PROCEEDINGS OF RESEARCH WORKSHOP ON BANACH SPACE THEORY

The University of Iowa

June 29 - July 31, 1981

Edited by Bor-Luh Lin

NETS OF GRASSMANN MANIFOLD AND ORTHOGONAL GROUP

Stanislaw J. Szarek
Instytut Matematyczny PAN
Warszawa, Poland
and
Ohio State University
Columbus, Ohio 43210

0. INTRODUCTION

All the main results of this paper can be summarized as follows. Let M be either the orthogonal group $O(m)$ or the Grassmann manifold $G_{n,m}$, equipped with some natural metric (we admit metrics generated by unitary ideal norms on $O(m)$ and their quotients on $G_{n,m}$, the operator metric on $O(m)$ and the usual Hausdorff distance of spheres on $G_{n,m}$—see Remark 5—being the most standard examples). Let, for some $\epsilon \in (0,\text{diam } M]$, $\mathcal{N}$ be an ϵ-net of M of minimal cardinality. Then

$$(*) \qquad (c \text{ diam } M/\epsilon)^{\dim M} \le \# \mathcal{N} \le (c' \text{ diam } M/\epsilon)^{\dim M},$$

where c and c' are universal constants (independent of M and ϵ). Of course universality of the constants of M is the crucial point.

In view of Lemma 2 below, (*) is equivalent to saying that the (normalized) Haar measure of a ball of radius ϵ is roughly $(C\epsilon/\text{diam } M)^{\dim M}$. Similar problems are extensively studied in the case of Riemannian manifolds, but little seems to be known in the case of non-Riemannian metrics. One may expect (*) to hold in much more general circumstances—perhaps under some geometrical assumptions on a manifold M.

A special case of (*) with $M = G_{n,m}$ and the Hausdorff distance of spheres as a metric (in fact its slightly weaker version, cf. Remark 11) was used in [5] to solve the finite-dimensional basis problem or, more precisely, to prove

Theorem 0 (1.1 in [5]): There is a constant $C > 0$ such that, for every n, there exists a $2n$-dimensional normed space B such that, for every projection P on B of rank n,

$$\|P : B \to B\| > C\sqrt{n}.$$

A similar result, giving slightly weaker estimate for $\|P\|$, was obtained independently and somewhat earlier by E. D. Gluskin (see [3]).

This paper is an extended version of section 8 from [5], where most of its results were announced without proof. The author does not know about any application of the general form of (*), but believes that it can be useful in many finite dimensional constructions involving ideal norms.

The following problem is not immediately related to the ones considered here, but may be of some interest.

Problem: Let M be the Banach-Mazur compact (i.e., the set of all n-dimensional normed spaces), d—the Banach-Mazur distance and $\epsilon > 1$. What is the minimal cardinality of an ϵ-net of M with respect to d?

All spaces and manifolds considered here are real. All arguments carry over to the complex case (i.e., $U(m)$, complex $G_{k,n}$, etc.)—word by word or with some simplifications. One must only remember to enter the topological (i.e., real) dimension of M into (*).

The author would like to thank the participants of the Banach Space Workshop, especially Professor G. Pisier and Professor J. Schäffer, for their interest and helpful remarks and the organizers of the Workshop for making that meeting possible.

1. NOTATION AND PRELIMINARY RESULTS

If K is a bounded subset of some metric space and ρ the corresponding metric, we shall denote by $N(K,\rho,\epsilon)$ the minimal cardinality of an ϵ-net of K with respect to ρ. As a rule, we shall consider only $\epsilon \in (0,\text{diam } K]$. If the metric space is actually a normed space with a norm $\|\cdot\|$ generated by an absolutely convex set K_0, we may write $N(K,\|\cdot\|,\epsilon)$ of $N(K,K_0,\epsilon)$ instead of $N(K,\rho,\epsilon)$.

The following properties of $N(\cdot,\cdot,\cdot)$ are either well known or easy to prove. The first two of them express $N(\cdot,\cdot,\cdot)$ in terms of a "volume ratio."

<u>Lemma 1</u>: Let K_0, $K \subset R^d$, K_0 absolutely convex. Then, for any $\epsilon > 0$,

$$N(K,K_0,\epsilon) \geq (\frac{1}{\epsilon})^d \frac{\text{vol } K}{\text{vol } K_0}.$$

Moreover, if $K_0 \subset K$ and K is convex, then, for every $\epsilon \in (0,3]$,

$$N(K,K_0,\epsilon) \leq (\frac{3}{\epsilon})^d \frac{\text{vol } K}{\text{vol } K_0}.$$

In particular $(1/\epsilon)^d \leq N(K,K,\epsilon) \leq (3/\epsilon)^d$ for absolutely convex K and $\epsilon \in (0,3]$.

<u>Lemma 2</u>: Let $M = G/H$ be a compact homogeneous space, μ—the normalized Haar measure on it, ρ—a G-invariant metric on M (not necessarily

Riemannian). Fix $x_0 \in M$ and let, for $\delta \geq 0$, $\varphi(\delta) \overset{\mathrm{df}}{=} \mu(\{x \in M : \rho(x,x_0) \leq \delta\})$ (clearly $\varphi(\delta)$ does not depend on x_0). Then, for every $\epsilon > 0$,

$$[\varphi(\epsilon)]^{-1} \leq N(M,\rho,\epsilon) \leq [\varphi(\epsilon/2)]^{-1}.$$

Lemma 3: If (K,ρ_0) and (M,ρ) are metric spaces, K_0 is a subset of K and $\Phi : K \longrightarrow M$ a map such that, for some ℓ, $L > 0$,

(a) $\Phi(K) = M$

(b) $\rho(\Phi(x),\Phi(y)) \leq L\rho_0(x,y)$ for $x,y \in K$

(c) $\rho(\Phi(x),\Phi(y)) \geq \ell\rho_0(x,y)$ for $x,y \in K_0$.

Then, for every $\epsilon > 0$,

$$N(K_0,\rho_0,2\epsilon/\ell) \leq N(M,\rho,\epsilon) \leq N(K,\rho_0,\epsilon/L).$$

If X is a normed space and $\|\cdot\|$ the corresponding norm, we shall denote by $B(X)$ or $B(\|\cdot\|)$ its unit ball, by $B_s(X)$, $B_s(\|\cdot\|)$ or simply B_s—the ball of radius s and center at 0. The following fact follows immediately from Lemmas 1 and 3.

Corollary 4: Let X be a normed space, $\dim X = d$, and (M,ρ)—a metric space. Let $\Phi : X \longrightarrow M$ satisfy, for some $R,L,r,\ell > 0$,

(a) $\Phi(B_R) = M$

(b) $\rho(\Phi(x),\Phi(y)) \leq L\|x-y\|$ for $x,y \in B_R$

(c) $\rho(\Phi(x),\Phi(y)) \geq \ell\|x-y\|$ for $x,y \in B_r$.

Then, for every $\epsilon \in (0,\operatorname{diam} M]$,

$$(c_0/\epsilon)^d \leq N(M,\rho,\epsilon) \leq (c_0'/\epsilon)^d,$$

where $c_0 = \frac{1}{2} r\ell$, $c_0' = 3RL$.

Lemma 5: Let (K_1,ρ_1) and (K_2,ρ_2) be metric spaces, $(K_1\times K_2,\rho_1\times\rho_2)$—their product with $(\rho_1\times\rho_2)((x_1,x_2)(y_1,y_2)) \overset{df}{=} \max\{\rho_1(x_1,y_1),\rho_2(x_2,y_2)\}$. Then, for every $\epsilon > 0$,

$$N(K_1,\rho_1,2\epsilon)N(K_2,\rho_2,2\epsilon) \le N(K_1\times K_2,\rho_1\times\rho_2,\epsilon) \le N(K_1,\rho_1,\epsilon)N(K_2,\rho_2,\epsilon).$$

Lemma 6: Let G be a compact group, H—its subgroup, ρ—an invariant metric on G, and ρ'—the corresponding quotient metric on G/H. Then, for every $\epsilon > 0$,

$$\frac{N(G,\rho,2\epsilon)}{N(H,\rho,\epsilon)} \le N(G/H,\rho',\epsilon) \le \frac{N(G,\rho,\epsilon/2)}{N(H,\rho,\epsilon)}.$$

As usually we shall denote by $\|\cdot\|_2$ the standard ℓ_2^m-norm on R^m, by $O(m)$ the group of orthogonal operators on R^m, by $SO(m)$ the subgroup of $O(m)$ consisting of all operators of determinant 1. Let $e_1,e_2,\cdots,e_m$ be the standard basis of R^m and let, for some $n \le m$, $F = \mathrm{span}\{e_1,e_2,\cdots,e_n\}$. Let $O(n,m) \overset{df}{=} \{V \in O(m) : VF = F\}$. Now define the Grassmann manifold $G_{n,m}$ as $O(m)/O(n,m)$—the set of left cosets of $O(n,m)$—and identify it with the set of all n-dimensional subspaces of R^m via $VO(n,m) \sim VF$.

Let $L(R^m)$ be the space of linear operators on R^m. Then every $T \in L(R^m)$ can be written in the form

$$(1) \qquad T = \sum_{i=1}^{m} \lambda_i \langle h_i,\cdot\rangle h_i'$$

where (h_i) and (h_i') are orthonormal bases of R^m with respect to the standard scalar product $\langle\cdot,\cdot\rangle$ and (λ_j) a nonincreasing sequence of nonnegative numbers. We shall refer to (1) as "the polar decomposition of T" and to $\lambda_1, \lambda_2, \cdots, \lambda_m$ as s-numbers of T.

Now let α be any unitary ideal norm on $L(R^m)$ (i.e., a norm such that $\alpha(UTV) = \alpha(T)$ for $T \in L(R^m)$, $U,V \in O(m)$ and $\alpha(S) = \|S\|_{op}$ if rank $S = 1$). It is clear that $\alpha(T)$ depends only on the s-numbers of T. Typical examples are the operator norm $\|\cdot\|_{op}$, the nuclear norm ν and the Hilbert-Schmidt norm. Let, for some fixed $n \le m$, ρ_α be the quotient metric on $G_{n,m}$. More specifically, we have, for $H_1, H_2 \in G_{n,m}$,

$$\rho_\alpha(H_1,H_2) = \inf_{V \in O(m), VH_1=H_2} \alpha(I-V), \tag{2}$$

regardless of whether we consider H_1, H_2 to be subspaces of R^m or cosets of $O(n,m)$ in $O(m)$. We shall skip the subscript in ρ_α if $\alpha = \|\cdot\|_{op}$.

It turns out that, for given H_1 and H_2, V in the above definition can be chosen independently of α. Moreover, we are able to describe s-numbers of such a V more precisely.

Let, for every subspace H of R^m, P_H denote the orthogonal projection onto H. Fix H_1, H_2 and consider the operator $P_{H_2^\perp} P_{H_1}$—let $\lambda_1, \lambda_2, \cdots, \lambda_m$ be its s-numbers (of course $\lambda_j = 0$ for $j > m/2$). Then the "optimal"—and independent of α—choice of V is such that the s-numbers of I-V are $\mu_1, \mu_1, \mu_2, \mu_2, \cdots$, where

$$\mu_j = \{2[1-(1-\lambda_j^2)^{\frac{1}{2}}]\}^{\frac{1}{2}}. \tag{3}$$

To show this let us consider the polar decomposition

$$P_{H_2^\perp}|H_1 = \sum_{j=1}^{n} \lambda_j \langle h_j, \cdot \rangle h'_j$$

and define, for $x \in H_1$, $Vx = \sum_{j=1}^{n} \langle h_j, x\rangle h'_j$. Similarly, considering the polar decomposition of $P_{H_2}|H_1^{\perp}$ (which has the same nonzero s-numbers as $P_{H_2^{\perp}}|H_1$) we may define V on $H_1^{\perp}$ and extend to R^m by linearity. It is not difficult to see that then the s-numbers of $I-V$ are $\mu_1, \mu_1, \mu_2, \mu_2, \cdots$, (μ_j) defined by (3).

On the other hand it is not difficult to see that such a V minimizes $\alpha(I-V)$ for any α. Notice that if, for some $\lambda \geq 0$ and some $x \in H_1$ (resp. $x \in H_1^{\perp}$), $\|x\|_2 = 1$,

$$\|P_{H_2^{\perp}} x\|_2 \geq \lambda \quad (\text{resp. } \|P_{H_2} x\| \geq \lambda),$$

then, for every $V \in O(m)$ such that $VH_1 = H_2$,

$$\|x - Vx\|_2 \geq \{2[1-(1-\lambda^2)^{\frac{1}{2}}]\}^{\frac{1}{2}},$$

which shows that the sequence of s-numbers of $I-V$ dominates $(\mu_1, \mu_1, \mu_2, \mu_2, \cdots)$.

Remark 5: We can define another metric on $G_{n,m}$ by $\rho'_\alpha(H_1, H_2) = \alpha(P_{H_2^{\perp}} P_{H_1})$. Since (3) clearly implies $\lambda_j \leq \mu_j \leq \sqrt{2}\,\lambda_j$, we have, for every α,

$$\rho'_\alpha \leq \rho_\alpha \leq 2\sqrt{2}\,\rho'_\alpha .$$

In the special case $\alpha = \|\cdot\|_{op}$, $\rho(H_1, H_2)$ is the same as the Hausdorff distance of $S_{m-1} \cap H_1$ and $S_{m-1} \cap H_2$, while $\rho'(H_1, H_2)$ is the same as the Hausdorff distance of the corresponding Euclidean balls. In particular it is clear from geometric considerations that $\rho' \leq \rho \leq \sqrt{2}\,\rho'$.

Consider the standard exponential map $\exp : L(R^m) \longrightarrow L(R^m)$ defined by $\exp T = \sum_{k=1}^{\infty} T^k/k!$ Let $A(R^m)$ and $S(R^m)$ denote the spaces of skew-symmetric and symmetric operators on R^m respectively. It is well known that $\exp(A(R^m)) = SO(m)$. Moreover, a more subtle argument shows that

$$\exp[B_\pi(A(R^m), \|\cdot\|_{op})] = SO(m). \tag{4}$$

To show (4), consider $U \in SO(m)$ and let $T_0 \in A(R^m)$ be such that $\exp T_0 = U$. Then T_0 can be written in the form

$$\begin{bmatrix} 0 & \lambda_1 & & & & & & & & \\ -\lambda_1 & 0 & & & & & & & 0 & \\ & & 0 & \lambda_2 & & & & & & \\ & & -\lambda_2 & 0 & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & 0 & \lambda_k & & & \\ & & & & & -\lambda_k & 0 & & & \\ & 0 & & & & & & 0 & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & 0 \end{bmatrix} \tag{5}$$

with respect to some orthonormal basis of R^m, where $k \le m/2$ and $\lambda_1, \cdots, \lambda_k$ are reals.

For any real λ let $\tilde{\lambda}$ be the (unique) real number satisfying (i) $(\lambda-\tilde{\lambda})/2\pi$ is an integer, (ii) $-\pi < \tilde{\lambda} \le \pi$. Define $T_1 \in A(R^m)$ to be represented by the matrix obtained from (5) by replacing each λ_j by $\tilde{\lambda}_j$. Then, in view of (i) and (ii), we have

(i)′ $\exp T_1 = \exp T_0 = U$,

(ii)′ $\|T_1\|_{op} = \max_{j\le k}|\lambda_j| \le \pi$,

as required.

2. THE ORTHOGONAL GROUP

Throughout this and the next section $\|\cdot\|$ will always stand for the operator norm with respect to the ℓ_2-norm. We start with the following

Proposition 6: There exist universal constants $c_1, c_1' > 0$ such that, for every positive integer m and $\epsilon \in (0,2]$,

$$(c_1/\epsilon)^d \le N(O(m),\|\cdot\|,\epsilon) \le (c_1'/\epsilon)^d,$$

where $d = m(m-1)/2 = \dim O(m)$.

Proof: Since $O(m)$ is geometrically a disjoint union of two copies of $SO(m)$, it is enough to prove Proposition 6 with $O(m)$ replaced by $SO(m)$. By Corollary 4 to do this it is sufficient to show that the conditions (a),(b),(c) of Corollary 4 are satisfied for $M = SO(m)$ and appropriate Φ, X, $\dim X = d$, with constants r, R, ℓ and L independent of m.

We choose $X = (A(R^m),\|\cdot\|)$, $\Phi = \exp$ and $R = \pi$, $L = e^{\pi}$, $r = .4$, $\ell = .5$. Then (a) follows immediately from (4). To prove (b) and (c) observe that, immediately from the definition of exp, we have

$$(6) \qquad (2-e^a)\|S-T\| \le \|\exp S - \exp T\| \le e^a\|S-T\|$$

for any $a \ge 0$ and $S,T \in L(R^m)$, $\|S\|,\|T\| \le a$. This shows Proposition 6. □

The following statement generalizes Proposition 6.

Proposition 7: There exist universal constants C_1, C_1' such that, for every unitary ideal norm α, positive integer m and $\epsilon \in (0, 2\alpha(I)]$,

$$(C_1\alpha(I)/\epsilon)^d \le N(O(m),\alpha,\epsilon) \le (C_1'\alpha(I)/\epsilon)^d, \tag{7}$$

where $d = m(m-1)/2$.

Proof: Observe first that the following generalization of (6) holds

$$(2-e^a)\alpha(S-T) \le \alpha(\exp S - \exp T) \le e^a\alpha(S-T) \tag{8}$$

for $S,T \in L(R^m)$, $\|S\|,\|T\| \le a$.

Hence, by Lemma 3 and similarly as in the proof of Proposition 5, it is enough to show that the condition (7) is satisfied with $O(m)$ replaced by $B \overset{df}{=} B(A(R^m),\|\cdot\|)$.

The right hand side inequality is easy. Indeed, for every $T \in L(R^m)$ we have

$$\alpha(T) \le \alpha(I)\|T\|.$$

In particular every $\epsilon/\alpha(I)$-net of B with respect to $\|\cdot\|$ is an ϵ-net with respect to α and it remains to apply Proposition 6 with $\epsilon/\alpha(I)$ instead of ϵ.

To prove the left hand side inequality we shall estimate from below

$$\text{vol } B/\text{vol } B(A(R^m),\alpha) \tag{9}$$

and then use the first part of Lemma 1. We have, for all $T \in L(R^m)$, $\alpha(I)\nu(T)/m \le \alpha(T)$ (ν is the nuclear form). Hence $B(A(R^m),\alpha) \subset m/\alpha(I)\ B_1$, where $B_1 \overset{df}{=} B(A(R^m),\nu)$. Therefore (9) is not less than

$$\frac{\text{vol } B}{\text{vol}(\frac{m}{\alpha(I)} B_1)} = \alpha(I)^d \frac{\text{vol } B}{\text{vol}(m\ B_1)}. \tag{10}$$

Now observe that $L(R^m) = A(R^m) \oplus S(R^m)$ and the coordinate projections are of norm 1 with respect to any unitary ideal norm (in particular, ν and $\|\cdot\|$). Hence, by Proposition 1.3, [6],

$$\text{vol } B \cdot \text{vol } B(S(R^m), \|\cdot\|) \geq 2^{-m^2} \text{ vol } B(L(R^m), \|\cdot\|).$$

On the other hand, also using Proposition 1.3, [6], we have

$$\text{vol } B_1 \cdot \text{vol } B(S(R^m), \nu) \leq \text{vol } B(L(R^m), \nu).$$

Combining these two estimates we get

$$(\tfrac{1}{2})^{m^2} \frac{\text{vol } B(L(R^m), \|\cdot\|)}{\text{vol}[m\ B(L(R^m), \nu)]} \leq \frac{\text{vol } B}{\text{vol}(m\ B_1)} \frac{\text{vol } B(S(R^m), \|\cdot\|)}{\text{vol}[m\ B(S(R^m), \nu)]} \leq \frac{\text{vol } B}{\text{vol}(m\ B_1)}, \tag{11}$$

the last inequality following trivially from the fact that $B(L(R^m), \|\cdot\|) \subset m\ B(L(R^m), \nu)$.

On the other hand, combining Proposition 3.1, [6], and Proposition, [2], (cf. the end of the proof of Proposition 8), one gets

$$\frac{\text{vol } B(L(R^m), \|\cdot\|)}{\text{vol}[m\ B(L(R^m), \nu)]} \geq b^{m^2}, \tag{12}$$

where $b \in (0,1)$ does not depend on m. Setting (10), (11), and (12) together we see that (9) is not less than $(b/2)^{m^2} \geq [(b/2)^4]^d$. This shows Proposition 7. □

3. THE GRASSMANN MANIFOLD

In this section we shall prove the following

Proposition 8: There exist universal constants c_2, c_2' such that, for every unitary ideal norm α, positive integers $n \le m$ and $\epsilon \in (0, D_\alpha]$,

$$(c_2 D_\alpha/\epsilon)^d \le N(G_{n,m}, \rho_\alpha, \epsilon) \le (c_2' D_\alpha/\epsilon)^d,$$

where $d = n(m-n) = \dim G_{n,m}$ and D_α is the diameter of $G_{n,m}$ with respect to ρ_α.

Remark 9: Let $k = \min\{n, m-n\}$ and let $G = \mathrm{span}\{e_1, \cdots, e_{2k}\} \subset R^m$. It follows easily from the consideration following (2) that $D_\alpha = \sqrt{2}\,\alpha(P_G)$.

Proof of Proposition 8: We shall proceed similarly as in the proofs of Propositions 6 and 7. Let $F = \mathrm{span}\{e_1, \cdots, e_n\}$ and let $A(n,m) = \{T \in A(R^m) : TF \subset F,\ TF^\perp \subset F^\perp\}$ (i.e., $T = (t_{ij}) \in A(n,m)$ iff $t_{ij} = 0$ for $i \le n < j$ and $j \le n < i$). It is well known that $\exp A(m,n) \subset O(m,n) \cap SO(m)$. Now let X be the orthogonal complement of $A(n,m)$ in $A(R^m)$ (i.e., $(t_{ij}) \in X$ iff $t_{ij} = 0$ for $i,j \le n$ and $i,j > n$) equipped with the operator norm and $\Phi : X \longrightarrow G_{n,m}$ be defined by $\Phi = q \circ \exp|_X$, where $q : O(m) \longrightarrow O(m)/O(n,m) = G_{n,m}$ is the quotient map.

I. Case $\alpha = \|\cdot\|$. We show that, for X and Φ just defined, the conditions (a)-(c) of Corollary 4 hold with $R = \pi$, $L = e^\pi$, $r = .05$, $\ell = .25$.

(a) First observe that, by Ex. 2(i), page 266, [4], $\exp X \exp A(n,m) = SO(m)$. Hence also $\exp X\, O(n,m) = O(m)$—in other words, $\Phi(X) = G_{n,m}$. Now (a) follows immediately if we observe that the construction from the proof of (4) yields $T_1 \in X$ if we start with $T_0 \in X$.

(b) Follows immediately from (6) and the fact that q, being a quotient map, is a contraction.

(c) We need some more properties of the exponential map. First observe that, immediately from the definition of exp,

$$\|\exp(S_1+S_2)-\exp S_1 \exp S_2\| \le 2e^{\|S_1\|+\|S_2\|}\|S_1\|\cdot\|S_2\| \tag{13}$$

for $S_1,S_2 \in L(R^m)$.

It is well known that exp admits a local inverse in the neighborhood of I, call it Ln. If $\|I-V\| < 1$, then $\operatorname{Ln} V$ is given by

$$\operatorname{Ln} V = \sum_{k=1}^{\infty} \frac{(I-V)^k}{k}$$

and hence

$$\|\operatorname{Ln} V\| \le \sum_{k=1}^{\infty} \|I-V\|^k/k = -\ln(1-\|I-V\|). \tag{14}$$

To prove (c) we must show that if $T_1,T_2 \in X$, $\|T_1\|,\|T_2\| \le r = .05$, then

$$\|(\exp T_1)V - \exp T_2\| \ge .25\|T_1-T_2\| \tag{15}$$

for all $V \in O(n,m)$. Given such T_1,T_2 fix $V \in O(n,m)$, for which the left hand side of (2) is minimal. Of course we must have

$$\|(\exp T_1)V - \exp T_2\| \le \|\exp T_1 - \exp T_2\| \le e^r\|T_1-T_2\|$$

by (6). Hence

$$\begin{aligned} \|I-V\| &\le \|I - \exp(-T_1)\exp T_2\| + \|V - \exp(-T_1)\exp T_2\| \\ &= \|\exp T_1 - \exp T_2\| + \|(\exp T_1)V - \exp T_2\| \le 2e^r\|T_1-T_2\|. \end{aligned} \tag{16}$$

In particular

$$\|I-V\| \le 2e^r \cdot 2r < .22. \tag{17}$$

Let $T = \operatorname{Ln} V$. Of course $V = \exp T$ and $T \in A(n,m)$. By (14),

$$\|T\| \le -\ln(1-\|I-V\|).$$

This combined with (17) shows that

$$\|T\| \le -\ln .78 < .25, \tag{18}$$

while on the other hand yields

$$\|T\| \le \ln \frac{1}{1-\|I-V\|} \le \frac{\|I-V\|}{1-\|I-V\|} \le \frac{\|I-V\|}{.78},$$

also with the use of (17). This and (16) imply

$$\|T\| \le \frac{2e^r}{.78} \|T_1-T_2\| \le 2.7\|T_1-T_2\|. \tag{19}$$

Coming back to (15) we have

$$\begin{aligned} \|(\exp T_1)V - \exp T_2\| &= \|\exp T_1 \ \exp T - \exp T_2\| \\ &\ge \|\exp(T_1+T) - \exp T_2\| - \|\exp(T_1+T) - \exp T_1 \ \exp T\|. \end{aligned} \tag{20}$$

Since $\|T_1\| \le .05$ and, by (18), $\|T_1+T\| \le \|T_1\| + \|T\| \le .05 + .25 = .3$, applying (6) with $a = .3$ we get

$$\begin{aligned} \|\exp(T_1+T) - \exp T_2\| &\ge (2-e^{.3})\|T_1+T-T_2\| \ge .65\|T_1-T_2+T\| \\ &\ge .65\|T_1-T_2\|, \end{aligned} \tag{21}$$

the last inequality following from the fact that $\|T'\| \le \|T'+T''\|$ for $T' \in X$, $T'' \in A(n,m)$.

On the other hand we have, by (13),

$$(22) \qquad \|\exp(T_1+T) - \exp T_1 \exp T\| \le 2e^{\|T_1\|+\|T\|} \cdot \|T_1\| \cdot \|T\|$$
$$\le 2e^{.3} \cdot .05\|T\| \le, \text{ by (19)}, \; 2e^{.3} \cdot .05 \cdot 2.7\|T_1-T_2\|$$
$$\le .4\|T_1-T_2\|.$$

Combining (20), (21) and (22) we get

$$\|(\exp T_1)V - \exp T_2\| \ge .65\|T_1-T_2\| - .4\|T_1-T_2\| = .25\|T_1-T_2\|.$$

This shows (15) and hence (c), completing the proof of Proposition 8 in the case $\alpha = \|\cdot\|$.

II. The case of general α. We proceed similarly as in the proof of Proposition 7. First we prove analogues of (a)-(c) from the case $\alpha = \|\cdot\|$ with $\|\cdot\|$ replaced by α where necessary (the balls B_r and B_R must be taken with respect to $\|\cdot\|$, not α, however). The argument is an almost exact translation of the case $\alpha = \|\cdot\|$: we must use (8) instead of (6), $\alpha(\exp(S_1+S_2) - \exp S_1 \exp S_2) \le 2e^{\|S_1\|+\|S_2\|}\|S_1\|\alpha(S_2)$ instead of (13), etc. One must only remember that the minimum in the definition of ρ_α is attained simultaneously for all α and hence one can work with both norms α, $\|\cdot\|$ and at the same time with just one $V \in O(n,m)$.

Hence, by Lemma 3 and similarly as in the proof of Proposition 7, it is enough to prove Proposition 8 with $G_{n,m}$ replaced by $B(X)$ (the unit ball in X in the operator norm $\|\cdot\|$). Once again the upper estimate is easy—it is enough to observe that $\alpha(T) \le \alpha(P_G)\|T\| = D_\alpha/\sqrt{2}\,\|T\|$, for $T \in X$, where P_G is the same as in Remark 9 and $D_\alpha = \text{diam}\, G_{n,m}$. To prove the lower estimate we must notice that, for $T \in X$,

$$\alpha(T) \geq \frac{\alpha(P_G)}{\nu(P_G)} \nu(T) = \frac{D_\alpha}{\sqrt{2}\cdot 2k} \nu(T),$$

where $k = \min\{n, m-n\}$, which—by Lemma 1 (cf. the proof of Proposition 7)—reduces the problem to showing that

$$\frac{\text{vol } B(X, \|\cdot\|)}{\text{vol}[k\cdot B(X,\nu)]} \geq \beta^d,$$

where $d = n(m-n) = \dim X$ and $\beta > 0$ is independent of m and n. Identifying an $m \times m$ matrix (t_{ij}) representing $T \in X$ with its $n \times (m-n)$ submatrix $(t_{ij})_{i \leq n, j > n}$ we see that in fact we need to estimate

$$\text{(23)} \qquad \frac{\text{vol}[B(L(R^n, R^{m-n}), \|\cdot\|)]}{\text{vol}[kB(L(R^n, R^{m-n}), \nu)]}$$

where $L(R^q, R^p)$ denotes the space of linear operators from R^q to R^p, $\|\cdot\|$ and ν—the operator and the nuclear norms with respect to the ℓ_2-norms. The problem is essentially the same as estimating (12). One could attempt to use the methods of [6] and [2] to prove analogues of Proposition 3.1 of [6] and Proposition of [2], but we choose another approach.

The following result is due to Santaló ([7]).

<u>Lemma 10</u>: Let $Y = (R^d, \|\cdot\|_0)$ be a normed space and Y^*—its dual identified with R^d via the standard scalar product. Then

$$\text{vol } B(Y) \cdot \text{vol } B(Y^*) \leq [\text{vol } B(\ell_2^d)]^2.$$

Lemma 10 shows that any estimate from below for $\text{vol } B(Y)$ always yields some estimate from above for $\text{vol } B(Y^*)$. Since the norms $\|\cdot\|$ and ν on $L(R^q, R^p)$ are dual with respect to the standard duality

$\langle S,T\rangle = \mathrm{tr}(S^*T)$, an easy computation shows that to conclude the proof of Proposition 8 it is enough to estimate from below the numerator of (23) in an appropriate way. In turn this can be done by using Lemma 3.1(2) from [1], as noticed by S. Kwapień and N. Tomczak-Jaegermann (cf. [5], proof of Fact 5.1). □

Remark 11: A slightly weaker version of Proposition 8 follows immediately from Proposition 7: We write $G_{n,m} = O(m)/O(n,m)$, observe that $O(n,m) \sim O(n) \times O(m-n)$ and then apply Lemmas 5 and 6 to estimate $N(G_{n,m},\rho_\alpha,\cdot)$ in terms of $N(O(k),\alpha,\cdot)$ for appropriate integers k. In the particular case of operator norm this shows that $N(G_{n,m},\rho,\epsilon)$ is, roughly speaking, of order $C^{m^2}\epsilon^{-n(m-n)}$—a result, which is not precise if n or $m-n$ is much smaller than m, but which was sufficient to prove Theorem 0 mentioned in section 0 (see [5] for details).

REFERENCES

1. S. Chevet, Series de variables aleatoires gaussiennes a valeurs dans $E \hat{\otimes}_\epsilon F$, Séminaire sur la geometrie des espaces de Banach 1977-78, Exposé No. XIX, École Polytechnique, Palaiseau.

2. E.D. Gluskin, The diameter of the Minkowski compactum is roughly equal to n, Functional Anal. Appl. 15(1981), 72-73.

3. E.D. Gluskin, Finite dimensional analogues of spaces without a basis, Doklady ANSSSR, to appear.

4. S. Helgason, Differential Geometry, Lie Groups and Symmetric Spaces, Academic Press, 1978.

5. S.J. Szarek, The finite dimensional basis problem with an appendix on nets of Grassmann manifold, to appear.

6. S.J. Szarek and N. Tomczak-Jaegermann, On nearly Euclidean decomposition for some classes of Banach spaces, Compositio Math. 40(1980), 367-385.

7. L.A. Santaló, Un invariante afin para los cuerpos convexos del espacio de n dimensiones, Port. Math. 8(1949), 155-161.