MATH 424 – SPRING 2021 – Set #7

Extra Exercise A Let $\mathcal{H}, \mathcal{K}$ be separable Hilbert spaces, so that orthonormal bases (ONB) of $\mathcal{H}$ and $\mathcal{K}$ are coutable. An operator $T \in L(\mathcal{H}, \mathcal{K})$ is called a *Hilbert-Schmidt* operator if, for some ONB (u_k) of $\mathcal{H}$ we have $\sum_k \|Tu_k\|^2 < \infty$. The quantity $\left(\sum_k \|Tu_k\|^2\right)^{1/2}$ is then called "the Hilbert-Schmidt norm" of T and denoted $\|T\|_{HS}$.

a. Show that $\|\cdot\|_{HS}$ is indeed a norm on the vector space $\{T \in L(\mathcal{H}, \mathcal{K}) : \|T\|_{HS} < \infty\}$.

b. Show that the expression $\sum_k \|Tu_k\|^2$ does not depend on the choice of the basis (u_k) and that $\|T\|_{HS} = \|T^*\|_{HS}$.
Hint: Let (u_k), (v_k) be two orthonormal bases of $\mathcal{H}$ and (e_j) an ONB of $\mathcal{K}$. Express $\|T\|_{HS}$ in terms of $\langle Tu_k, e_j\rangle$, and similarly for the basis (v_k).

c. Show that for any Hilbert-Schmidt operator T we have $\|T\| \leq \|T\|_{HS}$, where $\|\cdot\|$ is the usual *operator norm* defined on p.154.
Note: In the definition of a Hilbert-Schmidt operator we assumed boundedness, the above inequality effectively shows that that hypothesis is superfluous.

d. Show that every Hilbert-Schmidt operator T is compact.
Hint: You may use without proof the following rather easy fact (which really belongs to topology): *If (S_n) is a sequence in $L(\mathcal{X}, \mathcal{Y})$ which converges in norm (i.e., operator norm) to S, and if each S_n is compact, the so is S.* Now, if P_n is the orthogonal projection on to the space spanned by $u_1, u_2, \ldots, u_n$ (where (u_k) is the ONB dfrom the definition of $\|T\|_{HS}$), then TP_n is finite rank, hence compact (justify) and $\|T - TP_n\| \to 0$ (again, justify).

Extra Exercise B Let $\mathcal{H}, \mathcal{K}$ be separable Hilbert spaces, $B_{\mathcal{H}}$ – the unit ball of $\mathcal{H}$ and let $T \in L(\mathcal{H}, \mathcal{K})$ be a compact operator. Show that $T(B_{\mathcal{H}})$ is compact.
Hint: Use similar reasoning to the one that showed that such T attains its norm.

Extra Exercise C Let $\mathcal{H} = L_2([0,1], m)$ and let $T : \mathcal{H} \mapsto \mathcal{H}$ be defined by $(Tf)(t) = tf(t)$ (for $f \in \mathcal{H}$, $t \in [0,1]$). Show that T is an Hermitian operator and that T does not achieve its norm. (See p.172-173 for the definition of $L_2(X, \mu)$.)