Computational Neuroscience

aka MATH 378/478, BIOL 378/478, EECS 478, NEUR 478, EBME 478, COGS 378

Homework Problems # 2

Due Tuesday 2/26/08 in class or via electronic drop box at the course Bb site.

1. A simple model for a passive membrane as an RC circuit obeys the equation

$$C\frac{dV}{dt} = -g(V - V_0).$$

   Let $V_0 = -70$ mV and let $C/g = \tau$ ms. Suppose we implement a forward Euler numerical method to approximate trajectories of this system starting from $V(0) = -60$ mV. The Euler method gives us the following update rule with discrete time steps of length $\Delta t = h$:

$$v_{n+1} = v_n + h(-(v_n + 70)/\tau).$$

   We can view this system as a discrete-time dynamical system, and ask the following questions:

   (a) * What is the state space of this dynamical system?

   (b) * Determine the (unique) fixed point of the dynamical system.

   (c) * Find conditions on h relative to τ for which the fixed point is guaranteed to be stable.

   (d) ** Identify all bifurcations in the system as h varies (h is the control or bifurcation parameter; a bifurcation is a value of h at which the qualitative nature of the solutions changes character e.g. from stable to unstable or from nonoscillatory to oscillatory).

2. * 1-D system. Use the dfield tool (Matlab) or XPP to plot the direction fields for Izhikevich's Leak + Instantaneous $I_{Na,P}$ model:

$$\begin{aligned} C\dot{V} &= I - g_L(V - E_L) - g_{Na}m_\infty(V)(V - E_{Na}) \\ m_\infty(V) &= 1/(1 + \exp\{(V_{1/2} - V)/k\}) \end{aligned}$$

   with parameters given by Izhikevich on page 56. Plot the direction field for different values of I_0 to show how the number of fixed points changes across the bifurcation point. (You may use the XPP script from the course website).

3. * Consider a system of two RC elements (take R=1 and C=1) connected with a resistance of 1/20, as in the handout on numerical methods. The system of equations is

$$\begin{aligned} \dot{V}_1 &= -V_1 + 20(V_2 - V_1) \\ \dot{V}_2 &= -V_2 + 20(V_1 - V_2) \end{aligned}$$

Write this system in 2×2 matrix notation, find the eigenvalues and eigenvectors about the (unique) steady state, determine its stability, and relate the eigenvectors and eigenvalues to the behavior of the system as described in the handout and in class.

4. * Using either XPP, the PPlane tool for Matlab or another method for visualizing two-dimensional vector fields, produce at least two different versions of each of the following phase portraits by choosing different values of a, b, c and d. *Hint: Before you begin, read problem 5.*

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned}$$

   (a) A saddle point.
   (b) A stable focus (aka spiral sink).
   (c) An unstable node.
   (d) A center.

5. ** For each of the eight systems you described in problem 4, determine (either analytically or numerically) the eigenvalues of the equilibrium.

6. ** Izhikevich Chapter 3 problem 12: draw a bifurcation diagram for the I_{Kir} model using

   (a) I as the bifurcation parameter;
   (b) g_L as the bifurcation parameter; and
   (c) g_{Kir} as the bifurcation parameter.

7. Consider a version of the Fitzhugh Nagumo (FHN) equations:

$$\begin{aligned} v' &= v * (1 - v) * (v - a) - w + I_0 \\ w' &= \varepsilon(v - \gamma w) \end{aligned}$$

   with parameters a=.25, ε=.05, γ=1.

   (a) * Calculate the steady state for $I_0 = 0$ and for $I_0 = 0.25$.

(b) * For both values above, calculate the Jacobian of the system at the fixed point.

(c) * Find the eigenvalues and eigenvectors of the Jacobian (either by hand or using XPP, pplane (in Matlab) or another computer system), and determine the stability and local phase portrait of the fixed point.

(d) * Vary the "injected current" I_0 to get within $\pm$ 0.01 of the point where the eigenvalues become pure imaginary numbers. Produce global phase portraits and line fields for the system with I_0 roughly 20% above and 20% below this value. Describe the resulting phase portraits in words.

(e) ** Find parameters for a, ε and γ for which the FHN equations produce bistability, and demonstrate with appropriate phase diagrams.

8. *** Suppose

$$C\dot{V} = \sum_{k=1}^{K} g_k(V)(E_k - V) + I(t)$$

where $I(t)$ is integrable and bounded and the $g_k(V)$ are smooth and bounded, *i.e.* there exist constants g_k^+ such that for each k and for all V,

$$0 \leq g_k(V) \leq g_k^+.$$

Furthermore, assume that together, the set of voltages is bounded away from zero uniformly in V, *i.e.* for all V, $\min_k(g_k(V)) \geq g_- > 0$. This means that there is no voltage at which all the conductances are completely shut off.

(a) Let the current $I(t)$ be fixed at a constant value, I. Show that $V(t)$ must approach some steady-state value (which might depend on its initial condition) as $t \to \infty$.

(b) Now let the current $I(t)$ vary in time. Can we be sure that $V(t)$ is bounded for all time? Prove (and give explicit bounds) or else provide a counter example.

9. *** (Revised 2/15/08) Look up and state the Routh-Hurwitz conditions for the stability of systems of order 1, 2, 3 and 4. Describe how you would use these conditions to evaluate the stability of the full HH system at a given equilibrium point $v_{ss}, h_{ss}, m_{ss}, n_{ss}$?

10. *** (Revised 2/18/08) Application of the Implicit function theorem (IFT). Look up and state the implicit function theorem. Consider a smooth autonomous dynamical system

$$x' = f(x, \mu)$$

for $x \in \mathbb{R}^n$, and $\mu \in \mathbb{R}$. Let $\mathcal{C}$ be the set of equilbria in $\mathbb{R}^{n+1} = (x, \mu)$:

$$\mathcal{C} = \{(x, \mu) | f(x, \mu) = 0\}$$

Let $J(\mu)$ be the Jacobian

$$J(\mu) = \frac{\partial f(x, \mu)}{\partial x}.$$

Suppose $\mathcal{C}$ branches at a particular value of μ, say μ_c. That is, suppose there are two or more branches of the curve $\mathcal{C}$ (when viewed as a curve with x as a function of μ) and these multiple branches intersect each other at a particular value of μ, say $\mu = \mu_c$. Explain why $J(\mu_c)$ must have a zero eigenvalue.

* denotes problems to be done by all students.

** denotes problems to be done students enrolled in MATH 378 or any of the cross-lists numbered 478. Any math or statistics majors and any graduate students enrolled in any section should do these problems as well.

*** denotes problems to be done by students enrolled under MATH 478, or math graduate students enrolled under any course number.