# Saturating Constructions for Normed Spaces

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**Abstract**

We prove several results of the following type: given finite dimensional normed space $V$ there exists another space $X$ with $\log \dim X = O(\log \dim V)$ and such that every subspace (or quotient) of $X$, whose dimension is not "too small," contains a further subspace isometric to $V$. This sheds new light on the structure of such large subspaces or quotients (resp., large sections or projections of convex bodies) and allows to solve several problems stated in the 1980s by V. Milman.

## 1 Introduction

Much of geometric functional analysis revolves around the study of the family of subspaces (or, dually, of quotients) of a given Banach space. On the one hand, one wants to detect some possible regularities in the structure of subspaces which might have not existed in the whole space; on the other hand, one tries to determine to what degree the structure of the space can be recovered from the knowledge of its subspaces. In the finite dimensional case there is a compelling geometric interpretation: a normed space is determined by its unit ball, a centrally symmetric convex body, subspaces correspond to sections of that body, and quotients to projections. A seminal result in this direction was the 1961 Dvoretzky theorem: every convex body of (large) dimension $n$ admits central sections which are approximately ellipsoidal and whose dimension $k$ is at least of order $\log n$. (A proof giving the logarithmic order for $k$ was given by Milman in 1971 ([M1], or see [MS1].) In other words, the most regular $k$–dimensional structure, i.e., the Euclidean space, is present in every normed space of sufficiently high dimension. Another interpretation

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of the same phenomenon is as follows. From the information-theoretic point of view, the Euclidean spaces have minimal complexity. Thus by passing to a subspace or a quotient – of dimension much smaller than that of the original space, but still a priori arbitrarily large – one may, in a sense, lose all the information about a space.

It has been known that the $\log n$ estimate in the Dvoretzky theorem, while optimal in general, can be substantially improved for some special classes of spaces/convex bodies. Still, it was a major surprise when V. Milman ([M2]) discovered in the mid 1980's that *every* $n$-dimensional normed space admits a *subspace of a quotient* which is "nearly" Euclidean and whose dimension is $\geq \theta n$, where $\theta \in (0,1)$ is arbitrary (of course, the exact meaning of "nearly" depends on $\theta$). Moreover, as the first step of his approach he proved that every $n$-dimensional normed space admits a "proportional dimensional" quotient of bounded volume ratio, a volumetric characteristic of a body closely related to cotype properties (we refer to [MS1], [T] and particularly [P] for definitions of these and other basic notions and results that are relevant here). This showed that one can get a very essential regularity in a global invariant of a space by passing to a quotient or a subspace of dimension, say, approximately $n/2$ (while of course also losing a substantial part of the local information). It was thus natural to expect that similar statements may be true for other related characteristics. This line of thinking was exemplified in a series of problems posed by Milman in his 1986 ICM Berkeley lecture [M3]. A positive answer to any of them would have many important consequences in geometric functional analysis and high dimensional convexity.

Until the present work no techniques were available to approach those problems nor, more to the point, to determine what the correct questions were. In this paper we elucidate this circle of ideas and, in particular, we answer Problems 1-3 from [M3] negatively. Moreover, we ascertain the following phenomenon: passing to large subspaces or quotients can not, in general, erase $k$-dimensional features of a space if $k$ is below certain threshold value, which depends on the dimension of the initial space and the exact meaning of "large." For example, the threshold dimension is (at least) of order $\sqrt{n}$ in the context of "proportional" subspaces or quotients of $n$-dimensional spaces. "Impossibility to erase" may mean, for instance, that *every* subspace of our $n$-dimensional space of dimension $\geq n/2$ will contain a further 1-complemented subspace isometric to a preassigned (but a priori arbitrary) $k$-dimensional space $W$. In a sense, the original space is "saturated" with copies of $W$. However, this is really just a sample result; our methods are very flexible – one may say, canonical – and clearly can be used to treat other similar questions.

We employ probabilistic arguments, the most basic idea of which goes back to Gluskin [G] (see also [MT1] for the survey of other results and methods in this direction). Our technique introduces several novel twists as, for example, decoupling otherwise not-so-independent events. In particular, we exhibit families of spaces (or convex bodies) which enjoy appropriate properties with probability close to 1. However, among other similar constructions, ours appears to be a relatively good candidate for "derandomization" in the spirit of [KT] or [BS], i.e., for coming up with an argument that would yield an explicit space (resp., a convex body) with the same properties.

The organization of the paper is as follows. In the next section we state our principal results and their immediate consequences. Section 3 is devoted to the proof of our main result, a general saturation theorem which implies an answer to Problem 1 from [M3]. Section 4 contains the proof of (a version of) Theorem 2.2, relevant to Problems 2 and 3 from [M3]. Finally, in the Appendix we present a proof of Lemma 3.4, a result describing known phenomenon, for which we provide a compact argument yielding optimal or near optimal constants.

We use the standard notation of convexity and geometric functional analysis as can be found, e.g., in [MS1], [P] or [T]. The following useful jargon is possibly not familiar to a general mathematical reader. A normed space $X$ is completely described by its unit ball $K = B_X$ or its norm $\|\cdot\|_X$ and so we shall tend to identify these three objects. In particular, we will write $\|\cdot\|_K$ for the Minkowski functional defined by a centrally symmetric convex body $K \subset \mathbb{R}^n$ and denote the resulting normed space by $(\mathbb{R}^n, \|\cdot\|_K)$ or just $(\mathbb{R}^n, K)$. The standard Euclidean norm on $\mathbb{R}^n$ will be always denoted by $|\cdot|$. (*Attention:* the same notation may mean elsewhere cardinality of a set and, of course, the absolute value of a scalar.) We will write $B_2^n$ for the unit ball in $\ell_2^n$ and, similarly but less frequently, $B_p^n$ for the unit ball in $\ell_p^n$, $1 \leq p \leq \infty$.

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## 2 Description of results

Our first result is a general saturation theorem.

**Theorem 2.1** *Let $n$ and $m_0$ be positive integers with $\sqrt{n \log n} \leq m_0 \leq n$. Then, for every finite dimensional normed space $W$ with*

$$\dim W \leq c_1 \min\{m_0/\sqrt{n}, m_0^2/(n \log n)\}$$

*(where $c_1 > 0$ is a universal numerical constant) there exists an $n$-dimensional normed space $X$ such that every quotient $\tilde{X}$ of $X$ with $\dim \tilde{X} \geq m_0$ contains a 1-complemented subspace isometric to $W$.*

The theorem has bearing on Problem 1 from [M3]:

*Does every $n$-dimensional normed space admit a quotient of dimension $\geq n/2$ whose cotype 2 constant is bounded by a universal numerical constant?*

Let us start with several comments concerning the hypotheses on $k := \dim W$ and $m_0$ included in the statement of Theorem 2.1. If, say, $m_0 \approx n/2$, then $k$ of order $\sqrt{n}$ is allowed. Nontrivial (i.e., large) values of $k$ are obtained whenever $m_0 \gg \sqrt{n \log n}$; we included the condition $\sqrt{n \log n} \leq m_0$ to indicate for which values of the parameters the assertion of the Theorem is meaningful. Furthermore, for each "large" quotient $\tilde{X}$ of $X$, we can deduce quantitative information on $C_q(\tilde{X})$, the cotype $q$ constant of $\tilde{X}$. If, for example, $W = \ell_\infty^k$, then, for any corresponding $\tilde{X}$, $C_2(\tilde{X})$ is at least $\sqrt{k}$; in particular, if $m_0$ is "proportional" to $n$, then $C_2(\tilde{X})$ is at least of order $\sqrt[4]{n}$. Similarly, $C_q(\tilde{X})$ is at least of order $n^{1/2q}$ for any finite $q$. The problem from [M3] stated above is thus answered negatively in a very strong sense.

The dual $X^*$ of the space from Theorem 2.1 has the property that, under the same assumptions on $m$ and $k$, every $m$ dimensional *subspace* of $X^*$ contains a (1-complemented) subspace isometric to $W^* =: V$. If we choose, again, $W = \ell_\infty^k$, all "large" subspaces of $X^*$ contain isometrically $\ell_1^k$ and hence $X^*$ comes close to being a counterexample to Problems 2 and 3 from [M3]. Roughly speaking, those problems asked whether every space of nontrivial cotype $q < \infty$ contains a proportional subspace of type 2, or even just $K$-convex. This is well known to be true if $q = 2$ due to presence of nearly Euclidean subspaces. Although our space $X^*$ does not *a priori* have any cotype property, a relatively simple modification of our technique allows to correct this deficiency for sufficiently large $q$, which results in the following theorem.

**Theorem 2.2** *There exist $q_0 \in [2, \infty)$ such that, for any $q \in (q_0, \infty)$ there are constants $\alpha = \alpha_q \in (0, 1)$ and $c = c_q > 0$ so that the following holds: for any positive integers $n$ and $m_0$ with $n^\alpha \leq m_0 \leq n$ and for any normed space $V$ with*

$$\dim V \leq c_q \; m_0/n^\alpha$$

*there exists an $n$-dimensional normed space $Y$ whose cotype $q$ constant is bounded by a function of $q$ and $C_q(V)$ and such that every subspace $\tilde{Y}$ of $Y$ with $\dim \tilde{Y} \geq m_0$ contains a 1-complemented subspace isometric to $V$.*

Again, if – in the notation of Theorem 2.2 – $m_0$ is "proportional" to $n$ and $V = \ell_1^k$ is of the maximal dimension that is allowed (for fixed $q$), then the type 2 constant of any corresponding subspace $\tilde{Y}$ of $Y$ from the Theorem is at least of order $n^{(1-\alpha_q)/2}$ (and analogously for any nontrivial type $p > 1$). The $K$-convexity constant of any such $\tilde{Y}$ is at least of order $\sqrt{\log n}$ . Clearly, this answers in the negative Problems 2 and 3 from [M3].

We also remark that choosing $V = \ell_p^k$ (for some $1 < p < 2$) in Theorem 2.2 leads to a space $Y$ whose type $p$ and cotype $q$ constants are bounded by numerical constants and such that, for every $m$-dimensional subspace $\tilde{Y}$ of $Y$ and every $p < p_1 < 2$, the type $p_1$ constant of $\tilde{Y}$ is at least $k^{1/p-1/p_1}$. If $m_0$ is "proportional" to $n$, the type $p_1$ constant of $\tilde{Y}$ is at least of order $n^{(1-\alpha_q)(1/p-1/p_1)}$, in particular it tends to $+\infty$ as $n \to \infty$.

In Section 4 we will sketch a proof of Theorem 2.2 which gives $q_0 = 4$. The Theorem holds in fact even with $q_0 = 2$, but the proof, while still based on the same main idea, is much more subtle and will be presented elsewhere.

It has been realized in the last few years (cf. [MS2]) that many *local* phenomena (i.e., referring to subspaces or quotients of a normed space) have *global* analogues, expressed in terms of the entire space. For example, a "proportional" quotient of a normed space corresponds to the Minkowski sum of several rotations of its unit ball. Dually, a "proportional" subspace corresponds to the intersection of several rotations. (Such results were already implicit, e.g., in [K].) In the present paper we state (without proof) the following sample theorem about the Minkowski sum of two rotations of a unit ball.

**Theorem 2.3** *There exists a constant $c_2 > 0$ such that for any positive integers $n, k$ satisfying $k \leq c_2 n^{1/4}$ and for any $k$-dimensional normed space $W$, there exists a normed space $X = (\mathbb{R}^n, K)$ such that, for any rotation*

$\rho : \mathbb{R}^n \to \mathbb{R}^n$, *the normed space* $(\mathbb{R}^n, K + \rho(K))$ *contains a* 3*-complemented subspace* 3*-isomorphic to* $W$.

Theorem 2.3 answers a query directed to the authors by V. Milman. Its proof will be presented elsewhere.

It may be of interest that saturations similar to those described in Theorems 2.1-2.3 (and in Proposition 3.1 in the next section) may be implemented simultaneously for entire families of spaces $W$ whose dimension is uniformly bounded. For example, if $\mathcal{W}$ is such a family and $|\mathcal{W}|$ is bounded by a power of $n$ (the dimension of the space that is being saturated), the strengthening is straightforward: if every $W \in \mathcal{W}$ is of dimension allowed by a Theorem, then the space $X$ (or $Y$), whose existence is asserted in the Theorem, verifies the saturation condition in the assertion simultaneously for all $W \in \mathcal{W}$. [Only the values of the numerical constants $c_1, c_q, c_2$ included in the bound on the dimension $k$ will be affected.] In a slightly different direction, given $k \in \mathbb{N}$ and $\varepsilon \in (0,1)$, it is possible to construct a space of dimension $\leq \exp(C(\varepsilon)k)$ whose every "proportional" subspace (or quotient) contains a $(1+\varepsilon)$-complemented $(1+\varepsilon)$- isomorphic copy of *every* $k$-dimensional space. (The $\exp(C(\varepsilon)k)$ dimension estimate is already optimal for a super-space that is universal for all $k$-dimensional spaces.) We provide a sketch of the argument in a remark at the end of section 3.

## 3 The main construction

The statement of the main Theorem 2.1 is a particular case (namely, with $N \approx n^{3/2} \log n$) of the following technical proposition. Recall that for a normed space $W$, we denote by $\ell_1^N(W)$ the $\ell_1$-sum of $N$ copies of $W$.

**Proposition 3.1** *Let* $n$ *and* $N$ *be positive integers with* $n \log n \leq N \leq n^{3/2} \log n$. *Let* $m_0$ *be a positive integer with* $\max\{\sqrt{n \log n}, n^2 \log n/N\} \leq m_0 \leq n$. *Then, for every finite dimensional normed space* $W$ *with*

$$k := \dim W \leq c_1 \min\left\{\frac{m_0}{\sqrt{n}}, \frac{m_0^2}{n \log n}, \frac{m_0 N}{n^2 \log n}\right\}$$

*(where* $c_1 > 0$ *is an appropriate universal constant) there exists an* $n$*-dimensional normed space* $X$ *such that, for any* $m_0 \leq m \leq n$, *every* $m$*-dimensional quotient* $\tilde{X}$ *of* $X$ *contains a 1-complemented subspace isometric to* $W$. *Moreover,* $X$ *can be taken as a quotient of* $Z = \ell_1^N(W)$.

*Proof* Let $1 \leq k \leq m \leq n \leq kN$ be positive integers. More restrictions will be added on these parameters as we proceed. Notice that choosing the constant $c_1$ small makes the assertion vacuously satisfied for small values of $m_0$, and so we may and shall assume that $m_0, n$ and $N$ are large.

Let $W$ be a $k$-dimensional normed space. Identify $W$ with $\mathbb{R}^k$ in such a way that the Euclidean ball $B_2^k$ is the ellipsoid of minimal volume containing the unit ball $B_W$ of $W$. It follows in particular that $\frac{1}{\sqrt{k}} B_2^k \subset B_W \subset B_2^k$. Further, this allows to identify $Z = \ell_1^N(W)$ with $\mathbb{R}^{Nk}$.

Let $G = G(\omega)$ be a $n \times Nk$ random matrix (defined on some underlying probability space $(\Omega, \mathbb{P})$) with independent $N(0, 1/n)$-distributed Gaussian entries and set

$$K = B_{X(\omega)} := G(\omega)(B_Z) \subset \mathbb{R}^n. \tag{3.1}$$

The random normed space $X = X(\omega)$ is sometimes referred to as a random (Gaussian) quotient of $Z$, with $G(\omega)$ the corresponding quotient map. We shall show that, for appropriate choices of the parameters, $X(\omega)$ satisfies the assertion of Proposition 3.1 with probability close to 1. [The normalization of $G$ is not important; here we choose it so that the radius of the Euclidean ball circumscribed on $K$ is typically comparable to 1.] As usual in such arguments, we will follow the scheme first employed in [G] which consists of three steps.

*I.* For a fixed quotient map $Q : \mathbb{R}^n \to \mathbb{R}^m$, the assertion of the theorem holds outside of a small exceptional set $\Omega^Q$.

*II.* The assertion of the theorem is "essentially stable" under small perturbations of the quotient map, and so it is enough to verify it for an appropriate net in the set of all quotient maps.

*III.* There exists a net $\mathcal{Q}$ which works in step II such that the set $\bigcup_{Q \in \mathcal{Q}} \Omega^Q$ has small probability.

By combining steps I and II we see now that the assertion of the theorem holds on the complement of the union from step III and the theorem follows.

We start by introducing some basic notation. Denote by $F_1, \ldots, F_N$ the $k$-dimensional coordinate subspaces of $\mathbb{R}^{Nk}$ corresponding to the consecutive copies of $W$ in $Z$. In particular, $B_Z = \mathrm{conv}\,(F_j \cap B_Z : j = 1, \ldots, N)$. For $j = 1, \ldots, N$, we define subsets of $\mathbb{R}^n$ as follows: $E_j := G(F_j)$, $K_j := G(F_j \cap B_Z)$ and

$$K'_j := G(\mathrm{span}\,[F_i : i \neq j] \cap B_Z) = \mathrm{conv}\,(K_i : i \neq j). \tag{3.2}$$

Similarly, for $I \subset \{1, \ldots, N\}$, we let $K_I := \mathrm{conv}\,(K_i : i \in I) \subset \mathbb{R}^n$.

*Step I. Analysis of a single quotient map.* Since a quotient space is determined up to isometry by the kernel of a quotient map, it is enough to consider quotient maps which are *orthogonal* projections. Let, for the time being, $Q : \mathbb{R}^n \to \mathbb{R}^m$ be the natural projection on the first $m$ coordinates. In view of symmetries of our probabilistic model, all relevant features of this special case will transfer to an arbitrary rank $m$ orthogonal projection.

Let $\tilde{G} = QG$, i.e., $\tilde{G}$ is the $m \times Nk$ Gaussian matrix obtained by restricting $G$ to the first $m$ rows. Let $\tilde{K} = Q(K) = \tilde{G}(B_Z)$ and denote the space $(\mathbb{R}^m, \tilde{K})$ by $\tilde{X}$; the space $\tilde{X}$ is the quotient of $X$ induced by the quotient map $Q$. We shall use the notation of $\tilde{E}_j$, $\tilde{K}_j$, $\tilde{K}'_j$ and $\tilde{K}_I$ for the subsets of $\mathbb{R}^m$ defined in the same way as $E_j$, $K_j$, $K'_j$, $K_I$ above, using the matrix $\tilde{G}$ in place of $G$.

For any subspace $H \subset \mathbb{R}^m$, we will denote by $P_H$ the orthogonal projection onto $H$. We shall show that outside of an exceptional set of small measure there exists $j \in \{1, \ldots, N\}$ such that $P_{\tilde{E}_j}(\tilde{K}'_j) \subset \tilde{K}_j$. Since, for any given $i$, we *always* have $\tilde{K} = \mathrm{conv}\,(\tilde{K}_i, \tilde{K}'_i)$ and $\tilde{K}_i \subset \tilde{E}_i$, it follows that, for $j$ as above,

$$P_{\tilde{E}_j}(\tilde{K}) = \mathrm{conv}\,(\tilde{K}_j, P_{\tilde{E}_j}(\tilde{K}'_j)) = \tilde{K}_j = \tilde{E}_j \cap \tilde{K}_j. \tag{3.3}$$

As $\tilde{K}_j$ is an affine image of the ball $F_j \cap B_Z$, which is the ball $B_W$ on coordinates from $F_j$, we deduce that $\tilde{E}_j$ considered as a subspace of $\tilde{X}$ is then isometric to $W$ and, moreover, 1-complemented. We note in passing that for that subspace to be just isometric to $W$ (and not necessarily complemented), a weaker condition $\tilde{E}_j \cap \tilde{K}'_j \subset \tilde{K}_j$ suffices. This condition can also be analyzed by the methods of the present paper (it requires a generalization of Lemma 3.2), but no improvement results in the range of parameters that we are interested in.

The precise definition of the exceptional set will be somewhat technical. We start by introducing more subsets of $\mathbb{R}^m$. Let, for $j = 1, \ldots, N$, $\tilde{D}_j := \tilde{G}(F_j \cap B_2^{Nk})$ and

$$\tilde{D}'_j := \mathrm{conv}\,\left(\tilde{D}_i : i \neq j\right). \tag{3.4}$$

That is, $\tilde{D}'_j$ is obtained by replacing $\tilde{K}_i$ with $\tilde{D}_i$ in the definition of $\tilde{K}'_j$. We define similarly $\tilde{D}$, $\tilde{D}_I$ and, analogously, the subsets $D_j$, $D'_j$, $D_I$ and $D$ of

$\mathbb{R}^n$. So, for example, $D_j := G(F_j \cap B_2^{Nk})$, and $D := \mathrm{conv}\,(D_j : 1 \le j \le N)$. Note that since $\frac{1}{\sqrt{k}} B_2^k \subset B_W \subset B_2^k$, it follows that $\frac{1}{\sqrt{k}} D_j \subset K_j \subset D_j$. Consequently, analogous inclusions hold for all the corresponding $K$- and $D$-type sets as they are images (or convex hulls, or images of convex hulls) of the appropriate $K_j$'s and $D_j$'s. In particular, in order for the inclusion $P_{\tilde{E}_j}(\tilde{K}'_j) \subset \tilde{K}_j$ to hold it is enough to have

$$P_{\tilde{E}_j}(\tilde{D}'_j) \subset \frac{1}{\sqrt{k}} \tilde{D}_j, \tag{3.5}$$

and it is this seemingly "brutal" condition that we shall try to enforce. As a consequence, our results will not be always sharpest possible; improvements are possible if more geometric information about the space $W$ (notably its Banach-Mazur distance to the Euclidean space) is available. On the other hand, the construction of the space $X$ will be, in a sense, universal: we will obtain quotient maps $G = G(\omega) : \mathbb{R}^{Nk} \to \mathbb{R}^n$ which, simultaneously for all $W$, yield spaces verifying the assertion of Theorem 2.1 (that is, the resulting spaces are quotients of $\ell_1^N(W)$ via *the same* quotient map).

Let us go back to the definition of the exceptional set. We start by introducing, for $j \in \{1, \ldots, N\}$, the "good" sets

$$\Omega'_j := \left\{ \omega \in \Omega : P_{\tilde{E}_j}(\tilde{D}'_j) \subset \kappa B_2^m \right\} \tag{3.6}$$

$$\Omega'_{j,0} := \left\{ \frac{1}{2}\sqrt{\frac{m}{n}}(B_2^m \cap \tilde{E}_j) \subset \tilde{D}_j \subset 2\sqrt{\frac{m}{n}}(B_2^m \cap \tilde{E}_j) \right\}, \tag{3.7}$$

where $\kappa \in (0,1)$ will be specified at the end of this proof. Next we set

$$\Omega_j := \Omega'_j \cap \Omega'_{j,0}. \tag{3.8}$$

Now if $\kappa$, $k$, $m$ and $n$ satisfy

$$\kappa \le \frac{1}{\sqrt{k}} \cdot \frac{1}{2}\sqrt{\frac{m}{n}}, \tag{3.9}$$

then, for $\omega \in \Omega_j$, the inclusion (3.5) holds; *a fortiori,* $P_{\tilde{E}_j}(\tilde{K}'_j) \subset \tilde{K}_j$ and so the argument presented earlier applies. Thus outside of the exceptional set

$$\Omega^0 := \Omega \setminus \bigcup_{1 \le j \le N} \Omega_j = \bigcap_{1 \le j \le N} (\Omega \setminus \Omega_j) \tag{3.10}$$

there exists $j \in \{1, \dots, N\}$ such that $\tilde{E}_j$, considered as a subspace of $\tilde{X}$, is isometric to $W$ and 1-complemented.

It remains to show that the exceptional set $\Omega^0$ is appropriately small; this will be the most involved part of the argument. For $j, i \in \{1, \dots, N\}$ and $I \subset \{1, \dots, N\}$ we set

$$\Omega'_{j,I} := \left\{ \omega \in \Omega : P_{\tilde{E}_j}(\tilde{D}_I) \subset \kappa B_2^m \right\}, \quad \Omega'_{j,i} := \Omega'_{j,\{i\}}. \tag{3.11}$$

In particular, for any $j \in \{1, \dots, N\}$, $\Omega'_j = \bigcap_{i \in \{1,\dots,N\}\setminus\{j\}} \Omega'_{j,i}$ (cf. (3.4), (3.6)) and so, in view of (3.8),

$$\Omega_j = \bigcap_{i \in \{0,1,\dots,N\}\setminus\{j\}} \Omega'_{j,i}$$

This allows rewriting (3.10) as

$$\Omega^0 = \bigcap_{1 \le j \le N} \bigcup_{i \in \{0,1,\dots,N\}\setminus\{j\}} (\Omega \setminus \Omega'_{j,i}) = \bigcup_{(i_j)} \bigcap_{1 \le j \le N} \left( \Omega \setminus \Omega'_{j,i_j} \right), \tag{3.12}$$

where the second union is extended over all sequences $(i_1, \dots, i_N)$ satisfying $i_j \in \{0, 1, \dots, N\} \setminus \{j\}$ for $j = 1, \dots, N$. The following lemma enables us to handle components of that union by appropriately grouping them.

**Lemma 3.2** *Let $\Lambda = (\lambda_{ij})$ be an $N \times N$ matrix such that*
$1^\circ$ *all its elements are either 0 or 1*
$2^\circ$ *each column contains at most one 1*
$3^\circ$ *the diagonal consists of 0's.*
*Then there exists $J \subset \{1, \dots, N\}$ such that $|J| \ge N/3$ and*

$$i, j \in J \Rightarrow \lambda_{ij} = 0$$

*Proof* At the most fundamental level, this lemma can be deduced from Turán's theorem on existence of independent sets of vertices in graphs with few edges, cf. [AS], p. 82: consider a (directed) graph $G_\Lambda$ whose adjacency matrix is $\Lambda$, then $G_\Lambda$ has at most $N$ edges and the assertion means that the set of vertices $J$ is independent. The lower bound on $|J|$ follows by counting edges in the corresponding extremal Turán graphs.

More conveniently, the formulation we use here can be derived from the result of K. Ball on suppression of matrices presented and proved in [BT].

By Theorem 1.3 in [BT] applied to $\Lambda$, there exists a subset $J \subset \{1, \dots, N\}$ with $|J| \geq N/3$ such that $\sum_{i \in J} \lambda_{ij} < 1$ for $j \in J$, which is just a restatement of the condition in the assertion of the Lemma.

Finally, we present here a self-contained simple argument which yields a variant of the Lemma with $N/3$ replaced by $N/4$ in the estimate for $|J|$ (which would be sufficient for our purposes). Let $I := \{i \ : \ \sum_{j=1}^{N} \lambda_{ij} \leq 1\}$. If $|I^c| \geq N/4$, then a simple counting argument shows that at least $N/4$ rows of $\Lambda$ (indexed by a subset of $I$) consist only of 0's which clearly yields the assertion. If, on the other hand, $|I| > 3N/4$, consider a maximal subset $J \subset I$ for which the condition on entries given in the assertion holds. By maximality, for every $i \in I \setminus J$ one of the elements of $\{\lambda_{ij} \ : \ j \in J\} \cup \{\lambda_{ji} \ : \ j \in J\}$ equals 1 (note that $\lambda_{ii} = 0$ by hypothesis) and so there at least $|I \setminus J| > 3N/4 - |J|$ such elements. On the other hand, those elements are contained in the union of $|J|$ columns and $|J|$ rows each of which contains at most one nonzero element, which implies $2|J| > 3N/4 - |J|$ or $|J| > N/4$, as required. $\square$

We now return to our main argument. Let $(i_1, \dots, i_N)$ be any sequence satisfying $i_j \in \{0, 1, \dots, N\} \setminus \{j\}$, $j = 1, \dots, N$, and corresponding to a component in the second union in (3.12). Define a matrix $\Lambda = (\lambda_{ij})_{i,j=1}^{N}$ by $\lambda_{ij} = 1$ if $i = i_j$ for some $j \in \{1, \dots, N\}$ and $\lambda_{ij} = 0$ otherwise. Then $\Lambda$ satisfies the assumptions of Lemma 3.2 and let $J$ be the resulting set of indices such that $j \in J \Rightarrow i_j \notin J$ and that $|J| = \ell := \lceil N/3 \rceil$. It now follows directly from the definitions (3.11) that

$$\bigcap_{1 \leq j \leq N} \left(\Omega \setminus \Omega'_{j,i_j}\right) \subset \bigcap_{j \in J} \left(\Omega \setminus \Omega'_{j,i_j}\right) \subset \bigcap_{j \in J} \left(\Omega \setminus (\Omega'_{j,J^c} \cap \Omega'_{j,0})\right) =: \Omega_J.$$

(Here and in what follows the complement $\cdot^c$ is meant with respect to $\{1, \dots, N\}$, the index 0 playing a special role.) In combination with (3.12), the above inclusions show that

$$\Omega^0 \subset \bigcup_{J \in \mathcal{J}} \Omega_J, \tag{3.13}$$

where $\mathcal{J} := \{J \subset \{1, \dots, N\} \ : \ |J| = \ell\}$. Our next objective will be to estimate $\mathbb{P}(\Omega_J)$ for a fixed $J \in \mathcal{J}$. There is no harm in assuming that $J = \{1, \dots, \ell\}$. To keep the notation more compact, for $1 \leq j \leq N$ we let

$$\mathcal{E}_j := \Omega \setminus (\Omega'_{j,J^c} \cap \Omega'_{j,0}) = (\Omega \setminus \Omega'_{j,J^c}) \cup (\Omega \setminus \Omega'_{j,0}). \tag{3.14}$$

In particular, $\Omega_J = \bigcap_{j\in J} \mathcal{E}_j$. Let us now make the key observation – which is apparent from the definitions (3.7) and (3.11) – that the events $\mathcal{E}_j$, for $1 \le j \le \ell$, are *conditionally* independent with respect to $D_{J^c}$: once $D_{J^c}$ is fixed, and hence $\tilde{D}_{J^c}$ is fixed as well, each $\mathcal{E}_j$ depends only on the restriction $G_{|F_j}$. In fact, the ensemble $\{G_{|F_j} : 1 \le j \le \ell\} \cup \{D_{J^c}\}$ is independent since its distinct elements depend on disjoint sets of columns of $G$, and the columns themselves are independent. This and the symmetry in the indices $j \in \{1, \dots, \ell\}$ imply

$$\mathbb{P}(\Omega_J \mid D_{J^c}) = \mathbb{P}(\bigcap_{j\in J}(\mathcal{E}_j \mid D_{J^c}) = \prod_{j\in J} \mathbb{P}(\mathcal{E}_j \mid D_{J^c}) = \Big(\mathbb{P}(\mathcal{E}_1 \mid D_{J^c})\Big)^{\ell}. \quad (3.15)$$

As suggested above, somewhat informally we may think of the above conditional probabilities as being calculated "for fixed $D_{J^c}$" (and hence "for fixed $\tilde{D}_{J^c}$").

In the sequel we shall need the following auxiliary well-known result on rectangular Gaussian matrices (see, e.g., [DS], Theorem 2.13). We recall that for a $m \times k$ matrix $B$ with $1 \le k \le m$, its singular numbers $s_j(B)$ are the eigenvalues of $(B^*B)^{1/2}$ arranged in the non-increasing order. In particular, for all $x \in \mathbb{R}^k$,

$$s_k(B)|x| \le |Bx| \le s_1(B)|x|.$$

**Lemma 3.3** *Let $k, m$ be integers with $1 \le k \le m$ and let $A = (a_{ij})$ be an $m \times k$ random matrix with independent $N(0, \sigma^2)$-distributed Gaussian entries and let $t > 0$. Then*

$$\mathbb{P}\left(s_1(A) > (\sqrt{m} + \sqrt{k})\sigma + t\right) \le e^{-t^2/2\sigma^2},$$

$$\mathbb{P}\left(s_k(A) < (\sqrt{m} - \sqrt{k})\sigma - t\right) \le e^{-t^2/2\sigma^2}.$$

Returning to our main argument, we note that, by the definition of $\mathcal{E}_1$, $\mathbb{P}(\mathcal{E}_1 \mid D_{J^c}) \le \mathbb{P}(\Omega \setminus \Omega'_{1,J^c} \mid D_{J^c}) + \mathbb{P}(\Omega \setminus \Omega'_{1,0} \mid D_{J^c})$. Next, since $\Omega'_{1,0}$ is independent of $D_{J^c}$, the second term equals just $1 - \mathbb{P}(\Omega'_{1,0})$. Further, the condition from (3.7) defining $\Omega'_{1,0}$ can be restated as

$$\frac{1}{2}\sqrt{\frac{m}{n}} \le s_k(\tilde{G}_{|F_1}) \le s_1(\tilde{G}_{|F_1}) \le 2\sqrt{\frac{m}{n}}$$

and hence the probabilities involved can be estimated by using Lemma 3.3 with $\sigma = 1/\sqrt{n}$ and appropriate values of $t$. In particular, if $m \geq 16k$, we get

$$\begin{aligned} \mathbb{P}(\Omega \setminus \Omega'_{1,0} \mid D_{J^c}) &\leq \mathbb{P}\left( s_k(\tilde{G}_{|F_1}) < \frac{1}{2}\sqrt{\frac{m}{n}} \right) + \mathbb{P}\left( s_1(\tilde{G}_{|F_1}) > 2\sqrt{\frac{m}{n}} \right) \\ &\leq e^{-m/32} + e^{-9m/32}. \end{aligned} \tag{3.16}$$

To estimate the term $\mathbb{P}(\Omega \setminus \Omega'_{1,J^c} \mid D_{J^c})$, we need to introduce an auxiliary exceptional set

$$\Omega^1 := \{\omega : D \not\subset 2B_2^n\} \tag{3.17}$$

(recall that the set $D$ was defined after (3.4) and that $D \supset K$). An argument parallel to the one that led to (3.16) shows that if $k \leq n/16$, then

$$\mathbb{P}(\Omega^1) \leq N e^{-9n/32}. \tag{3.18}$$

(An alternative argument uses the Chevet-Gordon inequality – cf. the proof of Lemma 3.4 given in the Appendix – to estimate the expected value of the norm $\|G : \ell_1^N(\ell_2^k) \to \ell_2^n\|$, and then the Gaussian isoperimetric inequality to majorize the probablility that that norm is $> 2$, the condition equivalent to that in the definition of $\Omega^1$.) For future reference, we emphasize that $\Omega^1$ does not depend on $J$ nor on the projection $Q$.

Next we define

$$\Omega' := \{\omega : D_{J^c} \not\subset 2B_2^n\}.$$

Clearly $\Omega' \subset \Omega^1$. As we are dealing with conditional probabilities $\mathbb{P}(\cdot \mid D_{J^c})$, we may – by the remark following (3.15) – consider $D_{J^c}$ fixed. Furthermore, for the time being we shall restrict our attention to $\omega \notin \Omega'$, i.e., to sets $D_{J^c} \subset 2B_2^n$, which implies $\tilde{D}_{J^c} \subset 2B_2^m$. Given that $\Omega'$ is $D_{J^c}$-measurable, this restriction will not interfere with the conditional independence of $\mathcal{E}_j$ for $j \in J$.

The exceptional sets such as $\Omega'$ or $\Omega^1$ involve conditions on diameters of random images of sets. It is well known that such quantities are related to the functional $M^*(\cdot)$, defined for a set $S \subset \mathbb{R}^s$ via

$$M^*(S) := \int_{S^{s-1}} \sup_{y \in S} \langle x, y \rangle dx, \tag{3.19}$$

where the integration is performed with respect to the normalized Lebesgue measure on $S^{s-1}$ (this is 1/2 of what geometers call the mean width of $S$). It is elementary to verify that for $\omega \notin \Omega^1$ (hence, *a fortiori*, for $\omega \notin \Omega'$) and for any $j = 1, \dots, N$,

$$M^*(\tilde{D}_j) \leq 2M^*(\tilde{E}_j \cap B_2^m) = 2\int_{S^{m-1}} |Px| dx \leq 2\sqrt{k/m},$$

where $P = P_{\tilde{E}_j}$ is the orthogonal projection on $\tilde{E}_j$ (or, by rotational invariance, *any* orthogonal projection of rank $k$). It follows then via known arguments (most readily, by passing to Gaussian averages, cf. [LT], (3.6), or Lemma 8.1 in [MSz]) that, again for $\omega \notin \Omega^1$,

$$M^*(\tilde{D}_{J^c}) \leq C_0\sqrt{\frac{\log N}{m}} + \max_{j \in J^c} M^*(\tilde{D}_j) \leq (C_0+2)\sqrt{\frac{\max\{k, \log N\}}{m}}, \tag{3.20}$$

where $C_0 \geq 1$ is a universal constant.

Let $d, m$ be integers with $1 \leq d \leq m$ and let $G_{m,d}$ be the Grassmann manifold of $d$-dimensional subspaces of $\mathbb{R}^m$ endowed with the normalized Haar measure. The following lemma describes the behavior of the diameter of a random rank $d$ projection of a subset of $\mathbb{R}^m$.

**Lemma 3.4** *Let $a > 0$ and let $S \subset \mathbb{R}^m$ verify $S \subset aB_2^m$. Then, for any $t > 0$, the set $\left\{H \in G_{m,d} : P_H(S) \subset \left(a\sqrt{d/m} + M^*(S) + t\right) B_2^m\right\}$ has measure $\geq 1 - \exp(-t^2 m/2a^2 + 1)$.*

The phenomenon discussed in the Lemma is quite well known, at least if we do not care about the specific values of numerical constants (which are not essential for our argument) and precise estimates on probabilities. It is sometimes refered to as the "standard shrinking" of the diameter of a set, and it is implicit, for example, in probabilistic proofs of the Dvoretzky theorem, see [M1], [MS1]. For future reference, we provide in the Appendix a compact proof which yields the formulation given above.

Returning to our main argument, we now use the estimate from Lemma 3.4 for $S = \tilde{D}_{J^c}$, $d = k$ and $t = \kappa/2$. Since $\omega \notin \Omega'$, we may take $a = 2$ to deduce that the set $\{H \in G_{m,k} : P_H(\tilde{D}_{J^c}) \subset (2\sqrt{k/m} + M^*(\tilde{D}_{J^c}) + \kappa/2)B_2^m\}$ has measure larger than or equal to $1 - \exp(-\kappa^2 m/32 + 1)$. Taking into account (3.20) and requiring that, in addition to (3.9), $\kappa$ verifies the condition

$$C' \sqrt{\max\{k, \log N\}/m} \leq \kappa \tag{3.21}$$

with

$$C' \geq 2(C_0 + 4), \tag{3.22}$$

we see that

$$2\sqrt{k/m} + M^*(\tilde{D}_{J^c}) + \kappa/2 \leq \kappa,$$

and thus the measure of the set $\{H \in G_{m,k} : P_H(\tilde{D}_{J^c}) \subset \kappa B_2^m\}$ is also larger than or equal to $1 - \exp(-\kappa^2 m/32 + 1)$. Since, by the rotational invariance of the Gaussian measure, the distribution of $\tilde{E}_1$ on $G_{m,k}$ is uniform, we finally get that

$$\begin{aligned} \mathbb{P}\left(\Omega'_{1,J^c} \mid D_{J^c}\right) &= \mathbb{P}\left(P_{\tilde{E}_1}(\tilde{D}_{J^c}) \subset \kappa B_2^m \mid D_{J^c}\right) \\ &\geq 1 - \exp(-\kappa^2 m/32 + 1). \end{aligned} \tag{3.23}$$

The above estimate is valid outside of $\Omega'$, i.e., for $D_{J^c} \subset 2B_2^n$. Thus, recalling (3.14) and combining (3.23) and (3.16) we infer that, under the same restriction,

$$\mathbb{P}(\mathcal{E}_1 \mid D_{J^c}) \leq e^{-\kappa^2 m/32+1} + e^{-m/32} + e^{-9m/32}.$$

Plugging the above into (3.15) and recalling that, by (3.9), $\kappa \leq 1/2$, we obtain

$$\mathbb{P}(\Omega_J \mid D_{J^c}) \leq (2e\, e^{-\kappa^2 m/32})^\ell = (2e)^\ell e^{-\kappa^2 m\ell/32},$$

again, on the set $\Omega \setminus \Omega'$ which is $D_{J^c}$-measurable. Consequently, averaging over $\Omega \setminus \Omega'$,

$$\mathbb{P}(\Omega_J \mid \Omega \setminus \Omega') \leq (2e)^\ell e^{-\kappa^2 m\ell/32}.$$

Since (as was noted following the definition of $\Omega'$) $\Omega' \subset \Omega^1$, our argument shows that

$$\mathbb{P}(\Omega_J \setminus \Omega^1) \leq \mathbb{P}(\Omega_J \setminus \Omega') = \mathbb{P}(\Omega_J \mid \Omega \setminus \Omega')\mathbb{P}(\Omega \setminus \Omega') \leq (2e)^\ell e^{-\kappa^2 m\ell/32}. \tag{3.24}$$

Up to now the set $J$ was fixed, now is the time to allow it to vary over $\mathcal{J}$. Since $|\mathcal{J}| = \binom{N}{\ell}$, it follows that

$$\mathbb{P}\left(\bigcup_{J \in \mathcal{J}} \Omega_J \setminus \Omega^1\right) \leq \binom{N}{\ell}(2e)^\ell e^{-\kappa^2 m\ell/32},$$

and so, by (3.13),

$$\mathbb{P}(\Omega^0) \leq \mathbb{P}(\Omega^1) + \mathbb{P}\left(\bigcup_{J\in\mathcal{J}} \Omega_J \setminus \Omega^1\right) \leq Ne^{-9n/32} + \binom{N}{\ell}(2e)^\ell e^{-\kappa^2 m\ell/32}. \tag{3.25}$$

Later on we will make the two terms on the right hand side small, by appropriate assumptions on $k$ and appropriate choices of $\kappa$ and $N$. Then, recalling the definition of $\Omega^0$, we will deduce that $\tilde{X}$, which is the quotient of the random space $X$ via the map $Q$, with large probability contains a 1-complemented subspace isometric to $W$. However, our ultimate goal is to show that this is true for *all* $m$-dimensional quotients of $X$. As indicated earlier the strategy for such an argument depends on combining a sharp probability estimate for a single $Q$ (obtained in Step I above) with perturbation and discretization arguments involving an appropriate net in the set of all $Q$'s (shown in Steps II and III below).

Finally, we note that the condition $m \geq 16k$ that was used in deriving estimates (3.16) and (3.18) is implied by (3.21) combined with (3.22), and so in what follows we need to ensure only the latter two conditions.

*Step II. The perturbation argument* We first generalize the definition (3.10) of the exceptional set. Let $Q$ be *any* orthogonal projection of rank $m$ (which will remain fixed through the end of the present step). Let us denote by $\Omega^Q$ the set given by formally the same formulae as in (3.10) by the Gaussian operator $\tilde{G} = QG$ for this particular $Q$. By rotational invariance, all the properties we derived for $\Omega^0$ hold also for $\Omega^Q$. The object of Step II will be to show that – under appropriate hypotheses – properties just slightly weaker, but still adequate for our purposes, hold for projections sufficiently close to $Q$.

Recall that by the definition (3.17) of $\Omega^1$ for $\omega \notin \Omega^1$ we have, for $j \in \{1, \ldots, N\}$,

$$D'_j \subset D \subset 2B_2^n, \tag{3.26}$$

Let now $\omega \notin \Omega^1 \cup \Omega^Q$ and let $Q'$ be a rank $m$ projection such that $\|Q - Q'\| \leq \delta$, where $\|\cdot\|$ is the operator norm with respect to $|\cdot|$ and $\delta > 0$ a constant to be specified later. We recall that our objective is to show that, for some $j$, conditions just slightly weaker than those in (3.6) and (3.7) hold with $Q$ replaced by $Q'$. Consequently, one still will be able to deduce that the quotient of $X$ corresponding to $Q'$ also contains a 1-complemented subspace isometric to $W$, namely $Q'E_j$.

Since $\omega \notin \Omega^Q$, it follows from (3.8) and (3.10) that $\omega \in \Omega_j = \Omega'_j \cap \Omega'_{j,0}$ for some $j \in \{1, \dots, N\}$, and so the conditions from (3.6) and (3.7) hold for this particular $j$ and $Q$. (Note that $Q$ enters implicitly into definitions of all "tilde-objects".) Let us first analyze (3.7). It asserts that, for $x \in F_j$, we have two-sided estimates

$$\frac{1}{2}\sqrt{\frac{m}{n}}|x| \leq |\tilde{G}x| = |QGx| \leq 2\sqrt{\frac{m}{n}}|x|. \tag{3.27}$$

We want to show similar estimates for $Q'$. We have

$$\Big| \, |Q'Gx| - |QGx| \, \Big| \leq \|Q' - Q\| \, |Gx| \leq \delta |Gx|.$$

On the other hand, $\omega \notin \Omega^1$ is equivalent to $\|G_{|F_i}\| \leq 2$ for all $1 \leq i \leq N$ (as an operator with respect to the Euclidean norms; see (3.17) and the paragraph following (3.18)). Thus, by (3.27), we get $(1/2)\sqrt{m/n}|x| - 2\delta|x| \leq |Q'Gx|$, and an analogous upper estimate, for all $x \in F_j$. So if, say, $\delta \leq (1/8)\sqrt{m/n}$, we obtain an analogue of (3.27) – hence of the condition from (3.7) – for $Q'$, namely

$$\frac{1}{4}\sqrt{\frac{m}{n}}|x| \leq |Q'Gx| \leq \frac{9}{4}\sqrt{\frac{m}{n}}|x| \tag{3.28}$$

for all $x \in F_j$.

We now turn to (3.6). It asserts that for $y \in D'_j$ we have $|P_{\tilde{E}_j}(Qy)| \leq \kappa$ and, again, we want to prove a similar estimate for $Q'$. To make the notation more compact, set $\bar{E}_j := Q'G(F_j)$ for $j = 1, \dots, N$. We shall show first that

$$\|P_{\tilde{E}_j} - P_{\bar{E}_j}\| \leq 4\delta\sqrt{\frac{n}{m}} =: \delta_1. \tag{3.29}$$

It is a general and elementary fact that the difference $P_H - P_{H'}$ of two orthogonal projections of the same rank attains its norm on *each* of the ranges of the projections. (To see this, consider the Schmidt decomposition of the restriction $P_{H|H'} = \sum_i \lambda_i \langle \cdot, h'_i \rangle h_i$ and observe that the mutually orthogonal subspaces $\operatorname{span}(h_i, h'_i)$ are reducing for $P_H - P_{H'}$, and that $P_H - P_{H'}$ is a multiple of an isometry on each of these subspaces.) It is thus enough to estimate the norm of $P_{\tilde{E}_j} - P_{\bar{E}_j}$ restricted to the subspace $\tilde{E}_j$. If $u = QGx \in \tilde{E}_j$, with $x \in F_j$, then

$$(P_{\tilde{E}_j} - P_{\bar{E}_j})u = (I - P_{\bar{E}_j})(Q - Q')Gx.$$

Therefore, using (3.27), we get, for $u \in \tilde{E}_j$,

$$|(P_{\tilde{E}_j} - P_{\bar{E}_j})u| \leq \|Q - Q'\| \|G_{|F_j}\| |x| \leq 2\delta(2\sqrt{n/m})|u|,$$

and (3.29) immediately follows.

Now let $y \in D'_j$; then, by (3.26), $|y| \leq 2$ and

$$\begin{aligned} |P_{\bar{E}_j} Q'y| &\leq |(P_{\bar{E}_j} - P_{\tilde{E}_j})Q'y| + |P_{\tilde{E}_j}(Q' - Q)y| + |P_{\tilde{E}_j} Qy| \\ &\leq 2\delta_1 + 2\delta + \kappa. \end{aligned}$$

Thus if, say, $\delta_1 \leq \kappa/4$, then – since clearly $\delta < \delta_1$ – it follows that $|P_{\bar{E}_j} Q'y| \leq 2\kappa$, and so the analogue of (3.6) holds for $Q'$ with $2\kappa$ replacing $\kappa$.

We now set $\delta := 1/(8\sqrt{n})$. It is then easy to see that the above condition for $\delta_1$ holds, just combine (3.21) and (3.22); the requirement $\delta \leq (1/8)\sqrt{m/n}$, discussed earlier in the context of the analogue of (3.27), is then also trivially satisfied. We thus conclude that if $\omega \notin \Omega^1 \cup \Omega^Q$, $\|Q - Q'\| \leq \delta$ and

$$2\kappa \leq \frac{1}{\sqrt{k}} \cdot \frac{1}{4}\sqrt{\frac{m}{n}}, \tag{3.30}$$

then the quotient of $X$ corresponding to $Q'$ contains an isometric and 1-complemented copy of $W$. We note that (3.30) is just slightly stronger than (3.9), and as easy to satisfy.

*Step III. The discretization: a net in the set of quotients* The final step is now straightforward. If $\mathcal{Q}$ is any finite family of rank $m$ orthogonal projections, then

$$\begin{aligned} \mathbb{P}\left(\Omega^1 \cup \bigcup_{Q \in \mathcal{Q}} \Omega^Q\right) &= \mathbb{P}\left(\Omega^1 \cup \bigcup_{Q \in \mathcal{Q}} (\Omega^Q \setminus \Omega^1)\right) \qquad (3.31) \\ &\leq Ne^{-9n/32} + |\mathcal{Q}| \binom{N}{\ell} (2e)^\ell e^{-\kappa^2 m\ell/32} \end{aligned}$$

as in (3.25); just note that the exceptional set $\Omega^1$ does not depend on $Q$. If $8\log N \le n$ the first term is $< e^{-n/8}$. Now recall that the set of rank $m$ orthogonal projections on $\mathbb{R}^n$, endowed with the distance given by the operator norm, can be identified with $G_{n,m}$ endowed with the appropriate invariant metric. The latter set admits, for any $\delta > 0$, a $\delta$-net $\mathcal{N}$ of cardinality $|\mathcal{N}| \le (C_2/\delta)^{m(n-m)}$, where $C_2$ is a universal constant (see, e.g., [S]). For our choice of $\delta = 1/(8\sqrt{n})$, this does not exceed $e^{mn\log n}$, at least for sufficiently large $n$. We plug this estimate into (3.31).

Recall that $\ell = \lceil N/3 \rceil \ge N/3$ and let us assume that the constant $C'$ in the definition (3.21) of $\kappa$ satisfies also $C' \ge 20$, so that $\kappa^2 m \ge 400$. Then

$$\binom{N}{\ell}(2e)^{\ell} e^{-\kappa^2 m\ell/32} \le 2^N \left(2e^{1-\kappa^2 m/32}\right)^{\ell} \le e^{(1+4\log 2-\kappa^2 m/32)N/3} \le e^{-\kappa^2 mN/128}.$$

Thus the second term in (3.31) is less than or equal to $e^{mn\log n-\kappa^2 mN/128}$.

In conclusion, if $k$, $\kappa$, $m$, $n$ and $N$ satisfy

$$C'\sqrt{\max\{k,\log N\}/m} \le \kappa \le 1, \qquad 2^8\, mn\log n \le \kappa^2 mN, \qquad 8\log N \le n, \tag{3.32}$$

where

$$C' = \max\{20, 2(C_0+4)\}$$

(see (3.21) and (3.22)), then the set $\Omega \setminus (\Omega^1 \cup \bigcup_{Q\in\mathcal{Q}} \Omega^Q)$ has positive measure (in fact, very close to 1 for large $n$). If, additionally, (3.30) is satisfied, then any $\omega$ from this set induces an $n$ dimensional space $X$ whose *all* $m$-dimensional quotients contain an isometric 1-complemented copy of $W$. (Indeed, by the argument above, each such quotient is determined by a projection within $\delta = 1/(8\sqrt{n})$ of a certain $Q \in \mathcal{Q}$.) Then the assertion of Proposition 3.1 holds for that particular value of $m$.

To ensure that the conditions in (3.30) and (3.32) are consistent, it will be most convenient to let $\kappa = (m/(64\,nk))^{1/2}$, so that (3.30) holds (in fact with equality). The first inequality, $\kappa \ge C'\sqrt{\max\{k,\log N\}/m}$, becomes then $k \le c_1' \min\{m/\sqrt{n}, m^2/(n\log N)\}$, for some numerical constant $c_1' > 0$. The second condition in (3.32) is equivalent to $k \le 2^{-14}mN/(n^2\log n)$, and hence the requirements on $k$ may be summarized as

$$k \le c_1'' \min\left\{\frac{m}{\sqrt{n}}, \frac{m^2}{n\log N}, \frac{mN}{n^2\log n}\right\}, \tag{3.33}$$

where $c_1'' \in (0,1)$ is an appropriate absolute constant.

Only the cases when the fractions on the right hand side of (3.33) are at least 1 are of interest. In particular, $n^2 \log n / N \leq m \leq n$, which gives the lower estimate $N \geq n \log n$. (Lower bounds on $m$ from the hypotheses of the Proposition will follow similarly.) On the other hand, for $N \geq n^{3/2} \log n$ the right hand side of (3.33) does not depend on $N$ anymore, hence the interval $[n \log n, n^{3/2} \log n]$ does indeed include all interesting values of $N$. The third condition in (3.32) is then easily satisfied once $n$ is large enough.

Since $\log n \sim \log N$, (3.33) specified to $m = m_0$ reduces to the hypothesis of Proposition 3.1, and if it holds for $m = m_0$, it is necessarily true in the entire range $m_0 \leq m \leq n$. It follows that, under our hypothesis, the above construction can be implemented for each $m$ verifying $m_0 \leq m \leq n$. Moreover, since the estimates on the probabilities of the exceptional sets corresponding to different values of $m$ are exponential in $-m$ (as shown above), the sum of the probabilities involved is small. Consequently, the construction can be implemented *simultaneously* for all such $m$ with the resulting space satisfying the full assertion of Proposition 3.1 with probability close to 1. □

**Remark** We now sketch arguments that are behind the comments following Theorem 2.3 and concern simultaneous saturations. They are based on the following observation: the construction remains virtually unchanged if we add "not too many" additional arbitrary factors of dimension not exceeding $k$ to – for example – the $\ell_1$ sum defining $Z$; see the beginning of the proof of Proposition 3.1. This is because the number of such factors affects the estimates rather lightly; in the context of Proposition 3.1, its only effect on the argument is via $N$ in the second expression on the right hand side of formula (3.33). Accordingly, if we replace $Z = \ell_1^N(W)$ by $\left(\bigoplus_{W \in \mathcal{W}} \ell_1^N(W)\right)_1$ (a direct sum in the $\ell_1$-sense), the binomial coefficient $\binom{N}{\ell}$ in (3.31) and the argument following it needs to be replaced by $\binom{N|\mathcal{W}|}{\ell} < (eN|\mathcal{W}|/\ell)^\ell$. The corresponding upper bound on $k$ becomes then $cm^2/(n \log(N|\mathcal{W}|))$. This implies the first comment. For the second comment it is enough to invoke the known fact that the family of $k$-dimensional normed spaces admits a $(1+\varepsilon)$-net (in the Banach-Mazur distance) whose cardinality is at most $\exp(\exp(C(\varepsilon)k))$. [This estimate and its optimality are quite straightforward to derive employing, in particular, the methodology of [G], but in fact they have been shown earlier, see [B]; we thank the referee for providing us with this reference.] Finally, the optimality of the $\exp(C(\varepsilon)k)$ dimension estimate for a space containing a $(1+\varepsilon)$-isomorphic copy of every $k$-dimensional normed space follows from the known estimates on nets of Grassmann manifolds, already used in Step III earlier in this section, and from the optimality of the $\exp(\exp(C(\varepsilon)k))$ estimate mentioned above.

## 4 Subspaces of spaces with finite cotype

As mentioned in Section 2, we shall prove Theorem 2.2 for $q > 4$ only, in which case it will be an immediate consequence of the following technical statement.

**Proposition 4.1** *Let $q \in (4,\infty)$ and set $\beta := (q+2)/(2q-2) \in (1/2,1)$ and $\gamma := (q+1)/(2q-2)$. Let $n$ and $m_0$ be positive integers with $n^\beta(\log n)^\gamma \le m_0 \le n$. Let $V$ be any normed space with*

$$\dim V \le c_0 \min\Big\{\frac{m_0}{n^\beta(\log n)^{\gamma-1/2}}, \frac{m_0^2}{n^{2\beta}(\log n)^{2\gamma}}\Big\}$$

*(where $c_0 \in (0,1)$ an appropriate numerical constant). Then there exists an $n$-dimensional normed space $Y$ whose cotype $q$ constant is bounded by a function of $q$ and $C_q(V)$ and such that, for any $m_0 \le m \le n$, every $m$-dimensional subspace $\tilde{Y}$ of $Y$ contains a 1-complemented subspace isometric to $V$.*

In fact we know how to show a version of Proposition 4.1 under the hypothesis $\dim V \le c_q\, m_0/(n^{1-\eta}(\log n)^{(1-2\eta)/3})$, where $\eta := (q-2)/(2q+2)$ ($\in (0,1/2)$) and $c_q > 0$ is a constant depending on $q$ only. This gives a nontrivial outcome for any $q > 2$, and thus proves Theorem 2.2 in the full generality. However, the argument is much more subtle and involved than the one presented below.

*Proof of Proposition 4.1* The idea behind the argument is as follows. The starting point is a dual $X^*$ of the space from Theorem 2.1 which, in the present notation, is a subspace of $\ell_\infty^N(V)$, where, in particular, $N < n^2$. The trick is to consider the same subspace, but this time endowed with the norm inherited from $\ell_q^N(V)$ (which gives control of the cotype property). Since the ratio between the two norms is at most $N^{1/q}$, and since – as may be verified – the margin of error in conditions ensuring the assertion of Theorem 2.1 was also a (small but fixed) power of $n$, the rest of the proof carries over to the new setting if $q$ is large enough.

And here are the main points of the argument. Fix $q > 4$ and let $p = q/(q-1)$ be the conjugate exponent. Let $1 \le k \le m \le n \le kN$ be positive integers verifying conditions (3.32). We shall specify $N$ (depending on $q$) at the end of the proof when we shall also derive additional restrictions on $k$ and $m$. If $V$ is such that $\dim V = k$, we let $Z_p = \ell_p^N(V^*)$ and let $X_p = X_p(\omega)$ be the Gaussian quotient of $Z_p$ obtained analogously as in the proof of Proposition 3.1 with $W = V^*$; denote its unit ball by $K_p$. Otherwise, we shall use the same notation as that of the proof of Proposition 3.1, and we shall make the same choice of $\ell$. The dual spaces $X_p^* = X_p(\omega)^*$ are thus isometric to subspaces of $Z_p^* = \ell_q^N(V)$ and so their cotype $q$ constants are uniformly bounded (depending on $q$ and the cotype $q$ constant of $V$). We claim that once all the parameters are properly chosen then, outside of a small exceptional set, $Y = X_p(\omega)^*$ satisfies also the (remaining) condition of the assertion of Proposition 4.1 involving the subspaces isometric to $V$. That condition can be restated as follows: every quotient of $X_p(\omega)$ of dimension $m \ge m_0$ contains a 1-complemented subspace isometric to $V^*$ for values of $k$ described in Proposition 4.1. Thus we have a very similar problem to the one encountered in Proposition 3.1, and the strategy is to show that simple modifications of the argument applied then do yield the result asserted in the present context.

To that end, observe that $F_j \cap B_{Z_p} = F_j \cap B_Z$ for all $j \in \{1, \ldots, N\}$ (recall that $F_1, \ldots, F_N$ are $k$-dimensional coordinate subspaces of $\mathbb{R}^{Nk}$, defined in the paragraph preceding (3.2)). Moreover, $B_Z \subset B_{Z_p} \subset N^{1/q} B_Z$. Consequently, the same inclusions hold for the images of these balls by $G$ and $\tilde{G}$ and their appropriate sections and projections. Similarly, for any $j \in \{1, \ldots, N\}$, the set

$$\tilde{L}'_j := \tilde{G}(\mathrm{span}\,[F_i : i \neq j] \cap B_{Z_p})$$

(cf. (3.2)) satisfies $\tilde{L}'_j \subset (N-1)^{1/q} \tilde{K}'_j$. Hence (cf. (3.6))

$$\Big\{ \omega \in \Omega : P_{\tilde{E}_j}(\tilde{L}'_j) \subset (N-1)^{1/q} \kappa B_2^m \Big\} \supset \Omega'_j,$$

where, as before, $\kappa$ satisfies (3.21). Accordingly, for the argument of Proposition 3.1 to carry over to the present setting (with the same exceptional set!), we need to require, in place of (3.30), a stronger inequality

$$(N-1)^{1/q} \cdot 2\kappa \leq \frac{1}{\sqrt{k}} \cdot \frac{1}{4}\sqrt{\frac{m}{n}}. \tag{4.1}$$

It remains to reconcile this condition with (3.32), which provides *lower* bounds on $\kappa$ that may be summarized as

$$\kappa^2 \geq C_0 \max\left\{\frac{n\log n}{N}, \frac{k}{m}, \frac{\log N}{m}\right\}, \tag{4.2}$$

where $C_0 > 1$ is a numerical constant. [These bounds come from the first two inequalities in (3.32); as before, the third inequality will be automatically satisfied.] We now choose $\kappa$ to verify (4.1), say, $\kappa = \sqrt{m/(64\,nk)}\,N^{-1/q}$, plug this value into (4.2) and solve the obtained inequalities for $k = \dim V$. This leads to the constraint

$$\dim V \leq c_0' \min\left\{\frac{mN^{1-2/q}}{n^2\log n}, \frac{m}{\sqrt{n}N^{1/q}}, \frac{m^2}{nN^{2/q}\log N}\right\}, \tag{4.3}$$

where $c_0' \in (0,1)$ is a universal constant. It remains to choose $N$ to optimize this constraint. However, we are dealing with a range of possible values of $m$ and there is no universal optimal choice. Still, concentrating on the first two expressions in (4.3), for which the best value of $N$ is of order $(n^{3/2}\log n)^p$, gives a "near optimal" outcome asserted in Proposition 4.1: just replace everywhere $N$ by $(n^{3/2}\log n)^p$, note that then $\log N \sim \log n$, and that the most restrictive condition comes from the smallest value of $m$, namely $m = m_0$. (Ignoring the third expression in (4.3) when optimizing $N$ is justified by the fact that it is larger than the second one except in a narrow "logarithmic" range of $m$.) The lower bounds $n^\beta(\log n)^\gamma$ on $m$ and 4 on $q$ are included to indicate the values for which the Proposition is of interest. As is easily seen, if any of them is violated, then the only allowed value for $\dim V$ is 0. By contrast, if $q > 4$ and, say, $m_0$ remains a fixed proportion of $n$, then the upper bound on $\dim V$ grows without bound as $n \to \infty$, as needed for non-trivial applications to Theorem 2.2-like statements. □

# 5 Appendix

*Proof of Lemma 3.4* Let $A = (a_{ij})$ be a $d \times m$ random matrix with independent $N(0, 1/m)$-distributed Gaussian entries. We first show that an estimate very similar to the assertion of Lemma 3.4 holds if we replace $P_H$ by $A$, and then present a rather general argument which allows to deduce the estimates for $P_H$. (Other variants of a general argument connecting Gaussian and orthogonal settings were developed recently in [MT2].) We shall consider only the case $m > 1$ (the assertion of the Lemma holds trivially if $m = 1$), and we shall assume that $S$ is closed.

We begin by applying the Chevet-Gordon inequality (see, e.g., [Go], Theorem 5) to obtain

$$\mathbb{E} \max_{x\in S} |Ax| \le \sqrt{\frac{d}{m}}\, a + M^*(S); \tag{5.1}$$

[In fact a slightly stronger inequality holds: $\sqrt{d}$ and $M^*(S)$ may be replaced by the smaller quantities $\mathbb{E}|\cdot|$ and $\mathbb{E}\sup_{y\in S} \langle x, y\rangle/\sqrt{m}$, respectively. The expectations are being taken with respect to the standard Gaussian measure on the appropriate Euclidean space.] Next, since the function $A \to \max_{x\in S} |Ax|$ is $a$-Lipschitz with respect to the Hilbert-Schmidt norm, it follows from the Gaussian isoperimetric inequality (see, e.g., [LT]) that, for any $t > 0$,

$$\mathbb{P}\left( \max_{x\in S} |Ax| \ge \sqrt{\frac{d}{m}}\, a + M^*(S) + t \right) \le e^{-t^2 m/2a^2}. \tag{5.2}$$

(Notice that the variance of each entry of $A$ is $1/m$.) Denote the normalized Haar measure on $G_{m,d}$ by $\nu$. Lemma 3.4 will easily follow if we show that, for any $\tau > 0$,

$$\mathbb{P}\left( A : \max_{x\in S} |Ax| \ge \tau \right) \ge e^{-1}\, \nu\left( H \in G_{m,d} : \max_{x\in S} |P_H x| \ge \tau \right). \tag{5.3}$$

To that end, we note first that, in the calculation of the measure on the right hand side above, we may replace $P_H$ by $R_d U$ and $\nu$ by $\mu$, where $R_d : \mathbb{R}^m \to \mathbb{R}^d$ is the restriction to the first $d$ coordinates, $U \in O(m)$, and $\mu$ is the Haar measure on $O(m)$; this follows from the invariance of $\nu$ under the action of $O(m)$. Similarly, in the calculation of the probability we may replace $A$ by $(AA^*)^{1/2} R_d U$ and $\mathbb{P}$ by $\mathbb{P} \otimes \mu$. Indeed, by the invariance of the Gaussian measure under the action of $O(m)$, the radial and the polar parts of $A$ are independent and the latter must have the same distribution as $R_d U$. In other words, the inequality above can be rewritten as

$$\mathbb{P} \otimes \mu \left( \max_{x\in S} |(AA^*)^{1/2} R_d U x| \ge \tau \right) \ge e^{-1}\, \mu\left( \max_{x\in S} |R_d U x| \ge \tau \right). \tag{5.4}$$

This will follow from the Fubini theorem once we show a similar inequality for *conditional* probabilities for any fixed $U$. To that end, consider any $U \in O(m)$ verifying the condition on the right hand side of (5.4), i.e., such that there exists $y \in R_d US$ for which $|y| \geq \tau$. Then

$$\begin{aligned}\mathbb{P}\left(\max_{x\in S}|(AA^*)^{1/2}R_dUx| \geq \tau\right) &\geq \mathbb{P}\left(|(AA^*)^{1/2}y| \geq \tau\right) \\ &= \mathbb{P}\left(|A^*y| \geq \tau\right) \geq \mathbb{P}\left(|A^*y| \geq |y|\right).\end{aligned}$$

Again by rotational invariance of the Gaussian measure, the distribution of the random vector $A^*y$ depends only on $|y|$ and is an appropriate multiple of the standard Gaussian vector on $\mathbb{R}^m$. Accordingly, $\mathbb{P}\left(|A^*y| \geq s\right)$ may be expressed in terms of the appropriate $\chi^2$ or Gamma distributions. For example, for any $y \neq 0$,

$$\mathbb{P}\left(|A^*y| \geq |y|\right) = \mathbb{P}\left(\chi_m^2 \geq m\right) = (\Gamma(m/2))^{-1}\int_{m/2}^{\infty} e^{-u}u^{m/2-1}du =: \gamma_m.$$

It follows from the central limit theorem that $\gamma_m \to 1/2$ as $m \to \infty$. Moreover, it is easily verified that $\gamma_m \geq \gamma_2 = e^{-1}$ if $m \geq 2$. This shows (5.4) for $m > 1$ and concludes the proof of the Lemma. □

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