $(x_1, -y_1)$ is a point symmetric to (x_1, y_1). The foci $(x_1+iy_1, 0)$, $(x_1-iy_1, 0)$ lie on the x-axis.

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