MATH 424 – Assignment #1 – Extra Problems

Due Monday, February 8, 2021

A. Fix $n \in \mathbb{N}$. For $x = (x_j)_{j=1}^n \in \mathbb{R}^n$, we define three norms

$$\|x\|_1 := \textstyle\sum_{j=1}^n |x_j|, \qquad \|x\|_2 := \left(\textstyle\sum_{j=1}^n |x_j|^2\right)^{1/2}, \qquad \|x\|_\infty := \max_{1\le j\le n} |x_j|$$

(you do not need to prove that these are indeed norms). It was shown in class that the these norms are equivalent and that, more precisely, for every $x \in \mathbb{R}^n$,

(i) $\|x\|_\infty \le \|x\|_1 \le n\|x\|_\infty$
(ii) $\|x\|_2 \le \|x\|_1 \le \sqrt{n}\|x\|_2$
(iii) $\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\|x\|_\infty$

Show that the multiplicative constants implicit in the above inequalities are *optimal*.

This means, for example, that if $c > 1$, then the inequality $c\|x\|_\infty \le \|x\|_1$ will be false for some $x \in \mathbb{R}^n$. Likewise, if $C < n$ then the inequality $\|x\|_1 \le C\|x\|_\infty$ will be false for some $x \in \mathbb{R}^n$. (Equivalently, $\inf_{x\ne0} \frac{\|x\|_1}{\|x\|_\infty} = 1$ and $\sup_{x\ne0} \frac{\|x\|_1}{\|x\|_\infty} = n$.) Similarly for the inequalities in (ii) and (iii).

B. Let ℓ_∞ be the space of (infinite) real bounded sequences. That is, $x = (x_j)_{j=1}^\infty$ belongs to ℓ_∞ iff $\|x\|_\infty := \sup_{j\in\mathbb{N}} |x_j| < \infty$. It is well-known that $\|\cdot\|_\infty$ is a norm on ℓ_∞ (you do not need to prove that). Show that $(\ell_\infty, \|\cdot\|_\infty)$ is a *complete* normed space.