MATH 322/422 Spring 2022

Assignment #10, notes and hints

Problem 11.3.2(a) Use the definition of a closed set given in class: *E is closed if for every sequence (x_k) in E such that $x_k \to c$ we have $c \in E$.*
This is a simple problem, but we want the argument to be "waterproof."

Problem 11.3.3 We already observed in class that $\mathbb{R}^n$ and $\emptyset$ are both open and closed. Show that if $E \subset \mathbb{R}^n$ and $E \neq \emptyset, E \neq \mathbb{R}^n$, then E can not be simultaneously open and closed (or, equivalently, E, E^c can not be both closed).
You may follow the following plan:
(i) Explain why there exists a segment

$$[a, b] = \{(1-t)a + tb : t \in [0,1]\} =: \mathbf{x}_t$$

such that $a \in E$ and $b \in E^c$.
(ii) Set $t_0 = \sup\{t : (\mathbf{x}_t \in E\}$ and show that each of the possibilities $\mathbf{x}_{t_0} \in E$ and $\mathbf{x}_{t_0} \in E^c$ leads to a contradiction if we assume that E and E^c are both closed.
Note : This may require considering several cases (for example, $t_0 = 0, t_0 = 1, t_0 \in (0,1)$) and it may be helpful to focus first on the case $n = 1$.

Problem 11.3.6(a) The arguments for whether the sets "have isolated points or whether they are dense or nowhere dense" should be very succinct (no more than one line each). For example, if a set has an isolated point, it is enough to supply that point.

Problem 11.4.1 Show only the equivalence of (a) and (b) (i.e., ignore part (c)), and you may use Exercise 11.2.6(c).

Problem 11.4.8 Consider only the sets from parts (c),(e),(f) of Exercise 11.4.7.