Riemann sums and Riemann integrals

Notation and setup Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function (say, $|f| \leq M$ on $[a, b]$), $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ a partition of $[a, b]$, i.e.,

$$a = x_0 < x_1 < \ldots < x_n = b$$

and $\dot{\mathcal{P}}$ is a *tagged* partition, i.e.,

$$\dot{\mathcal{P}} = (\mathcal{P}, \{t_1, t_2 \ldots, t_n\}),$$

where each $t_j \in [x_{j-1}, x_j]$. As usual, one defines *Riemann sums*

$$S(f, \dot{\mathcal{P}}) := \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1})$$

and calls $\|\dot{\mathcal{P}}\| = \|\mathcal{P}\| := \max_j |x_j - x_{j-1}|$ the *norm* (or *mesh*) of the partitions $\mathcal{P}, \dot{\mathcal{P}}$.

Fact *The following are equivalent*

(i) $f \in \mathcal{R}[a,b]$ (f *is* **Riemann integrable**), *i.e.* $\exists I \in \mathbb{R} \;\; \forall \epsilon > 0 \;\; \exists \delta > 0 \;\; \forall \dot{\mathcal{P}} \;\; \|\dot{\mathcal{P}}\| < \delta \;\; \Rightarrow |S(f, \dot{\mathcal{P}}) - I| < \epsilon$

(ii) $\forall \epsilon > 0 \;\; \exists \delta > 0 \;\; \forall \dot{\mathcal{P}}, \dot{\mathcal{Q}} \;\; \|\dot{\mathcal{P}}\| < \delta, \|\dot{\mathcal{Q}}\| < \delta \;\; \Rightarrow |S(f, \dot{\mathcal{P}}) - S(f, \dot{\mathcal{Q}})| < \epsilon$

(iii) $\exists I \in \mathbb{R} \;\; \forall (\dot{\mathcal{P}}_n) \;\; \|\dot{\mathcal{P}}_n\| \to 0 \;\; \Rightarrow S(f, \dot{\mathcal{P}}_n) \to I$

(iv) $\forall (\dot{\mathcal{P}}_n) \;\; \|\dot{\mathcal{P}}_n\| \to 0 \;\; \Rightarrow (S(f, \dot{\mathcal{P}}_n))$ *is a Cauchy sequence*

Define the lower und upper sums

$$L(f,\mathcal{P}) \;:=\; \sum_{j=1}^{n} m_j(x_j - x_{j-1})$$

$$U(f,\mathcal{P}) \;:=\; \sum_{j=1}^{n} M_j(x_j - x_{j-1})$$

where $m_j := \inf\{f(x) \;:\; x_{j-1} \le x \le x_j\}$,
$M_j := \sup\{f(x) \;:\; x_{j-1} \le x \le x_j\}$.

If $\dot{\mathcal{P}}$ is any *tagged* partition with the same intervals as $\mathcal{P}$, then

$$L(f,\mathcal{P}) \le S(f,\dot{\mathcal{P}}) \le U(f,\mathcal{P})$$

and $L(f,\mathcal{P}) = \inf S(f,\dot{\mathcal{P}})$ and $U(f,\mathcal{P}) = \sup S(f,\dot{\mathcal{P}})$ (inf, sup over all *tagged* partitions with the same intervals as $\mathcal{P}$).

Proposition. *f is Riemann integrable iff*

$$\sup_{\mathcal{P}} L(f,\mathcal{P}) = \inf_{\mathcal{P}} U(f,\mathcal{P}) \tag{v}$$

One calls $L(f) := \sup_{\mathcal{P}} L(f,\mathcal{P})$ and $U(f) := \inf_{\mathcal{P}} U(f,\mathcal{P})$ *the lower and upper integrals* of f over $[a,b]$. If they coincide, the common value is the Riemann integral of f, or $\int_a^b f$ (same as I in the *Fact* above). As in Fact (iii), $U(f) = \lim_n U(f,\mathcal{P}_n)$ for any $(\mathcal{P}_n)$ verifying $\|\dot{\mathcal{P}}_n\| \to 0$, and similarly for $L(f)$. It follows that modifying f at a finite number of points does not affect $U(f)$ and $L(f)$ and hence the integrability and the value of the integral. For example, if $\|\mathcal{P}_n\| < \delta$, then modifying f at k points changes $U(f,\mathcal{P}_n)$ at most by $2k \times \delta \times M$ (since at most $2k$ intervals of $\mathcal{P}_n$ are affected) and this can be made as small as we please by letting $\delta \to 0$.