Preface

The quest to build a quantum computer is arguably one of the major scientific and technological challenges of the 21st century, and *quantum information theory* (QIT) provides the mathematical framework for that quest. Over the last dozen or so years, it has become clear that quantum information theory is closely linked to geometric functional analysis (Banach space theory, operator spaces, high-dimensional probability), a field also known as *asymptotic geometric analysis* (AGA). In a nutshell, asymptotic geometric analysis investigates quantitative properties of convex sets, or other geometric structures, and their approximate symmetries as the dimension becomes large. This makes it especially relevant to quantum theory, where systems consisting of just a few particles naturally lead to models whose dimension is in the thousands, or even in billions.

While the idea for this book materialized after we independently taught graduate courses directed primarily at students interested in functional analysis (at the University Lyon 1 and at the University Pierre et Marie Curie-Paris 6 in the spring of 2010), the final product goes well beyond enhanced lecture notes. The book is aimed at multiple audiences connected through their interest in the interface of QIT and AGA: at quantum information researchers who want to learn AGA or to apply its tools; at mathematicians interested in learning QIT, or at least the part of QIT that is relevant to functional analysis/convex geometry/random matrix theory and related areas; and at beginning researchers in either field. We have tried to make the book as user-friendly as possible, with numerous tables, explicit estimates, and reasonable constants when possible, so as to make it a useful reference even for established mathematicians generally familiar with the subject.

The first four chapters are of introductory nature. Chapter 0 outlines the basic notation and conventions with emphasis on those that are field-specific to AGA or to physics and may therefore need to be clarified for readers that were educated in the other culture. It should be read lightly and used later as a reference. Chapter 1 introduces basic notions from convexity theory that are used throughout the book, notably duality of convex bodies or of convex cones and Schatten norms. Chapter 2 goes over a selection of mathematical concepts and elementary results that are relevant to quantum theory. It is aimed primarily at newcomers to the area, but other readers may find it useful to read it lightly and selectively to familiarize themselves with the "spirit" of the book. Chapter 3 may be helpful to mathematicians with limited background in physics; it shows why various mathematical concepts appear in quantum theory. It could also help in understanding physicists talking about the subject and in seeing the motivation behind their enquiries. The choice of topics largely reflects the aspects of the field that we ourselves found not-immediately-obvious when encountering them for the first time.

Chapters 4 through 7 include the background material from the widely understood AGA that is either already established to be, directly or indirectly, relevant to QIT, or that we consider to be worthwhile making available to the QIT community. Even though most of this material can be found in existing books or surveys, many items are difficult to locate in the literature and/or are not readily accessible to outsiders. Here we have organized our exposition of AGA so that the applications follow as seamlessly as possible. Our presentation of some aspects of the theory is nonstandard. For example, we exploit the interplay between polarity and cone duality (outlined in Chapter 1 and with a sample application in Appendix D) to give novel and potentially useful insights. Chapters 4 (More convexity) and 5 (Metric entropy and concentration of measure) can be read independently of each other, but Chapters 6 and 7 depend on the preceding ones.

Chapters 8 through 12 discuss topics from the QIT proper, mostly via application of tools from the prior chapters. These chapters can largely be read independently of each other. For the most part, they present results previously published in journal articles, often (but not always) by the authors and their collaborators, most notably Cécilia Lancien, Elisabeth Werner, Deping Ye, Karol Życzkowski, and The Horodecki Group. A few results are byproducts of the work on this book (e.g., those in Section 9.4). The book also contains several new proofs. Some of them could arguably qualify as "proofs from *The Book*," for example the first proof of Størmer's Theorem 2.36 (Section 2.4.5) or the derivation of the sharp upper bound for the expected value of the norm of the complex Wishart matrix (Proposition 6.31).

Some statements are explicitly marked as "not proved here"; in that case the references (to the original source and/or to a more accessible presentation) are indicated in the "Notes and Remarks" section at the end of the chapter. Otherwise, the proof can be found either in the main text or in the exercises. There are over 400 exercises that form an important part of the book. They are diverse and aim at multiple audiences. Some are simple and elementary complements to the text, while others allow the reader to explore more advanced topics at their own pace. Still others explore details of the arguments that we judged to be too technical to be included in the main text, but worthwhile to be outlined for those who may need sharp versions of the results and/or to "reverse engineer" the proofs. All but the simplest exercises come with hints, collected in Appendix E. Appendices A to D contain material, generally of reference character, that would disrupt the narrative if included in the main text.

The back matter of the book contains material designed to simplify the task of the reader wanting to use the book as a reference: a guide to notation and a keyword index. The bibliography likewise contains back-references displaying page(s) where a given item is cited. For additional information and updates on or corrections to this book, we refer the reader to the associated blog at `https://aliceandbobmeetbanach.wordpress.com`. At the same time, we encourage—or even beg—the readers to report typos, errors, improvements, solutions to problems and the like to the blog. (An alternative path to the online post-publication material is by following the link given on the back cover of the book.)

While the initial impulse for the book was a teaching experience, it has not been designed, in its ultimate form, with a specific course or courses in mind. For starters, the quantity of material exceeds by far what can be covered in a single semester. However, a graduate course centered on the main theme of the book—the interface of QIT and AGA—can be easily designed around selected topics from Chapters 4–7, followed by selected applications from Chapters 8–12. While we assume at least a cursory familiarity with functional analysis (normed and inner product spaces, and operators on them, duality, Hahn–Banach-type separation theorems etc.), real analysis (L_p-spaces), and probability, deep results from these fields appear only occasionally and—when they do—an attempt is made to "soften the blow" by presenting some background via appropriately chosen exercises. Alternatively, most chapters could serve as a core for an independent study course. Again, this would be greatly facilitated by the numerous exercises and—mathematical maturity being more critical than extensive knowledge—the text will be accessible to sufficiently motivated advanced undergraduates.

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