MATH 322/422 Spring 2022

Assignment #8, additional problems and notes

ADDITIONAL PROBLEM A It was shown in class that the function E defined by $E(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ is a (strictly increasing) bijection E : verifying the initial value problem

$$E(0) = 1, \quad E'(x) = E(x). \tag{1}$$

It follows that E admits an inverse $L := E^{-1} : (0, \infty) \to \mathbb{R}$.

(a) State a general result that guarantees that L is a continuous strictly increasing function on its domain. You do not need to prove the result, just state it with the appropriate hypotheses.
Hint : A version of the result can be found in one of the exercises in Chapter 5.

(b) Derive a formula for $L'(x)$ from the above properties of $E(x)$. This should include citing the general result that that guarantees that L is differentiable. (Such result can be found in the book, but the best source is one of the files in the "MATH 321 handouts" Module on Canvas.)

ADDITIONAL PROBLEM B Define $e := E(1) = \sum_{k=0}^{\infty} \frac{1}{k!}$. Show rigorously that $2.71 < e < 2.72$.
Note : Your argument can use a calculator with 4 basic operations, but no more advanced functions.

ADDITIONAL PROBLEM C (a) Show that if $n \in \mathbb{N}$, then

$$0 < e\,n! - n! \sum_{k=0}^{n} \frac{1}{k!} < \frac{e}{n+1} \tag{2}$$

(b) Deduce that e is irrational.
Hints : For part (a), you may either use the Taylor-Lagrange theorem, or an elementary argument upper-bounding the remainder of the series defining e. For part (b), start by showing that if $r \in \mathbb{Q}$, then $rn!$ is an integer for all sufficiently large $n \in \mathbb{N}$.

Re Problem 10.5.7 *Note* : This is a problem about a function being analytic **at a particular point** (Definition 10.27). Another way to say that f is analytic at c is as follows: *There is $r > 0$ and a sequence of scalars (a_k) such that $f(x) = \sum_{n=0}^{\infty} a_k (x-c)^k$ if $|x - c| < r$.* In other words, f can be represented as a powers series (in powers of $(x - c)$) in some neighborhood of c. (*A posteriori*, that power series is necessarily the Taylor series for f at c.). This is a simpler concept than the definition of f being analytic on an open interval I that was given in class: *$f : I \to \mathbb{R}$ is analytic on I if it is analytic at every $c \in I$ (in the sense stated above).*