On the Homothety Conjecture *

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Abstract

Let K be a convex body in $\mathbb{R}^n$ and $\delta > 0$. The homothety conjecture asks: Does $K_\delta = cK$ imply that K is an ellipsoid? Here K_δ is the (convex) floating body and c is a constant depending on δ only. In this paper we prove that the homothety conjecture holds true in the class of the convex bodies B_p^n, $1 \le p \le \infty$, the unit balls of l_p^n; namely, we show that $(B_p^n)_\delta = cB_p^n$ if and only if $p = 2$. We also show that the homothety conjecture is true for a general convex body K if δ is small enough. This improvs earlier results by Schütt and Werner [16] and Stancu [20].

1 Introduction

Floating bodies appear in many contexts and have been widely studied (see e.g. [1, 2, 4, 5, 6, 8, 13, 14, 15, 19, 20, 21, 23, 26]). The homothety conjecture is among the problems related to floating bodies that was open for a long time. It asks:

Does K have to be an ellipsoid, if K is homothetic to K_δ for some $\delta > 0$?

In [16], Schütt and Werner obtained a (partial) positive solution to this conjecture. They showed that if there is a sequence $\delta_k \to 0$ such that K_{δ_k} is homothetic to K for all $k \in \mathbb{N}$ (with respect to the same center of homothety), then K is an ellipsoid.

Stancu [20] (see also [19]) proved that if for a convex body K with boundary of class C_+^2 there exists a positive number $\delta(K)$ such that K_δ is homothetic to K for some $\delta < \delta(K)$, then and only then K is an ellipsoid.

These results are not completely satisfactory for different reasons. The first one requires a sequence K_{δ_k} of convex bodies to be homothetic to the body K. The

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second one needs smoothness assumptions on ∂K and, in addition, δ has to be sufficiently small, but no estimates are given how small. Moreover, these results do not work even in the case of very basic convex bodies, such as B_p^n, $1 \leq p \leq \infty$, the unit balls of l_p^n.

If $\delta = \tau \ \mathrm{vol}_n(K)$ for some $0 \leq \tau < \frac{1}{2}$, and if K is centrally symmetric, Milman and Pajor [9] showed that the Banach-Mazur distance of K_δ to an ellipsoid is bounded (with constants depending only on τ, but not depending on n, which explode when $\delta \to 0$). This shows, in some sense, that if a centrally symmetric convex body is far from being an ellipsoid, then K_δ is not homothetic to K.

In this paper, we give a positive solution of the homothety conjecture. Namely we prove

Theorem 2 *Let K be a convex body in $\mathbb{R}^n$. There exists a positive number $\delta(K)$, such that the following are equivalent:*

(i) K_δ is homothetic to K for some $0 < \delta < \delta(K)$;

(ii) K is an ellipsoid.

Our proof is different from Stancu's proof. No smoothness assumptions are required. In fact, the main ingredient in our proof is to show that the homothety assumption implies that ∂K is C_+^2. This is done in Lemmas 2 and 4.

In addition, our proof of Theorem 2 gives a possibility to estimate the threshold $\delta(K)$ for convex bodies K in $\mathbb{R}^n$ that have sufficiently smooth boundary. This is done in Section 5.

We also show the following theorem, which provides a positive solution of the homothety conjecture for the B_p^n balls, $1 \leq p \leq \infty$, and their affine images without any requirements on the size of δ.

Theorem 1 *Let $B_p^n, 1 \leq p \leq \infty$ be the unit ball of l_p^n. Let $0 < \delta < \frac{|B_p^n|}{2}$. Then $(B_p^n)_\delta = cB_p^n$ for some $0 < c < 1$ if and only if $p = 2$.*

Throughout the paper we will use the following notations. For $1 \leq p \leq \infty$, $B_p^n = \{x \in \mathbb{R}^n : \|x\|_p \leq 1\}$ is the unit ball of $l_p^n = (\mathbb{R}^n, \| \ \|_p)$ where for $1 \leq p < \infty$

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}},$$

and for $p = \infty$

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|.$$

Let $u \in S^{n-1} = \partial B_2^n$, the boundary of the Euclidean unit ball B_2^n. Then $H(x,u) = \{y \in \mathbb{R}^n, \langle y, u\rangle = \langle x, u\rangle\}$ is the hyperplane through x with outer unit normal vector u. The two half-spaces generated by $H(x,u)$ are $H^-(x,u) = \{y \in \mathbb{R}^n, \langle y,u\rangle \geq \langle x,u\rangle\}$ and $H^+(x,u) = \{y \in \mathbb{R}^n, \langle y,u\rangle \leq \langle x,u\rangle\}$. $\| \ \|$ is the Euclidean norm on $\mathbb{R}^n$.

For a convex body K in $\mathbb{R}^n$ and $x \in \partial K$, the boundary of K, $N_K(x)$ denotes an outer unit normal vector to K and $\kappa_K(x)$ the Gauss curvature of K at x. $N_K(x)$ exists almost everywhere (see[11]). $\text{int}(K)$ is the interior of K. We write $|K|$ or $\text{vol}_n(K)$ for the volume of K. We say that K is in C^2_+, if ∂K is C^2 and has everywhere strictly positive Gauss curvature. Without loss of generality, we will assume throughout the paper that 0 is in the interiors of K and K_δ, and that K homothetic to K_δ is meant with 0 as the center of homothety.

The paper is organized as follows. In Section 2 we provide background and prove some properties of the convex floating body that are needed in the next sections. In Section 3 we prove that the homothety conjecture holds true for B_p^n, $1 \leq p \leq \infty$. In Section 4 we prove Theorem 2. Moreover, we give (partial) positive solutions of a generalized homothety conjecture. In Section 5 we give estimates on the threshold $\delta(K)$.

2 The convex floating body and homothety

Let K be a convex body with $0 \in \text{int}(K)$ and δ be a positive number such that $\delta < \frac{|K|}{2}$. The convex floating body is defined as follows [15].

Definition 1 *[15] Let K be a convex body in $\mathbb{R}^n$. The convex floating body K_δ is the intersection of all halfspaces H^+ whose defining hyperplanes H cut off a set of volume at most δ from K,*

$$K_\delta = \bigcap_{|H^- \cap K| \leq \delta} H^+.$$

Clearly, $K_0 = K$ and $K_\delta \subset K$ for all $\delta \geq 0$. Moreover, the convex floating body has the following property: for all (invertible) affine maps T on $\mathbb{R}^n$ and for all $\delta > 0$

$$(TK)_\delta = T\left(K_{\frac{\delta}{|det(T)|}}\right). \tag{1}$$

Here $|\det(T)|$ is the absolute value of the determinant of T. In particular, for an affine map T with $|\det(T)| = 1$, $(TK)_\delta = T(K_\delta)$ for all $\delta \geq 0$.

An ellipsoid $\mathcal{E}$ is the affine image of B_2^n, $\mathcal{E} = T(B_2^n)$, for some invertible affine map T on $\mathbb{R}^n$. It is easy to see that $(B_2^n)_\delta = cB_2^n$ for all $\delta \geq 0$ and for a constant

$c = c(\delta, n) < 1$ depending on δ and n only. Hence one gets with (1) that for all ellipsoids $\mathcal{E}$

$$\mathcal{E}_\delta = (T(B_2^n))_\delta = T\left((B_2^n)_{\frac{\delta}{|\det(T)|}}\right) = T\left(c\left(\frac{\delta}{|\det(T)|}\right)B_2^n\right) = c\left(\frac{\delta}{|\det(T)|}\right)\mathcal{E}$$

for all $\delta \geq 0$ and for some constant $c\left(\frac{\delta}{|\det(T)|}\right) < 1$. In other words, if the homethety conjecture holds true, then $K_\delta = cK$ for some $\delta > 0$ and some constant $0 < c < 1$ if and only if K is an ellipsoid in $\mathbb{R}^n$.

Now we make some general observations concerning homothety of K with one of its convex floating bodies. First, only a strictly convex body can be homothetic to one of its convex floating bodies. This is a consequence of the flowing lemma proved in [16].

Lemma 1 *[16] Let K be a convex body in $\mathbb{R}^n$, and let $0 < \delta < |K|/2$. Then K_δ is strictly convex.*

Thus, in particular, no polytope can be homothetic to one of its floating bodies. The next lemma and its proof are almost identical to Lemma 3 of [16]. We give the proof for completeness.

Lemma 2 *Let K be a convex body in $\mathbb{R}^n$, and let $\delta > 0$. If K_δ is homothetic to K, then ∂K is of class C^1.*

Proof. Suppose that ∂K is not of class C^1. Then there is $x_0 \in \partial K$ so that ∂K has two different supporting hyperplanes, H_1 and H_2, passing through x_0. We may assume that there are two sequences $(x_k^1)_{k\in\mathbb{N}}$ and $(x_k^2)_{k\in\mathbb{N}}$ on ∂K converging to x_0 so that we have:
(i) the supporting hyperplanes H_k^i through x_k^i, $k \in \mathbb{N}$, $i = 1, 2$ are unique;
(ii) the hyperplanes H_k^i converge to H_i, $i = 1, 2$;
(iii) $\lim_{k\to\infty} N_K(x_k^i)$ is orthogonal to H_i, $i = 1, 2$.

See the proof of Lemma 3 in [16] for the construction of these sequences. We choose a coordinate system such that $x_0 = 0$,

$$H_1 = \{x \in \mathbb{R}^n : x(n) = ax(n-1)\},$$

and

$$H_2 = \{x \in \mathbb{R}^n : x(n) = bx(n-1)\},$$

with $b < a$ where $x(l)$ denotes the l-th coordinate of the vector $x \in \mathbb{R}^n$. Let x_δ be the point on ∂K_δ that corresponds to x_0 by homothety. We can assume that in the chosen coordinate system $x_\delta = (0,\ldots,0,x_\delta(n))$ with $x_\delta(n) > 0$. Let

$$H = \left\{ x + x_\delta \in \mathbb{R}^n : x(n) = \frac{a+b}{2} x(n-1) \right\}.$$

By homothety, one sees that H is a support hyperplane of ∂K_δ. For $\alpha, \beta \in \mathbb{R}$, let

$$M(\alpha,\beta) = \left\{ x \in \mathbb{R}^n : x \in K \cap H^-, x(n-1) = \alpha, x(n) = \beta \right\},$$

and let M^* be such a set for which the $(n-2)$-dimensional volume is maximal

$$\mathrm{vol}_{n-2}(M^*) = \max_{(\alpha,\beta)\in\mathbb{R}^2} \mathrm{vol}_{n-2}\left(M(\alpha,\beta)\right).$$

We consider the set in the $(x(n-1), x(n))$-plane that consists of all points (α,β) so that $M(\alpha,\beta) \neq \emptyset$. This set is contained in the triangle T bounded by the lines $x(n) = ax(n-1)$, $x(n) = bx(n-1)$, and $x(n) = \frac{a+b}{2}x(n-1) + x_\delta(n)$. Therefore

$$\begin{aligned} \delta \leq |K \cap H^-| &= \int_{\mathbb{R}^2} M(\alpha,\beta) d(\alpha,\beta) \leq \mathrm{vol}_{n-2}(M^*)\mathrm{vol}_2(T) \\ &= \mathrm{vol}_{n-2}(M^*) \frac{2|x_\delta(n)|^2}{a-b}. \end{aligned} \tag{2}$$

Now we consider the sets $K \cap (x_\delta + H_i)^-$, $i = 1,2$. It follows from (i), (ii) and Lemma 2 of [16] that

$$|K \cap (x_\delta + H_i)^-| = \delta, \quad i = 1,2.$$

This is because $x_\delta + H_i, i = 1,2$ are tangent hyperplanes of K_δ. Let $\varepsilon > 0$ be given. By (iii) we may choose x_1 and x_2 in ∂K such that for $i = 1,2$

$$\begin{aligned} &\left(\left| x_i(n-1) - P_{H_i}(x_i)(n-1) \right|^2 + \left| x_i(n) - P_{H_i}(x_i)(n) \right|^2 \right)^{\frac{1}{2}} \leq \\ &\qquad \varepsilon \left(\left| P_{H_i}(x_i)(n-1) \right|^2 + \left| P_{H_i}(x_i)(n) \right|^2 \right)^{\frac{1}{2}}, \end{aligned} \tag{3}$$

where P_{H_i} denotes the orthogonal projection onto H_i, $i = 1,2$. As $K_\delta \subset K$, we can choose x_1 and x_2 in ∂K such that for $i = 1,2$, $P_{H_i + x_\delta}(x_i) \in K$ and (3) still holds.

Moreover, as K is strictly convex, by Lemma 1 and homothety, we have that for $i = 1,2$

$$\left(\left| x_i(n-1) - P_{H_i}(x_i)(n-1) \right|^2 + \left| x_i(n) - P_{H_i}(x_i)(n) \right|^2 \right)^{\frac{1}{2}} > 0.$$

It follows that there is a constant $\bar{c} > 0$ depending only on K so that

$$x_\delta(n) \leq \bar{c} \left(\left| x_i(n-1) - P_{H_i}(x_i)(n-1) \right|^2 + \left| x_i(n) - P_{H_i}(x_i)(n) \right|^2 \right)^{\frac{1}{2}}. \tag{4}$$

For $i = 1$ or $i = 2$ the convex hull

$$\left[M^*, [x_i, P_{H_i+x_\delta}(x_i)]\right] \subset K \cap (x_\delta + H_i)^-.$$

We may assume that $i = 1$. Then

$$\begin{aligned} \delta = |K \cap (x_\delta + H_1)^-| &\geq \left| \left[M^*, [x_1, P_{H_1+x_\delta}(x_i)]\right] \right| \\ &\geq \frac{1}{n^2} \text{vol}_{n-2}(M^*)\ d(x_1, H_{M^*})\ d\left(P_{H_1+x_\delta}(x_1), \tilde{H}\right), \end{aligned} \quad (5)$$

where H_{M^*} is the $(n-2)$-dimensional flat containing M^* and $\tilde{H}$ is the plane containing M^* and x_1. If ε is sufficiently small, we obtain by elementary computation

$$d(x_1, H_{M^*}) \geq c_1 \left(\left|P_{H_1}(x_1)(n-1)\right|^2 + \left|P_{H_1}(x_1)(n)\right|^2 \right)^{\frac{1}{2}},$$

and

$$d\left(P_{H_1+x_\delta}(x_1), \tilde{H}\right) \geq c_2 |x_\delta(n)|,$$

where c_1, c_2 depend only on K. Therefore we obtain by (2) and (5)

$$\frac{2|x_\delta(n)|}{a-b} \geq \frac{c_1\ c_2}{n^2} \left(\left|P_{H_1}(x_1)(n-1)\right|^2 + \left|P_{H_1}(x_1)(n)\right|^2 \right)^{\frac{1}{2}}.$$

By (3) we get with a new constant $\bar{c} > 0$

$$|x_\delta(n)| \geq \frac{\bar{c}}{\varepsilon} \left(\left|x_1(n-1) - P_{H_1}(x_1)(n-1)\right|^2 + \left|x_1(n) - P_{H_1}(x_1)(n)\right|^2 \right)^{\frac{1}{2}}.$$

If we choose ε sufficiently small we get a contradiction to (4).

The following symmetric matrix is closely related to the curvature of the floating body [4] (see also [16]). For $x \in \text{int}(K)$, $\xi \in S^{n-1}$, and an orthonormal coordinate system in the plane $H(x, \xi)$ with origin x, let $l(\eta)$ be the line through x with direction η and $y = l(\eta) \cap \partial K$. Let $r(\eta) = \|x - y\|$ and $\beta(\eta)$ be the angle between $l(\eta)$ and a tangent line to ∂K through y whose orthogonal projection onto $H(x, \xi)$ is $l(\eta)$. Define for $1 \leq i, j \leq n-1$

$$Q(i,j) = \frac{1}{|K \cap H(x,\xi)|} \int_{S^{n-2}} \eta_i \eta_j\ r^n(\eta)\ \cot(\beta(\eta))\ d\sigma_{n-2}(\eta), \quad (6)$$

where σ_{n-2} is the surface measure on S^{n-2}. For $x_\delta \in \partial K_\delta$, we will use $H(x_\delta, \delta)$ to denote the hyperplane through x_δ cutting off a set of volume δ from K. Such hyperplanes always exist by Lemma 2 of [16].

Then we have the following lemma [4] (see also [16])

Lemma 3 *[4] Let K be a convex body of class C^1 and let $\delta > 0$. Suppose that for every $x_\delta \in \partial K_\delta$ and every $H(x_\delta, \delta)$ the matrix Q is positive definite. Then ∂K_δ is of class C^2. Moreover, the Gauss curvature of K_δ at $x_\delta \in \partial K_\delta$ can be calculated by*

$$\frac{1}{\kappa_{K_\delta}(x_\delta)} = det(Q),$$

and hence, K_δ is of class C^2_+.

If K is strictly convex, C^1 and symmetric, then K_δ is C^2_+. This was shown in [6]. In the next lemma we show that this also holds in the non symmetric case, if δ is small enough. We will also use the following.

For every $x \in \partial K$, we define $\rho(x)$ to be the radius of the largest Euclidean ball that is contained in K and whose center lies on the line through x with direction $N_K(x)$, $\{x + tN_K(x) : t \in \mathbb{R}\}$. $\rho(x)$ is well defined because ∂K is of class C^1. Let $t(x) \geq \rho(x)$ be such that $x - t(x)N_K(x)$ is the center of the ball with radius $\rho(x)$ that is contained in K. $\rho(x)$ is a continuous, strictly positive function on ∂K because ∂K is of class C^1. By compactness there is $\rho_0 > 0$ so that we have for all $x \in \partial K$,

$$0 < \rho_0 \leq \rho(x). \tag{7}$$

Lemma 4 *If $K \in C^1$ is strictly convex, then there exists $\delta_0 > 0$, s.t., for all $0 < \delta < \delta_0$, the matrix Q is positive definite for every $x_\delta \in \partial K_\delta$. Hence, $K_\delta \in C^2_+$ for all $0 < \delta < \delta_0$.*

Moreover, if K is in addition in C^2_+, then the threshold δ_0 can be taken as

$$\delta_0 = \frac{\rho_0^{n-1} R |B_2^{n-1}|}{n\ 2^{n-1}} \left[1 - \left(1 - (\frac{\rho_0}{4R})^2\right)^{\frac{1}{2}}\right]^n, \tag{8}$$

where $R > 0$ (independent of δ and x) is such that $K \subset B_2^n(x - RN_K(x), R)$ for all $x \in \partial K$.

Proof. We want to show that there exists $\delta_0 > 0$, such that, for all $0 < \delta < \delta_0$, $\cot(\beta(\eta)) > 0$ for all $x_\delta \in \partial K_\delta$ and for all $\eta \in S^{n-2}$. This implies that (6) is a positive definite matrix and by Lemma 3 we get that ∂K_δ is of class C^2_+.

Let ρ_0 be as in (7). We first prove the following claim: Assume that K is strictly convex. Then there exists $\delta_1 > 0$, such that, for all $x_{\delta_1} \in \partial K_{\delta_1}$ and all $H(x_{\delta_1}, \delta_1)$, $\text{diam}\big(K \cap H(x_{\delta_1}, \delta_1)\big) \leq \frac{\rho_0}{4}$.

Suppose the claim does not hold. Then for all $\delta > 0$, there exist $x_\delta \in \partial K_\delta$ and $H(x_\delta, \delta)$, such that, $\text{diam}\big(K \cap H(x_\delta, \delta)\big) > \frac{\rho_0}{4}$. We denote by N_δ the normal vector of the hyperplane $H(x_\delta, \delta)$. In particular, for all $j \in \mathbb{N}$, there exists $x_j \in \partial K_{1/j}$, such that,

$$\mathrm{diam}\big(K \cap H(x_j, 1/j)\big) > \frac{\rho_0}{4}.$$

Therefore, for all $j \in \mathbb{N}$, there exist ξ_j and ζ_j in $\partial K \cap H(x_j, 1/j)$, such that, $\|\xi_j - \zeta_j\| \geq \frac{\rho_0}{8}$. Let $y_j = \frac{1}{2}(\xi_j + \eta_j)$. The sequences $\{y_j\}$, $\{\xi_j\}$ and $\{\zeta_j\}$ are in K and K is compact. Therefore, there exist subsequences (which we call again $\{y_j\}$, $\{\xi_j\}$ and $\{\zeta_j\}$), such that, $\lim_{j\to\infty} y_j = y_0$, $\lim_{j\to\infty} \xi_j = \xi_0$ and $\lim_{j\to\infty} \zeta_j = \zeta_0$. Clearly, y_0, $\xi_0, \zeta_0 \in \partial K$, $\xi_0 \neq \eta_0 \neq y_0$ and $y_0 = \frac{1}{2}(\xi_0 + \eta_0)$. This is a contradiction to the strict convexity of K. Hence the claim holds.

Let $\delta > 0$. Let $x_\delta \in \partial K_\delta$. Let $H(x_\delta, \delta)$ be a hyperplane through x_δ that cuts off exactly δ from K and let N_δ be the unit outer normal vector to $H(x_\delta, \delta)$. Let $x = x(\delta) \in \partial K$ be such that $N_K(x(\delta)) = N_\delta$. We now claim: Assume that K is C^1 and strictly convex. Then there exists $\delta_2 > 0$, such that for all x_{δ_2}, $\mathrm{dist}\,(x(\delta_2), H(x_{\delta_2}, \delta_2)) \leq \frac{\rho_0}{8}$. This second claim follows again by compactness.

Similar to above, we assume that the claim does not hold and find sequences $\{x_j\}_{j\in\mathbb{N}} \subset \partial K_{1/j} \cap H(x_j, \frac{1}{j})$ and $\{x(j)\}_{j\in\mathbb{N}} \subset \partial K$ with $\lim_{j\to\infty} x_j = x_0 \in \partial K$ and $\lim_{j\to\infty} x(j) = x_1 \in \partial K$ respectively. Let N_j be the outer unit normal vector to $H(x_j, \frac{1}{j})$. By assumption, $\|x_j - x(j)\| \geq \mathrm{dist}\left(x(j), H(x_j, \frac{1}{j})\right) > \frac{\rho_0}{8}$ for all j. Therefore, $\|x_0 - x_1\| \geq \frac{\rho_0}{8}$, hence $x_0 \neq x_1$. Passing to a subsequence if necessary, we get that $N_j \to N_0$, $N_0 \in S^{n-1}$. As ∂K is C^1 and as $N_K(x(j)) = N_j$, $N_0 = N_K(x_1)$ and $H(x_1, N_0)$ is a support hyperplane of K. $H(x_j, \frac{1}{j}) \to H(x_0, N_0)$, which is a hyperplane through x_0 that cuts off 0 of K. As K is strictly convex, $x_0 \notin H(x_1, N_0)$. Therefore, $H(x_0, N_0) \cap \mathrm{int}(K) \neq \emptyset$ and hence $|H(x_0, N_0)^- \cap K| > 0$, a contradiction.

The claims then imply that for all $\delta < \delta_0 = \min\{\delta_1, \delta_2\}$ the orthogonal projection of $K \cap H(x_\delta, \delta)$ is contained in

$$B_2^n\left(x(\delta) - t(x(\delta))N_K(x(\delta)), \frac{\rho_0}{4}\right) \cap H\left(x(\delta) - t(x(\delta))N_K(x(\delta)), N_K(x(\delta))\right).$$

From Figure 1 (with $x = x(\delta)$), it follows that $\cot(\beta(\eta)) \geq \cot(\alpha) \geq \frac{\rho_0}{4\,t(x)} > 0$. This proves the first part of the lemma.

If K is in addition in C^2_+, there exists $R > 0$, independent of δ and x, such that for all $x \in \partial K$, $K \subset B_2^n\left(x - RN_K(x), R\right)$. If we choose $s = R\left[1 - \left(1 - (\frac{\rho_0}{4R})^2\right)^{\frac{1}{2}}\right]$, then for all $x \in \partial K$, the orthogonal projection of

$$B_2^n\left(x - RN_K(x), R\right) \cap H\left(x - sN_K(x), N_K(x)\right)$$

is contained in $B_2^n\left(x - t(x)N_K(x), \frac{\rho_0}{4}\right) \cap H\left(x - t(x)N_K(x), N_K(x)\right)$.

We take δ_0 as in (8). Then for all $\delta \leq \delta_0$, one has

$$\left|H^-\left(x - sN_K(x), N_K(x)\right) \cap K\right| \geq \delta.$$

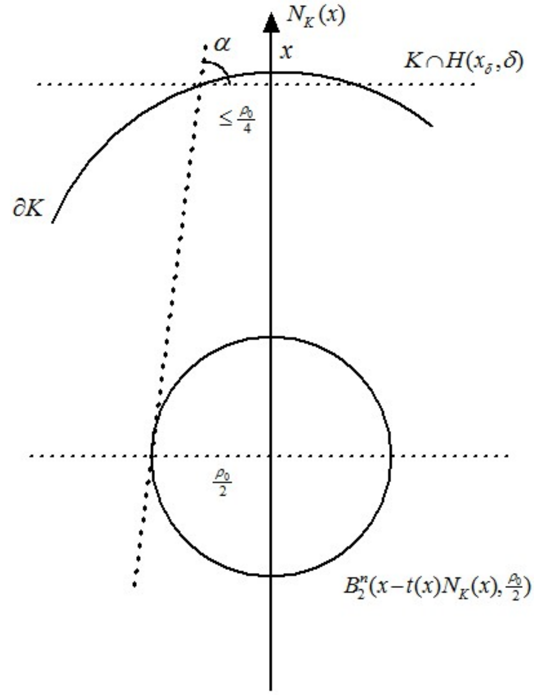

Figure 1:

Indeed, for $x \in \partial K$, let

$$C\left(t(x), \rho_0\right) = \text{conv}\big[x, H(x - t(x)N_K(x), N_K(x)) \cap B_2^n(x - t(x)N_K(x), \rho_0)\big].$$

Then,

$$\begin{aligned}\big|H^-\left(x - sN_K(x), N_K(x)\right) \cap K\big| &\geq \big|H^-\left(x - sN_K(x), N_K(x)\right) \cap C\left(t(x), \rho_0\right)\big| \\ &= \frac{\rho_0^{n-1} s^n |B_2^{n-1}|}{n\ t(x)^{n-1}} \geq \frac{\rho_0^{n-1} s^n |B_2^{n-1}|}{n\ 2^{n-1} R^{n-1}} \\ &= \frac{\rho_0^{n-1} R |B_2^{n-1}|}{n\ 2^{n-1}} \left[1 - \left(1 - (\frac{\rho_0}{4R})^2\right)^{\frac{1}{2}}\right]^n = \delta_0 \geq \delta.\end{aligned}$$

Similar to above, with $B_2^n(x - RN_K(x), R)$ instead of K in Figure 1, it follows that $\cot(\alpha) \geq \frac{\frac{\rho_0}{2} - \frac{\rho_0}{4}}{t(x)} > 0$ for all $0 < \delta < \delta_0$. From Figure 2, it follows that $\beta(\eta) \leq \gamma \leq \alpha$ and $\cot(\beta(\eta)) \geq \cot(\gamma) \geq \cot(\alpha) \geq \frac{\rho_0}{4\ t(x)} > 0$. This proves that Q is positive definite, and hence $K_\delta \in C_+^2$ for all $0 < \delta < \delta_0$.

Thus we have the following Proposition as a consequence of Lemmas 1, 2, 3, and 4.

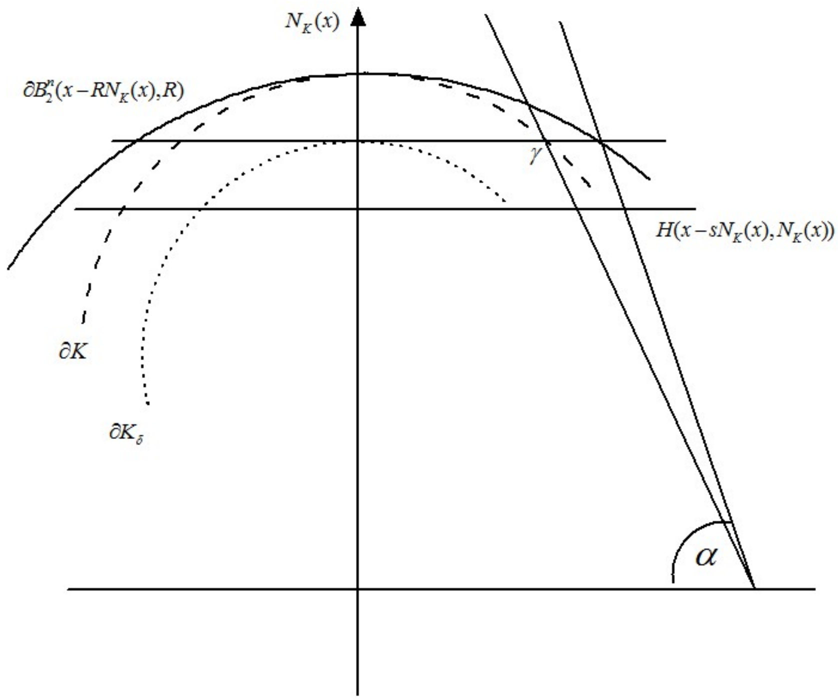

Figure 2:

Proposition 1 *A convex body K in $\mathbb{R}^n$ can only be homothetic to one of its floating bodies K_δ for small enough δ, if K is strictly convex and of class C^2_+.*

3 Homothety conjecture in the class $\mathcal{B}$=$\{B_p^n : 1 \le p \le +\infty\}$

In this section, we show that the homothety conjecture holds true in the class of the bodies B_p^n for all $1 \le p \le \infty$ and their affine images.

Theorem 1 *Let $B_p^n, 1 \le p \le \infty$ be the unit ball of l_p^n. Let $0 < \delta < \frac{|B_p^n|}{2}$. Then $(B_p^n)_\delta = c\ B_p^n$ for some $0 < c < 1$ if and only if $p = 2$.*

Remark. By (1), the same holds true for affine images $T\left(B_p^n\right)$ of B_p^n under an invertible linar map T on $\mathbb{R}^n$: *Let $K = T\left(B_p^n\right)$, $1 \le p \le \infty$. Let $0 < \delta < \frac{|K|}{2}$ be a constant. $K_\delta = c\ K$ for some constant $0 < c < 1$ if and only if K is an ellipsoid.*

Proof of Theorem 1. For the "if" part, it is enough to consider p with $1 < p < \infty$. Indeed, B_1^n and B_∞^n are polytopes and it was observed above that polytopes cannot be homothetic to any of their floating bodies.

We first consider the case when $1 < p < 2$. Then ∂B_p^n is of class C^1 but not of class C^2 at $e_n = (0, \cdots, 0, 1)$. If B_p^n were homothetic to $(B_p^n)_\delta$ for some δ, then Lemma 3 would imply that $(B_p^n)_\delta$ is C^2. Indeed, for all $x \in \partial(B_p^n)_\delta$, $0 < \beta(\eta) < \frac{\pi}{2}$ for all $\eta \in S^{n-2}$ and thus $\cot(\beta(\eta)) > 0$ or all $\eta \in S^{n-2}$. Therefore the matrix Q of (6) is positive definite and by Lemma 3 $(B_p^n)_\delta$ and thus, by homothety, B_p^n is C^2, a contradiction.

Now we consider the case $2 \leq p < \infty$. Then ∂B_p^n is of class C^2 and thus, assuming that B_p^n is homothetic to $\left(B_p^n\right)_\delta$ for some $\delta > 0$, $\left(B_p^n\right)_\delta$ is C^2.

It was shown in [18], that the curvature $\kappa_{B_p^n}$ at $x \in \partial B_p^n$ is

$$\kappa_{B_p^n}(x) = \frac{(p-1)^{n-1} \prod_{i=1}^n x_i^{p-2}}{\left(\sum_{i=1}^n |x_i|^{2(p-1)}\right)^{\frac{1}{2}}}. \tag{9}$$

Thus for $e_n = (0, \cdots, 0, 1)$, $\kappa_{B_p^n}(e_n) = 0$, if $p > 2$ and $\kappa_p(e_n) = 1$, if $p = 2$. Consequently, as we assume that $\left(B_p^n\right)_\delta = cB_p^n$ for some $\delta > 0$, the curvature at $ce_n \in \partial \left(B_p^n\right)_\delta$ is different from 0 only when $p = 2$ and for all $p > 2$

$$\kappa_{\left(B_p^n\right)_\delta}(ce_n) = 0. \tag{10}$$

By Lemma 3, $\kappa_{\left(B_p^n\right)_\delta}(ce_n) = \frac{1}{\det(Q)}$, which is 0 if and only if $\det(Q) = \infty$.

By (6), the matrix Q has entries

$$Q(i,j) = \frac{1}{|K \cap H(ce_n, e_n)|} \int_{S^{n-1}} \eta_i \eta_j \; r^n(\eta) \cot(\beta(\eta)) \; d\sigma_{n-2}(\eta),$$

$1 \leq i, j \leq n-1$. Again, for all $\eta \in S^{n-2}$, $0 < \beta(\eta) < \frac{\pi}{2}$. Therefore, $\det(Q) < \infty$ and thus the curvature at $ce_n \in \partial \left(B_p^n\right)_\delta$ is strictly positive which contradicts (10).

We point out that $(B_p^n)_\delta \in C_+^2$ for all $p \in (1, \infty)$ and all $\delta > 0$ can be obtained from results in [6].

More generally, in the same way as Theorem 1, one can prove

Proposition 2 *Let K be a convex body in $\mathbb{R}^n$. If K has a point on the boundary where the Gauss curvature is either 0 or ∞, then K is not homothetic to K_δ for any (sufficiently small) $\delta > 0$.*

Proof. Let x_0 in ∂K be such that $\kappa_K(x_0) = \infty$ and suppose that K is homothetic to K_δ for some $\delta > 0$. Then, by Proposition 1, K is strictly convex and in C_+^2. By Lemma 3, $\kappa_{K_\delta}(cx_0) = \frac{1}{\det(Q)}$, where cx_0 is the point on ∂K_δ corresponding to x_0 by homothety. This is ∞ if and only if $\det(Q) = 0$. But by Lemmas 3 and

4 (respectively Proposition 1), K is in C^2_+, thus Q is positive definite and thus $\det(Q) > 0$, a contradiction.

The case $\kappa_K(x_0) = 0$ is treated similarly.

4 Homothety conjecture for general K

In Section 2 we proved the homothety conjecture for $B^n_p, 1 \le p \le \infty$ and their affine images. The proof uses the fact that one only needs to examine one properly chosen direction in order to be able to conclude. In this section, we will use two directions to give a positive answer to the homothety conjecture. Moreover, our approach is robust and can be used to obtain (partial) positive solutions for *generalized homothety conjecture.*

Theorem 2 *Let K be a convex body in $\mathbb{R}^n$. There exists a positive number $\delta(K)$, such that the following are equivalent:*

(i) K_δ is homothetic to K for some $0 < \delta < \delta(K)$;

(ii) K is an ellipsoid.

In the next section we provide estimates for the threshold $\delta(K)$.

To prove this theorem, we need the following results. Proposition 3 was proved by Petty in [10] and Lemma 5 was proved in [15]. See also [26] for similar results.

Proposition 3 *[10] Let $K \in C^2_+$ be a convex body with boundary of class C^2 and everywhere strictly positive Gaussian curvature. Let $c(K,n)$ be a constant only depending on K and n. Then*

$$\kappa_K(x)\langle x, N_K(x)\rangle^{-(n+1)} = c(K,n), \quad \forall x \in \partial K$$

holds true if and only if K is an ellipsoid.

Lemma 5 *[15] Let $K \in C^2_+$ be a convex body with boundary of class C^2 and everywhere strictly positive Gaussian curvature. Then, for any $x \in \partial K$,*

$$\lim_{\delta \to 0} c_n \frac{\langle x, N_K(x)\rangle}{n\ \delta^{\frac{2}{n+1}}} \left[1 - \left(\frac{\|x_\delta\|}{\|x\|}\right)^n\right] = \big(\kappa_K(x)\big)^{\frac{1}{n+1}},$$

where $\{x_\delta\} = \partial(K_\delta) \cap [0,x]$ and $c_n = 2\left(\frac{|B^{n-1}_2|}{n+1}\right)^{\frac{2}{n+1}}$.

Proof of Theorem 2. Suppose that K is homothetic to K_δ for some $0 < \delta < \delta(K)$. $\delta(K)$ will be determined precisely in Section 5. By homothety and Lemmas 1, 2, 3, and 4, $K \in C^2_+$. Define, for $x \in \partial K$,

$$f_\delta(x) = \frac{c_n}{n\ \delta^{\frac{2}{n+1}}}\left[1 - \left(\frac{\|x_\delta\|}{\|x\|}\right)^n\right]. \tag{11}$$

Lemma 5 implies that for all $x \in \partial K$, $f_\delta(x) \to f(x) = \frac{\kappa_K(x)^{\frac{1}{n+1}}}{\langle x, N_K(x)\rangle}$ as $\delta \to 0$. If K is not an ellipsoid, then by Proposition 3, $f(x)$ is not constant. It follows that for δ small enough, $f_\delta(x)$ is not constant, which is impossible if K is homothetic to K_δ. Therefore, K must be an ellipsoid.

Note that, following the above proof, one gets

Remark. Let K be a convex body in $\mathbb{R}^n$. Suppose that there are two points $x, y \in \partial K$, such that, both $\kappa_K(x)$ and $\kappa_K(y)$ exist and are finite, and

$$\kappa_K(x)\langle x, N_K(x)\rangle^{-(n+1)} \neq \kappa_K(y)\langle y, N_K(y)\rangle^{-(n+1)}.$$

Then there is a constant $\delta(K, x, y)$ depending on K, x and y such that K_δ is not homothetic to K for all $\delta \leq \delta(K, x, y)$.

Analogously, we can ask the following *generalized homothety conjecture*.

Generalized Homothety Conjecture: *Let K be a convex body in $\mathbb{R}^n$. Does $K_s = cK$ for some $0 < s$ and some $c > 0$ imply that K is an ellipsoid? Here $\{K_s\}_{s\geq 0}$ is a family of convex bodies constructed from K.*

Besides the convex floating body K_δ, examples of such K_s include

1. the illumination body [22] $K^\delta = \{x \in \mathbb{R}^n : |\mathrm{conv}(x, K)| - |K| \leq \delta\}$;
2. the convolution body [3, 12] $C(K, t) = \left\{\frac{x}{2} \in \mathbb{R}^n : |K \cap (K + x)| \geq 2t\right\}$, defined for a symmetric convex body K in $\mathbb{R}^n$ and $t \geq 0$;
3. the Santaló-regions [7] $S(K, t) = \left\{x \in K : |K^x| \leq \frac{1}{t}\right\}$, where $K^x = (K - x)^\circ = \{z \in \mathbb{R}^n : \langle z, y - x\rangle \leq 1, \forall y \in K\}$ is the polar body of K with respect to $x \in K$.

We refer to [23, 24, 25] for more general constructions.

The following theorem provides (partial) positive solutions of the generalized homothety conjecture. Theorem 3 (i) was proved with a different method in [20].

Theorem 3 *Let K be a convex body in C^2_+.*

(i) [20] There exists a positive number $\tilde{\delta}(K)$ such that K^δ is homothetic to K for some $0 < \delta < \tilde{\delta}(K)$, if and only if K is an ellipsoid.

(ii) There exists a positive number $t(K)$ such that $C(K,t)$ is homothetic to K for some $0 < t < t(K)$, if and only if K is an ellipsoid.

(iii) There exists a positive number $\tilde{t}(K)$ such that $S(K,t)$ is homothetic to K for some $0 < t < \tilde{t}(K)$, if and only if K is an ellipsoid.

Remark. The proof of this theorem is same as the proof of Theorem 2. The proof of (i) also relies on Lemma 3 in [22]. For the proof of (ii), we refer to results similar to Lemma 5 in [12]. For (iii), one uses Lemma 13 in [7].

Estimates on the thresholds $\tilde{\delta}(K)$, $t(K)$ and $\tilde{t}(K)$ can be obtained similar to the one for $\delta(K)$. This is treated in the next section.

5 Estimates on the threshold $\delta(K)$

Our proof of Theorem 2 gives a possibility to estimate the threshold $\delta(K)$ for a convex body K in $\mathbb{R}^n$.

Let $f(x) = \frac{\kappa_K(x)^{\frac{1}{n+1}}}{\langle x, N_K(x)\rangle}$ be the function introduced in the previous section. We define points $x_M, x_m \in \partial K$ and numbers T_M, T_m and τ as follows.

$$T_M = f(x_M) = \max_{x\in\partial K} f(x), \qquad T_m = f(x_m) = \min_{x\in\partial K} f(x), \qquad \tau = \frac{T_M}{T_m}.$$

Note that the points x_m and x_M may not be uniquely determined. We just choose any two points satisfying the conditions. Let

$$r_m = \kappa_K(x_m)^{-\frac{1}{n-1}}, \quad r_M = \kappa_K(x_M)^{-\frac{1}{n-1}}$$

and

$$a = \min\left\{1 - \left(\frac{2}{1+\tau}\right)^{\frac{n+1}{n-1}}, \left(\frac{3\tau}{1+2\tau}\right)^{\frac{n+1}{n-1}} - 1\right\}. \tag{12}$$

Theorem 4 *Let K be a convex body in $\mathbb{R}^n$ with everywhere on ∂K strictly positive Gauss curvature and such that ∂K is C^3. Let a be as in (12). Then $\delta(K)$ of Theorem 2 can be chosen to be*

$$\delta(K) \quad = \quad \min\left\{\delta_0, \delta_1, \delta_2, \delta_m, \delta_M, \ \frac{(1-a)^n\ r_m^n\ |B_2^n|}{2}, \frac{(1-a)^n\ r_M^n\ |B_2^n|}{2}\right\},$$

where δ_0 is as in (8), and the expressions for $\delta_1, \delta_2, \delta_m, \delta_M$ are in the proof.

Proof. We first rewrite the proof of Theorem 2 in a more quantative way. Suppose hence that K is homothetic to K_δ for some $0 < \delta < \delta(K)$, $\delta(K) \le \delta_0$ with δ_0 given by Lemma 4, and suppose that K is not an ellipsoid. Then, by Proposition 3, $\tau > 1$. By Lemma 5, there exists $\delta_1(K) > 0$, such that,

$$f_\delta(x_M) \ge T_M \left(1 - \frac{\tau - 1}{3\tau}\right) = T_M \left(\frac{2\tau + 1}{3\tau}\right), \text{ for all } 0 < \delta \le \delta_1(K), \tag{13}$$

with f_δ as in (11). Again by Lemma 5, there exists $\delta_2(K) > 0$, such that,

$$f_\delta(x_m) \quad \le \quad T_m \left(1 + \frac{\tau - 1}{2}\right) = T_m \left(\frac{\tau + 1}{2}\right), \text{ for all } 0 < \delta \le \delta_2(K). \tag{14}$$

Let $\delta(K) = \min\{\delta_0, \delta_1(K), \delta_2(K)\}$. Then formulas (13) and (14) together with the homothety condition imply that $T_M \left(\frac{2\tau+1}{3\tau}\right) \le T_m \left(\frac{\tau+1}{2}\right)$ or, equivalently, that $\tau \le 1$, which is a contradiction.

Thus, to estimate $\delta(K)$, it is enough to estimate $\delta_1(K)$ and $\delta_2(K)$ which we will do now.

For $\alpha = (\alpha_1, \dots \alpha_n) \in \mathbb{N}^n$, and an $|\alpha|$-times continuously differentiable function g, let $D_i^{\alpha_i}$ be the α_i times product of D_i, and

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \quad \alpha! = \alpha_1! \dots \alpha_n!, \quad D^\alpha g = D_1^{\alpha_1} \dots D_n^{\alpha_n} g = \frac{\partial^{|\alpha|} g}{\partial t_1^{\alpha_1} \dots \partial t_n^{\alpha_n}}.$$

As determining $\delta(K)$ is invariant under affine transformations of determinant 1, we can assume that the ellipsoid approximating ∂K at x_m is a Euclidean ball and then have (see [17]): For $a > 0$ given as in (12) above, there exists $\Delta_{a,m}$ such that for all $\Delta \le \Delta_{a,m}$

$$\begin{aligned} B_2^n\left(x_m - \bar{r}_m N_K(x_m), \bar{r}_m\right) \cap H^-\left(x_m - \Delta N_K(x_m), N_K(x_m)\right) \\ \subseteq K \cap H^-\left(x_m - \Delta N_K(x_m), N_K(x_m)\right), \end{aligned} \tag{15}$$

where $\bar{r}_m = (1 - a) r_m$. In addition, we also choose $\Delta_{a,m} < \bar{r}_m$.

Assume now that $x_m = 0$, $N_K(x_m) = -e_n$, and the other $(n-1)$ axes of the approximating ellipsoid coincide with the remaining $(n-1)$ coordinate axes. Let $P_m(K)$ be the orthogonal projection of K onto $\mathbb{R}^{n-1}$ and ρ_m be such that $B_2^{n-1}(0, \rho_m) \subset \operatorname{int}\left(P_m(K)\right)$, the interior of $P_m(K)$. Locally, we can describe ∂K by a convex function $f_m : B_2^{n-1}(0, \rho_m) \to \mathbb{R}$, such that $t = (t_1, \dots, t_{n-1}) \to \big(t, f_m(t)\big) \in \partial K$.

As $\partial K \in C^3$, Taylor's theorem implies that for all $t \in B_2^{n-1}(0, \rho_m)$,

$$f_m(t) \le \frac{1}{2}\langle t, At\rangle + \frac{(n-1)^3}{6} D\|t\|^3 = \frac{1}{2}\frac{\|t\|^2}{r_m} + \frac{(n-1)^3}{6} D\|t\|^3, \tag{16}$$

with $A = \left(\frac{\partial^2 f_m}{\partial t_i \partial t_j}(0)\right)_{i,j=1}^{n-1} = \frac{1}{r_m} Id_{n-1}$ is the Hessian of f_m at 0.

In our chosen coordinate system, $\partial B_2^n\big(x_m - \bar r_m N_K(x_m), \bar r_m\big)$ is described (for $g_m(t) \le \bar r_m$) by

$$g_m(t) = \bar r_m \left[1 - \sqrt{1 - (\|t\|/\bar r_m)^2}\right] = \bar r_m - \sqrt{\bar r_m^2 - \|t\|^2}.$$

As $\frac{b}{2} \le 1 - \sqrt{1-b}$ for all $b \in [0,1]$, we get for all t with $\|t\| \le \bar r_m$

$$g_m(t) \ge \frac{1}{2} \frac{\|t\|^2}{(1-a) r_m}. \tag{17}$$

Thus by (16) and (17), $g_m(t) \ge f_m(t)$ for all t with $\|t\| \le t_{a,m} = \min\left\{\rho_m, \bar r_m, \frac{3a}{D \bar r_m (n-1)^3}\right\}$. We then let $\Delta_{a,m} = \bar r_m - \sqrt{\bar r_m^2 - t_{a,m}^2}$, and thus condition (15) holds true for all $\Delta \le \Delta_{a,m}$. We further let

$$\delta_{m,1} = |B_2^{n-1}| \int_{\bar r_m - \Delta_{a,m}}^{\bar r_m} \left(\bar r_m^2 - y^2\right)^{\frac{n-1}{2}} dy. \tag{18}$$

Denote by $\Delta_{\bar r_m}$ the height of a cap of $\bar r_m B_2^n$ of volume exactly δ. Recall that $\{x_{m,\delta}\} = \partial K_\delta \cap [0, x_m]$. Put $\Delta_{x_m} = \left\langle \frac{x_m}{\|x_m\|}, N_K(x_m)\right\rangle \|x_m - x_{m,\delta}\|$. By (15) and (18), one has $\Delta_{x_m} \le \Delta_{\bar r_m}$ for all $\delta \le \delta_{m,1}$. Thus

$$\frac{c_n}{\delta^{\frac{2}{n+1}}} \frac{\|x_m - x_{m,\delta}\|}{\|x_m\|} = \frac{c_n\ \Delta_{x_m}}{\delta^{\frac{2}{n+1}}\ \langle x_m, N_K(x_m)\rangle} \le \frac{c_n\ \Delta_{\bar r_m}}{\delta^{\frac{2}{n+1}}\ \langle x_m, N_K(x_m)\rangle}. \tag{19}$$

Let $\delta < \frac{|B_2^n| \bar r_m^n}{2}$. Then $\Delta_{\bar r_m} < \bar r_m$ and by definition of $\Delta_{\bar r_m}$,

$$\begin{aligned}
\delta &= |B_2^{n-1}| \int_{\bar r_m - \Delta_{\bar r_m}}^{\bar r_m} (\bar r_m - y)^{\frac{n-1}{2}}\ (\bar r_m + y)^{\frac{n-1}{2}}\ dy \\
&\ge |B_2^{n-1}|\ (2\bar r_m - \Delta_{\bar r_m})^{\frac{n-1}{2}} \int_{\bar r_m - \Delta_{\bar r_m}}^{\bar r_m} (\bar r_m - y)^{\frac{n-1}{2}}\ dy \\
&= 2^{\frac{n+1}{2}}\ \bar r_m^{\frac{n-1}{2}}\ \frac{|B_2^{n-1}|}{n+1} \left(1 - \frac{\Delta_{\bar r_m}}{2\bar r_m}\right)^{\frac{n-1}{2}} \Delta_{\bar r_m}^{\frac{n+1}{2}}.
\end{aligned}$$

In particular, using $\left(1 - \frac{\Delta_{a,m}}{2\bar r_m}\right) > \frac{1}{2}$ and $\Delta_{a,m} = \bar r_m \left(1 - \sqrt{1 - \frac{t_{a,m}^2}{\bar r_m^2}}\right) \ge \frac{t_{a,m}^2}{2\bar r_m}$,

$$\delta_{m,1} \ge \frac{t_{a,m}^{n+1}\ |B_2^{n-1}|}{2^{\frac{n-1}{2}}\ (n+1)\ \bar r_m} = \delta_m. \tag{20}$$

Similarly,

$$\delta \le |B_2^{n-1}|\ 2^{\frac{n-1}{2}}\ \bar{r}_m^{\frac{n-1}{2}} \int_{\bar{r}_m-\Delta_{\bar{r}_m}}^{\bar{r}_m} (\bar{r}_m - y)^{\frac{n-1}{2}}\, dy = 2^{\frac{n+1}{2}}\ \bar{r}_m^{\frac{n-1}{2}}\ \frac{|B_2^{n-1}|}{n+1}\ \Delta_{\bar{r}_m}^{\frac{n+1}{2}}.$$

Hence for all $\delta < \frac{\bar{r}_m^n |B_2^n|}{2}$,

$$\frac{\bar{r}_m^{-\frac{n-1}{n+1}}}{c_n} \ \le\ \frac{\Delta_{\bar{r}_m}}{\delta^{\frac{2}{n+1}}} \ \le\ \frac{\bar{r}_m^{-\frac{n-1}{n+1}}}{c_n}\left(1-\frac{\Delta_{\bar{r}_m}}{2\bar{r}_m}\right)^{-\frac{n-1}{n+1}}. \tag{21}$$

Recall that x and $x_{m,\delta}$ are colinear. Therefore, $1-\left(\frac{\|x_{m,\delta}\|}{\|x_m\|}\right)^n \le\ n\ \frac{\|x_m - x_{m,\delta}\|}{\|x_m\|}$. Together with (19), (20) and (21), we get for all $\delta < \min\{\frac{|B_2^n|\bar{r}_m^n}{2}, \delta_m\}$,

$$f_\delta(x_m) \le \frac{\bar{r}_m^{-\frac{n-1}{n+1}}}{\langle x_m, N_K(x_m)\rangle}\left(1-\frac{\Delta_{\bar{r}_m}}{2\bar{r}_m}\right)^{-\frac{n-1}{n+1}} = T_m\,(1-a)^{-\frac{n-1}{n+1}}\left(1-\frac{\Delta_{\bar{r}_m}}{2\bar{r}_m}\right)^{-\frac{n-1}{n+1}}.$$

Hence (14) will hold for all $\delta \le \min\{\delta_2, \frac{|B_2^n|\bar{r}_m^n}{2}, \delta_m\}$, if we choose δ_2 so that

$$\left(1-\frac{\Delta_{\bar{r}_m}}{2\bar{r}_m}\right)^{-\frac{n-1}{n+1}} \le (1-a)^{\frac{n-1}{n+1}}\ \left(\frac{\tau+1}{2}\right). \tag{22}$$

By (21) and (22),

$$\Delta_{\bar{r}_m} \le \delta^{\frac{2}{n+1}}\ \frac{\bar{r}_m^{-\frac{n-1}{n+1}}}{c_n}\left(1-\frac{\Delta_{\bar{r}_m}}{2\bar{r}_m}\right)^{-\frac{n-1}{n+1}} \le \delta^{\frac{2}{n+1}}\ \frac{\bar{r}_m^{-\frac{n-1}{n+1}}}{c_n}\ (1-a)^{\frac{n-1}{n+1}}\ \left(\frac{\tau+1}{2}\right).$$

To have (22), it is enough to have that

$$\left(1-\frac{\delta^{\frac{2}{n+1}}}{c_n \bar{r}_m^{\frac{2n}{n+1}}}\ \left(\frac{\tau+1}{4}\right)(1-a)^{\frac{n-1}{n+1}}\right)^{-1} \le (1-a)\left(\frac{\tau+1}{2}\right)^{\frac{n+1}{n-1}}.$$

Hence, we can let

$$\delta_2 = 2^{\frac{3(n+1)}{2}}\ \left(\frac{1-a}{1+\tau}\right)^{\frac{n+1}{2}}\ r_m^n\ \frac{|B_2^{n-1}|}{n+1}\ \left[1-(1-a)^{-1}\left(\frac{2}{\tau+1}\right)^{\frac{n+1}{n-1}}\right]^{\frac{n+1}{2}}. \tag{23}$$

Now we consider x_M. We let $r_M = \kappa_K(x_M)^{\frac{1}{1-n}}$, $\bar{r}_M = (1-a)r_M$, and $\bar{R}_M = (1+a)r_M$. Similar to the case of x_m, we can describe ∂K locally at x_M by a convex function $f_M : B_2^{n-1}(0,\rho_M) \to \mathbb{R}$, such that $t = (t_1,\dots,t_{n-1}) \to \big(t, f_M(t)\big) \in \partial K$.

Here, ρ_M is such that $B_2^{n-1}(0,\rho_M) \subset P_M(K)$, the corresponding orthogonal projection of K onto R^{n-1} at x_M. As in inequality (16), $f_M(t) \geq \frac{1}{2}\frac{\|t\|^2}{r_M} - \frac{(n-1)^3}{6} D\|t\|^3$. Let

$$t_{M,1} = \min\left\{\bar{r}_M, \frac{3a}{D\bar{r}_M(n-1)^3}, \rho_M\right\}, \text{ and } \Delta_{M,1} = \bar{r}_M - \sqrt{\bar{r}_M^2 - t_{M,1}^2}.$$

We repeat the previous argument and get for all $\Delta \leq \Delta_{M,1}$, that

$$\begin{aligned} B_2^n\left(x_M - \bar{r}_M N_K(x_M), \bar{r}_M\right) \cap H^-\left(x_M - \Delta N_K(x_M), N_K(x_M)\right) \\ \subseteq K \cap H^-\left(x_M - \Delta N_K(x_M), N_K(x_M)\right). \end{aligned} \tag{24}$$

In our chosen coordinate system, $\partial B_2^n\left(x_M - \bar{R}_M N_K(x_M), \bar{R}_M\right)$ is described (for $g_M(t) \leq \bar{R}_M$) by

$$g_M(t) = \bar{R}_M\left[1 - \sqrt{1 - \left(\|t\|/\bar{R}_M\right)^2}\right] = \bar{R}_M - \sqrt{\bar{R}_M^2 - \|t\|^2}.$$

Let $\xi = 1 + \frac{a}{2}$. Clearly, $1 - \sqrt{1-b} \leq \frac{\xi}{2} b$ for all $b \in [0, \frac{4(\xi-1)}{\xi^2}]$. Thus, for $\|t\| \leq \frac{2(\xi-1)^{\frac{1}{2}}}{\xi}\bar{R}_M$, we have $g_M(t) \leq \frac{\xi}{2}\frac{\|t\|^2}{\bar{R}_M}$.

Let $t_{M,2} = \min\left\{\frac{2(\xi-1)^{\frac{1}{2}}}{\xi}\bar{R}_M, \frac{3a}{2D\bar{R}_M(n-1)^3}, \rho_M\right\}$. Then, for $\|t\| \leq t_{M,2}$, we get that $f_M(t) \geq g_M(t)$ which implies that

$$\begin{aligned} B_2^n\left(x_M - \bar{R}_M N_K(x_M), \bar{R}_M\right) \cap H^-\left(x_M - \Delta N_K(x_M), N_K(x_M)\right) \\ \supseteq K \cap H^-\left(x_M - \Delta N_K(x_M), N_K(x_M)\right), \end{aligned} \tag{25}$$

for all $\Delta \leq \Delta_{M,2}$ with $\Delta_{M,2} = \bar{R}_M - \sqrt{\bar{R}_M^2 - t_{M,2}^2}$.

We let $\Delta_{a,M} = \min\{\Delta_{M,1}, \Delta_{M,2}\}$, and

$$\delta_{M,1} = |B_2^{n-1}| \int_{\bar{r}_M - \Delta_{a,M}}^{\bar{r}_M} \left(\bar{r}_M^2 - y^2\right)^{\frac{n-1}{2}} dy. \tag{26}$$

In particular, as above,

$$\delta_{M,1} \geq 2\frac{|B_2^{n-1}|}{n+1}\, \bar{r}_M^{\frac{n-1}{2}} \Delta_{a,M}^{\frac{n+1}{2}} = \delta_M, \tag{27}$$

where $\Delta_{a,M}$ can be taken as $\min\left\{\frac{t_{M,1}^2}{2(1-a)r_m}, \frac{t_{M,2}^2}{2(1-a)R_M}\right\}$.

Let $\Delta_{\bar{R}_M}$ be the height of a cap of $\bar{R}_M B_2^n$ of volume exactly δ. By (24), (25), (26), and (27), for all $\delta \leq \delta_M$, $\Delta_{\bar{R}_M} \leq \Delta_{x_M} = \left\langle \frac{x_M}{\|x_M\|}, N_K(x_M)\right\rangle \|x_M - x_{M,\delta}\|$. Recall $\{x_{M,\delta}\} = [0, x_M] \cap \partial K_\delta$. Similar to (21), for all $\delta < \min\{\frac{\bar{r}_M^n |B_2^n|}{2}, \delta_M\}$,

$$\frac{\bar{R}_M^{-\frac{n-1}{n+1}}}{c_n} \leq \frac{\Delta_{\bar{R}_M}}{\delta^{\frac{2}{n+1}}} \leq \frac{\bar{R}_M^{-\frac{n-1}{n+1}}}{c_n}\left(1 - \frac{\Delta_{\bar{R}_M}}{2\bar{R}_M}\right)^{-\frac{n-1}{n+1}}.$$

Again, similar to above, this implies that, for all $\delta < \min\{\frac{\bar{r}_M^n |B_2^n|}{2}, \delta_M\}$,

$$f_\delta(x_M) \geq (1+a)^{-\frac{n-1}{n+1}} T_M \left(1 - \frac{(n-1) T_M \delta^{\frac{2}{n+1}}}{2^{\frac{2}{n+1}} c_n (1+a)^{\frac{n-1}{n+1}}}\right).$$

Thus, to have (13), it is enough to have

$$(1+a)^{-\frac{n-1}{n+1}} \left(1 - \frac{(n-1)\ T_M\ \delta^{\frac{2}{n+1}}}{2^{\frac{2}{n+1}} c_n (1+a)^{\frac{n-1}{n+1}}}\right) \geq \frac{2\tau+1}{3\tau},$$

or, equivalently,

$$\delta \leq \frac{\left[1-(1+a)^{\frac{n-1}{n+1}}\left(\frac{2\tau+1}{3\tau}\right)\right]^{\frac{n+1}{2}}\ 2^{\frac{n+3}{2}}\ |B_2^{n-1}|(1+a)^{\frac{n-1}{2}}}{(n-1)^{\frac{n+1}{2}} T_M^{\frac{n+1}{2}} (n+1)} (=: \delta_1). \qquad (28)$$

Now we let the threshold $\delta(K)$ be

$$\delta(K) \quad = \quad \min\left\{\delta_0, \delta_1, \delta_2, \delta_m, \delta_M,\ \frac{(1-a)^n\ r_m^n\ |B_2^n|}{2}, \frac{(1-a)^n\ r_M^n\ |B_2^n|}{2}\right\},$$

where $\delta_0, \delta_1, \delta_2, \delta_m, \delta_M$ are as in (8), (28), (23), (20) and (27) respectively.

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