MATH 423 - Assignment #1 (revised 9/06/2022)

Due Tuesday, September 6, 2022

The primary objective (Exercises 1-4) of this assignmentis to recall and to connect the properties of the Riemann integral and the Riemann measure. One assumes all properties of the Riemann integral learned in an undergraduate analysis course and summarized on the attached handout. Some of the arguments were sketched in class, and you are supposed to provide the details.

For $A \subset [a,b]$, the (*outer*) Riemann measure of A is defined by

$$m^*(A) \quad = \quad \inf_{\bigcup J_i \supset A} \sum_i \ell(J_i), \tag{1}$$

where (J_i) is a *finite* family of closed intervals and $\ell(J)$ is the length of the interval J. One similarly defines the *inner* Riemann measure of A by

$$m_*(A) \quad = \quad \sup_{\bigcup J_i \subset A,\ J_i's \text{ disjoint}} \sum_i \ell(J_i),$$

If $m^*(A) = m_*(A)$, then A is said to be *Riemann measurable*, and the common value is then called the *Riemann measure* of A, or $m(A)$.

Note: One can show that the collection $\mathcal{R}$ of Riemann measurable subsets of $[a,b]$ is an algebra, and that m is a finitely additive measure on $\mathcal{R}$. (You are not asked to prove this.)

1. Show that the value of $m^*(A)$ remains unchanged if
(i) we require that the intervals (J_i) in the definition (1) are disjoint
(ii) we require that the intervals J_i in (1) are contained in $[a,b]$
(iii) we require that the family (J_i) in (1) is minimal, i.e., no proper subfamily of (J_i) covers A
(iv) we require that the intervals J_i are open (rather than closed).

2. Show that $m^*(A) = U(\chi_A)$ (see the handout) where χ_A is the indicator function of A (see p. 46 in the textbook). *Note* : This is relatively subtle; start by associating with a partition of $[a,b]$ the family of intervals of that partition that intersect A.

3. One can similarly show that $m_*(A) = L(\chi_A)$. (You do not need to justify this.) Explain why this implies that A is Riemann measurable if and only if $\chi_A \in \mathcal{R}[a,b]$, and that then $m(A) = \int_a^b \chi_A$.

4. Show that if $m^*(A) = 0$, then A is Riemann measurable.

5. Suppose $\mu : \mathcal{P}(\mathbb{R}) \to [0,\infty]$ is a *finitely* additive measure verifying
(1) $\mu(J) = \ell(J)$ for every interval $J \subset \mathbb{R}$
and which is
(2) translation invariant (i.e., $\mu(a + A) = \mu(A)$ for every $a \in \mathbb{R}$ and $A \subset \mathbb{R}$).
What is $\mu(N)$, where $N \subset [0,1)$ is the non-measurable set defined on p. 20 of the textbook (or better its variant defined in class on Aug. 29)?
Recall that it was shown that $\mu : \mathcal{P}(\mathbb{R}) \to [0,\infty]$ verifying (1) and (2) can not be *countably* additive, but the existence of such *finitely* additive measure was left open.