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On the Gram Matrices of Systems of Uniformly Bounded Functions

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Abstract—Let A_N, $N=1,2,\dots$, be the set of the Gram matrices of systems $\{e_j\}_{j=1}^N$ formed by vectors e_j of a Hilbert space H with norms $\|e_j\|_H \le 1$, $j=1,\dots,N$. Let $B_N(K)$ be the set of the Gram matrices of systems $\{f_j\}_{j=1}^N$ formed by functions $f_j \in L^\infty(0,1)$ with $\|f_j\|_{L^\infty(0,1)} \le K$, $j=1,\dots,N$. It is shown that, for any K, the set $B_N(K)$ is narrower than A_N as $N\to\infty$. More precisely, it is proved that not every matrix A in A_N can be represented as $A = B+\Delta$, where $B \in B_N(K)$ and Δ is a diagonal matrix.

In this paper we prove a theorem announced in [1]. First, we recall some well-known concepts and problems and introduce the notation used in what follows.

As usual, $\mathbb{R}^N$, $N\in\mathbb{N}$, is the N-dimensional Euclidean space with the inner product $\langle\cdot,\cdot\rangle$ and Euclidean norm $|\cdot|$. Let

$$B^N = \{x\in\mathbb{R}^N : |x|\le 1\}, \qquad S^{N-1} = \{x\in\mathbb{R}^N : |x| = 1\}.$$

For a set of vectors $\mathcal{Z} = \{z_i\}_{i=1}^p \subset \mathbb{R}^N$, denote by $G_\mathcal{Z}$ the Gram matrix

$$G_\mathcal{Z} = \{\langle z_i, z_k\rangle\}_{i,k=1}^p.$$

Let O^N be the rotation group of $\mathbb{R}^N$. Clearly, for $\sigma\in O^N$ and $\sigma\mathcal{Z} = \{\sigma z_i\}_{i=1}^p$, we have

$$G_{\sigma\mathcal{Z}} = G_\mathcal{Z}. \tag{1}$$

Below, we consider the Hilbert space $L^2(0,1)$ and the smaller space $L^\infty(0,1)$. Denote by $(\cdot,\cdot)$ the inner product

$$(f,g) = \int_0^1 fg\,d\mu, \qquad f,g\in L^2(0,1),$$

where μ is the Lebesgue measure on $(0,1)$.

Finally, for $N=1,2,\dots$, we introduce an N-dimensional space of piecewise constant functions:

$$D_N = \left\{ f\in L^2(0,1)\colon\ f(x) = \mathrm{const} = c_j \text{ for } x \in \left(\frac{j-1}{N}, \frac{j}{N}\right),\ 1\le j\le N \right\}.$$

The following problem arises naturally in analysis, geometry, and applied mathematics: Given a set

$$\mathcal{Z} = \{z_1,\dots,z_N\} \subset B^N \tag{2}$$

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of N vectors in $\mathbb{R}^N$ ($N = 1, 2, \dots$), determine whether there exists a set $\{f_j\}_{j=1}^N \subset L^\infty(0,1)$ of functions such that their uniform norms are bounded by an absolute constant and

$$(f_j, f_k) = \langle z_j, z_k \rangle, \qquad 1 \le j, k \le N. \tag{3}$$

Using the well-known inequality

$$\mathcal{M}\left\{x \in S^{N-1} \colon \|x\|_{\ell_\infty^N} \ge \frac{C \ln^{1/2} N}{N^{1/2}}\right\} \le \frac{1}{N^2},$$

where $C > 0$ is an absolute constant, $\mathcal{M}$ is the normalized Lebesgue measure on the sphere S^{N-1}, and $\|x\|_{\ell_\infty^N} = \max_{1\le i\le N} |x(i)|$ for $x = \{x(i)\}_{i=1}^N \in \mathbb{R}^N$, we can immediately conclude that, for any set of the form (2), there exists a rotation $\sigma \in O^N$ such that

$$\|\sigma z_j\|_{\ell_\infty^N} \le \frac{C \ln^{1/2}(N+1)}{N^{1/2}}, \qquad j = 1, 2, \dots, N. \tag{4}$$

Due to the obvious relationship between the Euclidean spaces $\mathbb{R}^N$ and D_N, we directly find from (4) that, for any set of vectors of the form (2), there exist functions $\{f_j\}_{j=1}^N \subset D_N \subset L^\infty(0,1)$ such that (3) holds and

$$\max_{1\le j\le N} \|f_j\|_{L^\infty} \le C \ln^{1/2}(N+1). \tag{5}$$

Estimate (5) is order-sharp. More precisely, if the vectors $\{z_i\}_{i=1}^N$ form a 1-net (with respect to the norm $|\cdot|$) on a sphere of a subspace $L \subset \mathbb{R}^N$ of dimension $\dim L \ge c \ln N$, then each function set $\{f_j\}_{j=1}^N$ with property (3) satisfies

$$\max_{1\le j\le N} \|f_j\|_{L^\infty} \ge c_1 \ln^{1/2} N \tag{6}$$

(see, e.g., Lemma 3 in [1]).

It is of interest to find conditions on the set (2) under which inequality (5) can be replaced by a stronger estimate

$$\|f_j\|_{L^\infty} \le K, \qquad j = 1, 2, \dots, N, \tag{7}$$

where K is an absolute constant.

The first partial results of this kind were obtained by Menchoff as early as the 1930s in connection with extending a system of functions defined on the interval $[-1, 0]$ to a uniformly bounded orthonormal system on $[-1, 1]$ (see [2]). Note that the following question, raised by Olevskii [3, p. 58], still remains open: Does there exist a function system $\{f_j\}_{j=1}^N$ with properties (3) and (7) if the Gram matrix of the set (2) satisfies

$$I_N - G_{\mathcal{Z}} \ge 0$$

(here, I_N is the identity matrix of order N, and $B \ge 0$ means that the quadratic form generated by a symmetric matrix B is positive semidefinite)?

The problems considered here are directly related to the classical results on the factorization of linear operators in functional analysis. For example, the Grothendieck and Pietsch theorems imply that, for any set $\mathcal{Z}$ of the form (2), there exists a function system $\mathcal{F} = \{f_j\}_{j=1}^N$ with $\|f_j\|_{L^\infty} \le \sqrt{\pi/2}$ for $j = 1, \dots, N$ and such that

$$G_{\mathcal{Z}} = G_{\mathcal{F}} - \Delta, \qquad \Delta \ge 0$$

(for more detail, see [4, Ch. 5] and, in particular, Theorem 5.10 there). In view of this result, it is natural to consider whether the Gram matrix of an arbitrary set (2) can be represented in the form

$$G_{\mathcal{Z}} = G_{\mathcal{F}} - \Delta, \qquad \Delta \text{ is a diagonal matrix},$$

where the functions in $\mathcal{F}$ satisfy estimate (7). This question was also raised by Megretski in connection with applied problems in quadratic programming (see [5]). It turns out that the answer to this question is negative.

Theorem. *For $N = 1, 2\dots$, there exists a set $\mathcal{Z} = \mathcal{Z}(N)$ of the form (2) such that any function system $\{f_j\}_{j=1}^N$ with the property*

$$\langle z_j, z_k\rangle = (f_j, f_k) \qquad \textit{for} \quad 1 \le j < k \le N \tag{8}$$

satisfies the inequality

$$\max_{1\le j\le N} \|f_j\|_{L^\infty} \ge c(\ln N)^{1/4},$$

where $c > 0$ is an absolute constant.

Proof. For $N = 1, 2, \dots$, define K_N as the infimum of the numbers K such that, for any set of vectors of the form (2), there exist functions $\{f_j\}_{j=1}^N$ with $\|f_j\|_{L^\infty} \le K$, $j = 1, 2, \dots, N$, that satisfy (8). We need to verify that $K_N \ge c(\ln N)^{1/4}$. For $N = 1, 2, \dots$, we also define M_N as the infimum of the numbers M such that, for any set of vectors of the form (2), there exist functions $\{f_j\}_{j=1}^N$ with $\|f_j\|_{L^\infty} \le M$, $j = 1, \dots, N$, that satisfy (8) and

$$\langle z_j, z_j\rangle \le (f_j, f_j), \qquad j = 1, 2, \dots, N. \tag{9}$$

Let us verify the inequality

$$K_{2N} \ge M_N, \qquad N = 1, 2, \dots, \tag{10}$$

and then show that

$$M_N \ge c'(\ln N)^{1/4}, \qquad c' > 0, \quad N = 1, 2, \dots. \tag{11}$$

This will prove the theorem. To verify (10), for a given set $\mathcal{Z}$ of the form (2), we construct a set of vectors

$$\mathcal{W} = (w_1, \dots, w_{2N}) = (z_1, z_1, z_2, z_2, \dots, z_N, z_N) \subset B^N \subset B^{2N}$$

and (see the definition of K_N), for a given $\varepsilon > 0$, find functions $g_1, \dots, g_{2N}$ such that $(g_i, g_k) = \langle w_i, w_k\rangle$ if $i \ne k$ and $\|g_i\|_{L^\infty} \le K_{2N} + \varepsilon$ for $i = 1, \dots, 2N$. Then, in particular, we have

$$|z_k|^2 = \langle z_k, z_k\rangle = \langle w_{2k-1}, w_{2k}\rangle = (g_{2k-1}, g_{2k}) \le \|g_{2k-1}\|_{L^2} \cdot \|g_{2k}\|_{L^2}$$

for $k = 1, 2, \dots, N$; i.e.,

$$\max\{\|g_{2k-1}\|_{L^2}, \|g_{2k}\|_{L^2}\} \ge |z_k|, \qquad k = 1, 2, \dots, N.$$

Hence, taking g_{2k-1} or g_{2k} as f_k, we ensure that condition (9), together with (8) and the estimate $\|f_k\|_{L^\infty} \le K_{2N} + \varepsilon$, is satisfied. Thus, we have proved inequality (10).

Let us prove (11). Below, for a set of vectors $\{v_j\}_{j=1}^N$ in a Euclidean space, we denote by $\operatorname{span}\{v_1, \dots, v_N\}$ and $\{v_1, \dots, v_N\}^\perp$ the linear span and its orthogonal complement, respectively. We fix a set of the form (2) (arbitrary for the time being) and an $\varepsilon \in (0, 1)$ and choose functions $\{f_j\}_{j=1}^N$ with

$$\|f_j\|_{L^\infty} \le M_N + \varepsilon, \qquad j = 1, \dots, N, \tag{12}$$

so that (8) and (9) hold. Due to the canonical isometric embeddings $\mathbb{R}^N \subset \mathbb{R}^{2N}$ and $B^N \subset B^{2N}$, we can find vectors u_j, $j=1,\dots,N$, in $\mathbb{R}^{2N}$ such that

$$u_j \in \{z_1,\dots,z_N\}^{\perp}, \quad j=1,\dots,N, \qquad \langle u_i,u_j\rangle = 0 \quad \text{for} \quad i\neq j \tag{13}$$

and

$$\langle u_j,u_j\rangle + \langle z_j,z_j\rangle = (f_j,f_j), \qquad j=1,\dots,N.$$

Setting $w_j = z_j + u_j$, $j=1,\dots,N$, and taking into account (13), we obtain

$$\langle w_i,w_j\rangle = (f_i,f_j) \qquad \forall (i,j), \tag{14}$$

which implies that the mapping

$$\mathbb{R}^{2N} \supset \operatorname{span}\{w_j,\ j=1,\dots,N\} \xrightarrow{\varphi} \operatorname{span}\{f_j,\ j=1,\dots,N\} \subset L^2(0,1)$$

is well defined by the equalities $\varphi(w_j) = f_j$, $j=1,\dots,N$, and is an isometry. Obviously, this mapping can be extended to an isometry (also denoted by φ) between $\mathbb{R}^{2N}$ and a $2N$-dimensional subspace of $L^2(0,1)$.

Let

$$g_j = \varphi(z_j), \qquad h_j = \varphi(u_j), \qquad j=1,\dots,N.$$

Then,

$$g_j, h_j \in L^2, \qquad g_j + h_j = f_j \in L^\infty(0,1), \qquad j=1,\dots,N. \tag{15}$$

Moreover,

$$(g_i,h_j) = 0 \quad \forall (i,j), \qquad (h_i,h_j) = 0 \quad \text{if} \quad i\neq j. \tag{16}$$

Let $L = \operatorname{span}\{g_1,\dots,g_N\}$. Then, φ isometrically maps the subspace

$$E = \operatorname{span}\{z_1,\dots,z_N\} \subset \mathbb{R}^{2N}$$

onto L. The set $\{z_j\}_{j=1}^N \subset B^N \subset B^{2N}$ has been arbitrary thus far. Now, we specify z_j, $j=1,\dots,N$, assuming (without loss of generality) that N is sufficiently large.

Let $E\subset \mathbb{R}^N \subset \mathbb{R}^{2N}$ be a subspace of dimension $\ell = [c_0 \ln N]$. Here, the absolute constant $c_0>0$ is chosen so small that there exists a 1-net $\mathcal{V} = \{v_\nu\}_{\nu=1}^{\nu_0}$ on the sphere S_E (with respect to the Euclidean norm $|\cdot|$), where the number of elements ν_0 satisfies

$$\nu_0 \le 3^\ell \le N^{1/4} \tag{17}$$

($[x]$ stands for the integer part of x). Then, for the convex hull, we have

$$\operatorname{conv}\{v_\nu\}_{\nu=1}^{\nu_0} \supset \frac{1}{2}\, B_E \equiv \frac{1}{2}\,(B^{2N}\cap E). \tag{18}$$

The required set $\mathcal{Z} = \{z_j\}_{j=1}^N$ of type (2) is constructed by repeating each element v_ν of $\mathcal{V}$ s times, $s = [N^{3/4}]$, and by supplementing (if necessary) the resulting set in an arbitrary manner to obtain a system of N elements in B_E. Using the notation introduced earlier, by virtue of (18), we conclude that

$$\operatorname{conv}\{g_j\}_{j=1}^N \supset \frac{1}{2}\,\{f\in L\colon\ \|f\|_{L^2}\le 1\} \equiv \frac{1}{2}\,B_2^L \equiv \frac{1}{2}\,(B_2\cap L), \tag{19}$$

where B_2 is the unit ball in $L^2(0,1)$. Moreover, relations (12), (15), and (16) guarantee that

$$g_j \subset (M_N+\varepsilon)P_L(B_\infty), \qquad j=1,\dots,N, \tag{20}$$

where P_L is the orthogonal projection from $L^2(0,1)$ onto L and

$$B_\infty = \{f\in L^2(0,1)\colon\ \|f\|_{L^\infty}\le 1\}.$$

It follows from (20) and (19) that

$$B_2^L \subset 2(M_N+\varepsilon)P_L(B_\infty). \tag{21}$$

In turn, (21) implies that

$$\|g\|_{L^1}\le \|g\|_{L^2}\le 2(M_N+\varepsilon)\|g\|_{L^1} \tag{22}$$

for any $g\in L$ (the left inequality in (22) is true for any $g\in L^2$). Indeed (see (21)), if $g\in L$ with $\|g\|_{L^2}=1$ can be represented as

$$g=f+h, \qquad \|f\|_{L^\infty}\le 2(M_N+\varepsilon), \quad h\perp g,$$

then

$$0=(g,h)=(g,g-f)=1-(g,f).$$

This implies that $\|g\|_{L^1}\cdot\|f\|_{L^\infty}\ge 1$. Hence, $\|g\|_{L^1}\ge[2(M_N+\varepsilon)]^{-1}$, which proves (22).

Thus, we have a subspace L of $L^2(0,1)$ with $\dim L=\ell=[c_0\ln N]$ such that

(i) if $g\in L$, then (22) is true;

(ii) for each element ω of the 1-net Ω on the sphere $S_2^L=\{g\in L\colon \|g\|_{L^2}=1\}$, there exist functions $h_{j_1},\dots,h_{j_s}\in L^\perp$ satisfying $(h_{j_p},h_{j_q})=0$ for $p\ne q$ and such that

$$\|\omega+h_{j_p}\|_{L^\infty}\le M_N+\varepsilon, \qquad p=1,\dots,s. \tag{23}$$

Moreover, for $p=1,\dots,s$, we have

$$\|h_{j_p}\|_{L^2}^2=\|\omega+h_{j_p}\|_{L^2}^2-\|\omega\|_{L^2}^2\le (M_N+\varepsilon)^2-1\le (M_N+\varepsilon)^2 \tag{24}$$

(the set $\varphi(\mathcal{V})$ should be used as Ω). It follows from (23) that

$$\left\|\omega+\frac{1}{s}\,(h_{j_1}+\ldots+h_{j_s})\right\|_{L^\infty}\le M_N+\varepsilon, \tag{25}$$

while the pairwise orthogonality of the functions h_j and estimate (24) imply

$$\left\|\frac{1}{s}\,(h_{j_1}+\ldots+h_{j_s})\right\|_{L^2}\le \frac{M_N+\varepsilon}{\sqrt{s}}. \tag{26}$$

It now follows from (25) and (26) that

$$\omega\in (M_N+\varepsilon)B_\infty+\frac{M_N+\varepsilon}{\sqrt{s}}\,B_2 \tag{27}$$

for all $\omega\in\Omega$. Inclusion (27) can be transferred to the convex hull of Ω. Finally, we have (see also (22))

$$g\in L, \quad \|g\|_{L^2}\le 1 \quad\Longrightarrow\quad \begin{cases} \|g\|_{L^1}\ge[2(M_N+\varepsilon)]^{-1}\|g\|_{L^2}, \\ g\in 2(M_N+\varepsilon)B_\infty+2\,\dfrac{M_N+\varepsilon}{\sqrt{s}}\,B_2. \end{cases} \tag{$*$}$$

Let us derive a lower estimate for M_N from $(*)$. To this end, we need the following elementary and easy-to-check result.

Lemma. *Let $g \in L^2(0,1)$ and $f \in L^\infty(0,1)$; let $\alpha > 0$ and $\beta > 0$ be such that*

$$\alpha \le \|g\|_{L^1} \le \|g\|_{L^2} \le 1, \qquad \beta \le \|f\|_{L^1} \le \|f\|_{L^\infty} \le 1.$$

Set

$$A = A(g,\alpha) = \left\{t\colon\ \frac{\alpha}{3} \le |g(t)| \le \frac{3}{\alpha}\right\}, \qquad B = B(f,\beta) = \left\{t\colon\ |f(t)| \ge \frac{\beta}{2}\right\}.$$

Then,

$$\int\limits_A |g|\,d\mu \ge \frac{\alpha}{3}, \qquad \mu(A) \equiv \operatorname{meas} A \ge \left(\frac{\alpha}{3}\right)^2, \tag{28}$$

$$\int\limits_B |f|\,d\mu \ge \frac{\beta}{2}, \qquad \operatorname{meas} B \ge \frac{\beta}{2}.$$

Let $\psi_1,\dots,\psi_\ell$ be an arbitrary orthonormal basis in L. Applying the above lemma with $\alpha = [2(M_N+\varepsilon)]^{-1}$ to the functions ψ_r, $r = 1,\dots,\ell$, from $L^2(0,1)$, we find the corresponding sets A_r, $r = 1,2,\dots,\ell$ (see (28)). Let

$$\Psi = \sum_{r=1}^{\ell} |\psi_r|\,\chi_{A_r},$$

where χ_A denotes the characteristic function of a set A. Then,

$$\|\Psi\|_{L^\infty} \le 6(M_N+\varepsilon)\ell, \qquad \int\limits_0^1 \Psi\,d\mu \ge \frac{\ell}{6(M_N+\varepsilon)}\,.$$

Applying the second part of the lemma to the function $f = \Psi[6(M_N+\varepsilon)\ell]^{-1}$ and $\beta = [36(M_N+\varepsilon)^2]^{-1}$, we find a set $B \subset (0,1)$ with $\operatorname{meas} B \ge [72(M_N+\varepsilon)^2]^{-1}$ such that

$$\sum_{r=1}^{\ell} |\psi_r(t)| \ge \frac{\ell}{12(M_N+\varepsilon)} \tag{29}$$

for $t \in B$ and, moreover,

$$\int\limits_B \sum_{r=1}^{\ell} |\psi_r(t)|\,d\mu \ge \frac{\ell}{12(M_N+\varepsilon)}\,.$$

It follows from (29) that there exists a subset $B_0 \subset B$ with

$$\operatorname{meas} B_0 \ge 2^{-\ell+1} \operatorname{meas} B \ge 2^{-\ell} \frac{1}{[6(M_N+\varepsilon)]^2}$$

and a set of signs $\varepsilon_r = \pm 1$, $r = 1,\dots,\ell$, such that

$$\left|\frac{1}{\sqrt{\ell}} \sum_{r=1}^{\ell} \varepsilon_r \psi_r(t)\right| \ge \frac{\sqrt{\ell}}{12(M_N+\varepsilon)}\,, \qquad t \in B_0. \tag{30}$$

Let

$$B_2^L \ni g(t) = \frac{1}{\sqrt{\ell}} \sum_{r=1}^{\ell} \varepsilon_r \psi_r(t).$$

Note that, by virtue of (30), every function $f \in 2(M_N+\varepsilon)B_\infty$ satisfies

$$|g(t)-f(t)| \geq \frac{\sqrt{\ell}}{12(M_N+\varepsilon)} - 2(M_N+\varepsilon) \equiv \gamma \qquad \text{for} \ \ t \in B_0. \tag{31}$$

Now, employing (31), we show that the inequality

$$(M_N+\varepsilon) \leq \frac{1}{10}\,[c_0 \ln N]^{1/4} = \frac{1}{10}\,\ell^{1/4} \tag{32}$$

contradicts $(*)$; hence,

$$M_N \geq \frac{1}{10}\,[c_0 \ln N]^{1/4}, \qquad N \geq N_0,$$

which completes the proof of estimate (11). Indeed, inequality (32) implies that $\gamma \geq \frac{1}{2}\,[c_0 \ln N]^{1/4}$. Therefore, for each $f \in 2(M_N+\varepsilon)B_\infty$,

$$\|g-f\|_{L^2} \geq \gamma\,(\operatorname{meas} B_0)^{1/2} \geq \frac{1}{2}\,2^{-\ell/2} > \frac{2(M_N+\varepsilon)}{\sqrt{s}}\,. \tag{33}$$

This contradicts the second relation in $(*)$. While verifying the last inequality in (33), we took into account that $s=[N^{3/4}]$, $2^\ell < 3^\ell \leq N^{1/4}$ (see (17)), and, hence (see also (5)),

$$s > 20(C \ln^{1/2} N)^2 \cdot 2^\ell \geq [4(M_N+\varepsilon)]^2 \cdot 2^\ell$$

if the number N is sufficiently large.

The theorem is proved.

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