MATH 423, Fall 2022 Assignment #3 – additional problem and guidelines

Additional problem. Complete the details of the proof (sketched in class) of the following fact.

Let $[a, b] \subset \mathbb{R}$ be a (bounded closed) interval. Then

$$\inf \sum_i \ell(I_i) \;=\; |b-a|,$$

the infimum being over all sequences $(I_i)_i$ of open intervals such that $\bigcup_i I_i \supset [a, b]$ and $\ell(I)$ denoting the usual length of the interval I.

Section 1.3, Exercise 14. Here is an outline of a possible argument. Justify all steps!

1. Argue by contradiction, which leads to existence of $E \in \mathcal{M}$ with $\mu(E) = \infty$ such that $M := \sup\{\mu(F) : F \in \mathcal{M}, F \subset E, \mu(F) < \infty\}$ verifies $0 < M < \infty$.
2. Show that the *sup* in step 1. is attained on some $F_0 \subset E$.
3. What is $\mu(E \setminus F_0)$?
4. Derive a contradiction with the definition of M.

Section 1.5, Exercise 29. For the purpose of this exercise we assume the basic properties of the Lebesgue measure that were stated (but not fully proved) in class: m is a countably additive measure which is (i) defined on some σ-algebra $\mathcal{M} \supset \mathcal{B}_{\mathbb{R}}$ (ii) m is translation invariant (iii) if J is an interval with endpoints a, b, then $m(J) = |b - a|$. Next, you may find more convenient to replace the non-measurable set $N \subset [0, 1)$ defined on p. 20 of the textbook by its variant defined in class on Aug. 29, which verifies

$$[0,1) \subset \bigcup_{r\in\mathbb{Q}, |r|<1} (r+N) \subset (-1,2) \tag{1}$$

with the union being disjoint. You do not need to justify the property (1).
In part **b.**, restrict your attention to $E \subset [0, 1)$.