Additional Problem The objective of this problem is to show that any norm $\|\cdot\|'$ on $\mathbb{R}^n$ is equivalent to the Euclidean norm $\|\cdot\|$, that is, there exist constants $C, c > 0$ such that

$$\forall x \in \mathbb{R}^n \ c\|x\| \leq \|x\|' \leq C\|x\|. \qquad (EQ)$$

The constants C, c may depend on n and on the particular norm $\|\cdot\|'$, but they must be independent of x.

You may follow the steps indicated below. From this point on n and $\|\cdot\|'$ are considered fixed.

1° Let $S := \{x \in \mathbb{R}^n : \|x\| = 1\}$ (the *unit sphere*, often denoted S^{n-1}). Show that S is compact.

2° Let $(e_k)_{k=1}^n$ be the standard unit vector basis. To prove the second inequality in (EQ), we need to show that, for any sequence of coefficients (t_k) we have

$$\Big\| \sum_{k=1}^n t_k e_k \Big\|' \leq C\Big(\sum_{k=1}^n |t_k|^2\Big)^{1/2}$$

Show this by applying first the triangle inequality for the norm $\|\cdot\|'$ (in the form $\|\sum_{i=1}^N z_i\|' \leq \sum_{i=1}^N \|z_i\|'$, for any finite sequence (z_i) in $\mathbb{R}^n$) and then the Cauchy-Schwarz inequality (in the form $|\sum_{i=1}^N \alpha_i\beta_i| \leq (\sum_{i=1}^N |\alpha_i|^2)^{1/2}(\sum_{i=1}^N |\beta|^2)^{1/2}$, for any sequences of scalars (α_i), (β_i)).

3° Deduce that $f(x) := \|x\|'$ is a continuous function on $\mathbb{R}^n$ and hence that f attains its minimum and maximum on S.

4° Show that if $m := \min_{x\in S} f(x)$, then $m > 0$ (f is the same as in step 3°; remember that $\|\cdot\|'$ must satisfy the conditions in the definition of a norm).

5° Show that $c = m$ (the same m as in step 4°) works in (EQ). Consider separately the case $x = 0$ and $x \neq 0$; in the latter case consider $u := \frac{x}{\|x\|} \in S$.