NON-ADDITIVITY OF RÉNYI ENTROPY AND DVORETZKY'S THEOREM

GUILLAUME AUBRUN, STANISŁAW SZAREK, AND ELISABETH WERNER

Abstract. The goal of this note is to show that the analysis of the minimum output p-Rényi entropy of a typical quantum channel essentially amounts to applying Milman's version of Dvoretzky's Theorem about almost Euclidean sections of high-dimensional convex bodies. This conceptually simplifies the counterexample by Hayden–Winter to the additivity conjecture for the minimal output p-Rényi entropy (for $p > 1$).

1. Introduction. Many major questions in quantum information theory can be formulated as additivity problems. These questions have received considerable attention in recent years, culminating in Hastings' work showing that the minimal output von Neumann entropy of a quantum channel is not additive. He used a random construction inspired by previous examples due to Hayden and Winter, who proved non-additivity of the minimal output p-Rényi entropy for any $p > 1$. In this short note, we show that the Hayden–Winter analysis can be simplified (at least conceptually) by appealing to Dvoretzky's theorem. Dvoretzky's theorem is a fundamental result of asymptotic geometric analysis, which studies the behaviour of geometric parameters associated to norms in $\mathbb{R}^n$ (or equivalently, to convex bodies) when n becomes large. Such connections between quantum information theory and high-dimensional convex geometry promise to be very fruitful.

2. Notation. If $\mathcal{H}$ is a Hilbert space, we will denote by $\mathcal{B}(\mathcal{H})$ the space of bounded linear operators on $\mathcal{H}$, and by $\mathcal{D}(\mathcal{H})$ the set of *density matrices* on $\mathcal{H}$, i.e., positive semi-definite trace one operators on $\mathcal{H}$ (or *states* on $\mathcal{H}$, or – more properly – states on $\mathcal{B}(\mathcal{H})$). Most often we will have $\mathcal{H} = \mathbb{C}^n$ for some $n \in \mathbb{N}$, and we will then write $\mathcal{M}_n$ for $\mathcal{B}(\mathbb{C}^n)$.

For $p \geq 1$, the *p-Rényi entropy* of a state ρ is defined as

$$S_p(\rho) = \frac{1}{1-p}\,\mathrm{tr}\rho^p.$$

(For $p = 1$, this should be understood as a limit and coincides with the von Neumann entropy.)

A linear map $\Phi : \mathcal{M}_m \to \mathcal{M}_d$ is called a *quantum channel* if it is completely positive and trace-preserving. The *minimal output p-Rényi entropy* of Φ is then defined as

$$S_p^{\min}(\Phi) = \min_{\rho \in \mathcal{D}(\mathbb{C}^m)} S_p(\Phi(\rho)).$$

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3. The Additivity Conjecture. The *Additivity Conjecture* [1] asserted that the following equality held for every pair Φ, Ψ of quantum channels

$$S_p^{\min}(\Phi \otimes \Psi) \stackrel{?}{=} S_p^{\min}(\Phi) + S_p^{\min}(\Psi). \tag{1}$$

The most important case, $p = 1$, has been shown to be equivalent to a number of central questions in quantum information theory [22]. Of course, had the conjecture been true for every $p > 1$, it would have held also for $p = 1$ by continuity.

The conjecture has been recently disproved for *all* values of $p \geq 1$, mostly using nonconstructive methods. Early (explicit) counterexamples for $p > 4.79$ are due to Holevo and R. F. Werner [13]. The case $p > 1$ was settled by Hayden and Winter in [12]. Finally Hastings found a counterexample to the additivity conjecture for $p = 1$ [11].

We want to show in this note that a large part of the analysis by Hayden and Winter is actually a fallout of Dvoretzky's theorem, a classical result in high-dimensional convex geometry dating to the 1960s [4, 17]. We note that this approach, at least in its present form, does not cover Hastings' construction.

4. Multiplicative form. It will be more convenient to consider a multiplicative version of the conjecture. Instead of the Rényi entropy, we work with the Schatten p-norm $\|\sigma\|_p = \big(\operatorname{tr}(\sigma^\dagger\sigma)^{p/2}\big)^{1/p}$. (The limit case $\|\cdot\|_\infty$ is the operator, or "spectral," norm.)

If $p > 1$ and ρ is a state, then $S_p(\rho) = \frac{p}{1-p}\log\|\rho\|_p$, and so the study of $S_p^{\min}(\Phi)$ is replaced by that of $\max_{\rho\in\mathcal{D}(\mathbb{C}^m)}\|\Phi(\rho)\|_p$, or the *maximum output p-norm.* The latter quantity has a nice functional-analytic interpretation: it equals $\|\Phi\|_{1\to p}$, i.e., the norm of Φ as an operator from $(\mathcal{M}_m, \|\cdot\|_1)$ to $(\mathcal{M}_d, \|\cdot\|_p)$. This allows to rewrite conjecture (1) in a multiplicative form

$$\|\Phi \otimes \Psi\|_{1\to p} \stackrel{?}{=} \|\Phi\|_{1\to p}\|\Psi\|_{1\to p}. \tag{2}$$

The inequality " $\geq$ " is trivial, so the conjecture asked if " $\leq$ " was always true.

We point out that the argument that follows deals directly with the maximum output p-norm and not with $\|\Phi\|_{1\to p}$, so the knowledge that the two are equal is not really needed. Note that it is only obvious that the maximum output p-norm is equal – for any linear map Φ – to the norm of the restriction of Φ to the $\mathbb{R}$-linear space $\mathcal{M}_m^H$ of $m \times m$ Hermitian matrices. The fact that it coincides – for quantum channels, or even for all 2-positive maps – with the *a priori* larger norm $\|\Phi\|_{1\to p}$ is also elementary, but less immediate; see [24] for a short proof and references.

Finally, let us observe that $\|\Phi\|_{1\to p}$ never exceeds 1 if Φ is a quantum channel. If, additionally, Φ is $\mathcal{M}_d$-valued, then $\|\Phi\|_{1\to p}$ is at least $d^{1/p-1}$ (the p-norm of the maximally mixed state in $\mathcal{M}_d$).

5. Channels as subspaces. Let $\mathcal{W}$ be a subspace of $\mathbb{C}^d \otimes \mathbb{C}^d$ of dimension m. Then $\Phi : \mathcal{B}(\mathcal{W}) \to \mathcal{M}_d$ defined by $\Phi(\rho) = \operatorname{tr}_2(\rho)$ (*partial trace* in the second factor, i.e., $\operatorname{tr}_2(\rho_1 \otimes \rho_2) = \operatorname{tr}(\rho_2)\,\rho_1$) is a quantum channel. Alternatively (and perhaps more properly), we could identify $\mathcal{W}$ with $\mathbb{C}^m$ via an isometry $V : \mathbb{C}^m \to \mathbb{C}^d \otimes \mathbb{C}^d$ whose range is $\mathcal{W}$ and set, for $\rho \in \mathcal{M}_m$, $\Phi(\rho) = \operatorname{tr}_2(V\rho V^\dagger)$; then Φ goes from $\mathcal{M}_m$ to $\mathcal{M}_d$. (One could also consider here subspaces $\mathcal{W} \subset \mathbb{C}^d \otimes \mathbb{C}^r$, where possibly $r \neq d$; this would allow to preserve full generality, but would lead to more involved notation.)

By convexity, the maximum output p-norm, and hence also $\|\Phi\|_{1\to p}$, is attained on pure states. In other words

$$\|\Phi\|_{1\to p} = \max_{x\in\mathbb{C}^m, |x|=1} \|\Phi(|x\rangle\langle x|)\|_p,$$

where $|\cdot|$ is the Euclidean norm. A standard and well-known argument shows that eigenvalues of $\Phi(|x\rangle\langle x|)$ are exactly squares of $s_j(x)$, the "Schmidt coefficients" of x, so

$$\|\Phi\|_{1\to p} = \max_{x\in\mathcal{W},|x|=1} \Big(\sum_{j=1}^{d} s_j(x)^{2p}\Big)^{1/p} = \max_{x\in\mathcal{W},|x|=1} \|x\|_{2p}^2 ,$$

where in the last expression we identify $x \in \mathcal{W} \subset \mathbb{C}^d \otimes \mathbb{C}^d$ (or, to be more precise, $\mathbb{C}^d \otimes \overline{\mathbb{C}^d}$ — a distinction we will ignore) with an element of $\mathcal{M}_d$ via the canonical map induced by $u\otimes v \to |u\rangle\langle v|$. (Schmidt coefficients of an element of $\mathbb{C}^d\otimes\overline{\mathbb{C}^d}$ become singular values of the corresponding element of $\mathcal{M}_d$.)

In other words, $\|\Phi\|_{1\to p}$ is the square of the maximum of the ratio $\|x\|_{2p}/\|x\|_2$ over the m-dimensional subspace of $\mathcal{M}_d$ that corresponds to $\mathcal{W}$ under the canonical identification, and that we will still call $\mathcal{W}$,

$$\|\Phi\|_{1\to p} = \max_{x\in\mathcal{W}} \left(\|x\|_{2p}/\|x\|_2\right)^2 . \tag{3}$$

6. The Hayden–Winter counterexample. The Hayden–Winter construction can be described as follows. Let $V : \mathbb{C}^m \to \mathbb{C}^d \otimes \mathbb{C}^d$ be a random isometry (chosen with respect to the Haar measure) and $\Phi : \rho \mapsto \mathrm{tr}_1(V\rho V^\dagger)$ be the corresponding quantum channel from $\mathcal{M}_m$ into $\mathcal{M}_d$. We show in the next section that Dvoretzky's theorem implies that for $m \sim d^{1+1/p}$, such random quantum channel typically satisfies

$$\|\Phi\|_{1\to p} \sim d^{1/p-1}. \tag{4}$$

Here, and throughout the remainder of the paper, $\sim$ means "equivalent up to a universal multiplicative constant."

Take as the second channel the (complex) conjugate channel $\bar\Phi$ and let $|\psi\rangle$ be the maximally entangled state in $\mathbb{C}^m \otimes \mathbb{C}^m$. It is shown in [12] (Lemma 3.3) that $(\Phi \otimes \bar\Phi)(|\psi\rangle\langle\psi|)$ has an eigenvalue $\geq m/d^2$, which implies that with the above choice of m,

$$\|\Phi \otimes \bar\Phi\|_{1\to p} \geq \|\Phi \otimes \bar\Phi\|_{1\to\infty} \geq m/d^2 \sim d^{1/p-1}.$$

On the other hand, again with the same choice of m, by (4)

$$\|\Phi\|_{1\to p} = \|\bar\Phi\|_{1\to p} \sim d^{1/p-1},$$

and thus

$$\|\Phi\|_{1\to p}\, \|\bar\Phi\|_{1\to p} \sim \left(d^{1/p-1}\right)^2 \ll d^{1/p-1}, \tag{5}$$

so that we obtain a violation of the multiplicativity provided that $d^{1/p-1} \leq 1/C$, i.e., $d \geq C^{p/(p-1)}$, where C is the absolute constant hidden behind the $\sim$ symbol. Moreover, this violation is asymptotically extremal. Indeed, while the inequality $\|\Phi \otimes \bar\Phi\|_{1\to p} \leq \|\Phi\|_{1\to p} = \|\bar\Phi\|_{1\to p}^{-1}\|\Phi\|_{1\to p}\|\bar\Phi\|_{1\to p}$ always holds (this follows from results from [1]), in this example the reverse inequality also holds up to an absolute multiplicative constant. At the same time, $\|\bar\Phi\|_{1\to p}^{-1} \sim d^{1-1/p}$ is of largest possible order in the class of $\mathcal{M}_d$-valued quantum channels, see the observation at the end of section 4.

The lower estimate for $\|\Phi \otimes \bar{\Phi}\|_{1\to p}$ was the relatively simple part of the argument from [12]; the authors refer to the proof of their Lemma 3.3 as "an easy calculation." Of course, it is "easy" only after the fact, and the crucial point was coming up with the right pair of channels to analyze.

Finally, as pointed out in [11], the random approach allows working initially with real spaces ($\mathbb{R}^m, \mathbb{R}^d \otimes \mathbb{R}^d$ etc.) and producing channels Φ fitting into the Hayden–Winter scheme, whose representation in the computational basis is real. In particular, $\Phi = \bar{\Phi}$, so we have channels (acting on complex spaces) for which $\|\Phi \otimes \Phi\|_{1\to p} \gg \|\Phi\|_{1\to p}^2$ and $S_p^{\min}(\Phi \otimes \Phi) < 2S_p^{\min}(\Phi)$.

7. Dvoretzky's Theorem. By (3), $\|\Phi\|_{1\to p} = \max_{x\in\mathcal{W}} (\|x\|_{2p}/\|x\|_2)^2$, where $\mathcal{W} \subset \mathcal{M}_d$ is an m-dimensional subspace. The behavior of the ratio between the Euclidean norm and some other norm on subspaces of given dimension is a quantity that has been extensively studied in geometry of Banach spaces. The most classical result in this direction is Dvoretzky's theorem:

Given $m \in \mathbb{N}$ and $\varepsilon > 0$ there is $N = N(m, \varepsilon)$ such that, for any norm on $\mathbb{R}^N$ (or $\mathbb{C}^N$) there is an m-dimensional subspace on which that ratio is (approximately) constant, up to a multiplicative factor $1 + \varepsilon$.

This reveals a striking geometric phenomenon: any high-dimensional convex body, no matter how peaked it may be, has sections which are close to Euclidean balls.

For specific norms this statement can be made much more precise, both in describing the dependence $N = N(m, \varepsilon)$ and in identifying the constant of (approximate) proportionality of norms. The version of Dvoretzky's theorem that is relevant here is due to Milman [17]. (Alternative good expositions are, for example, [5, 20] and [21]; the last one presents a proof based on Gaussian analysis, which allows to bypass the — deep and not so easy to prove — spherical isoperimetric inequality.)

Dvoretzky's theorem (Tangible version) *Consider the n-dimensional Euclidean space (real or complex) endowed with the Euclidean norm $|\cdot|$ and some other norm $\|\cdot\|$ such that, for some $b > 0$, $\|\cdot\| \le b|\cdot|$. Denote $M = \mathbb{E}\|X\|$, where X is a random variable uniformly distributed on the unit Euclidean sphere. Let $\varepsilon > 0$ and let $m \le c\varepsilon^2(M/b)^2 n$, where $c > 0$ is an appropriate (computable) universal constant. Then, for most m-dimensional subspaces E (in the sense of the invariant measure on the corresponding Grassmannian) we have*

$$\forall x \in E, \quad (1-\varepsilon)M|x| \le \|x\| \le (1+\varepsilon)M|x|.$$

Remarks. (i) The above result is usually stated with the hypothesis $a^{-1}|\cdot| \le \|\cdot\| \le b|\cdot|$ (for some $a, b > 0$). However, the parameter a does not enter into the assertion; lower bounds on $\|\cdot\|$ are related to lower bounds on M, needed to obtain non-trivial values of m (and the function $N(m, \varepsilon)$ mentioned earlier) in the abstract setting.
(ii) Standard and most elementary proofs yield the assertion only for $m \le c\varepsilon^2/\log(1/\varepsilon)(M/b)^2 n$; the dependence on ε of order ε^2 was obtained in the important paper [7]. However, for our purposes it is enough to have, say, $\varepsilon = \frac{1}{2}$, so this aspect of the story is not important.

8. Dvoretzky's theorem for Schatten classes. In the Hayden–Winter construction, $\mathcal{W} \subset \mathcal{M}_d$ is a random m-dimensional subspace distributed according to the Haar measure on the Grassmann manifold and (cf. (3), (4)) we want to control the ratio $\|x\|_{2p}/\|x\|_2$ uniformly on $\mathcal{W}$, where $2p =: q > 2$. Thus the context in which one needs to apply Dvoretzky's theorem is the Schatten q-norm on the complex space $\mathcal{M}_d$ for $q > 2$, in particular $n = d^2$, $\|\cdot\| = \|\cdot\|_q$ and

$|\cdot| = \|\cdot\|_2$, the Hilbert–Schmidt norm. This has been done, e.g., in the 1977 paper [5] (see Example 3.3 there; [5] focuses on real spaces, but it is noted that all proofs carry over to the complex case). The conclusion is that if $m \sim d^{1+2/q} = d^{1+1/p}$, then the inequality

$$d^{1/q-1/2}\|x\|_2 \leq \|x\|_q \leq C d^{1/q-1/2}\|x\|_2 \tag{6}$$

holds (for some constant $C \geq 1$ that does not depend on d nor — less crucially — on q) for all x in a typical m-dimensional subspace of $\mathcal{M}_d$. (If we used the normalized trace to define Schatten norms, the powers of d would disappear.) Combining (6) with (3) yields that when $m \sim d^{1+1/p}$, then $\|\Phi\|_{1\to p} \sim d^{2/q-1} = d^{1/p-1}$ for a typical Φ, which are exactly the values needed for the Hayden–Winter example.

For completeness, let us comment on the details of the derivation of (6) from Dvoretzky's theorem. What we need is to find (or estimate) the quantities b, M appearing in the theorem. Clearly, for all $x \in \mathcal{M}_d$,

$$d^{1/q-1/2}\|x\|_2 \leq \|x\|_q \leq \|x\|_2, \tag{7}$$

which yields the value of the parameter $b = 1$, the lower (trivial, and not actually needed for the multiplicativity problem) estimate from (6) and, *a fortiori*, the bound $M \geq d^{1/q-1/2}$. The upper estimate in (6) will now follow once we establish that M is *precisely* of order $d^{1/q-1/2}$. Indeed, using (the tangible version of) Dvoretzky's theorem with $\varepsilon = \frac{1}{2}$ we are then led to $m \geq c(\frac{1}{2})^2(M/b)^2 n \sim (d^{1/q-1/2})^2 d^2 = d^{1+2/q}$, and to an upper estimate $\frac{3}{2}M \sim d^{1/q-1/2}$ in (6) (i.e., on a "typical" m-dimensional subspace).

As we mentioned above, the fact that $M \sim d^{1/q-1/2}$ is implicit in the argument from [5].[1] However, it is instructive to note that it may also be obtained by many other "standard" methods developed in geometric functional analysis and in random matrix theory. One (*by far* not the easiest, but most precise, at least in the appropriate asymptotic regime) was used by Collins–Nechita [2]. A simple argument to get an upper bound for M goes as follows. Let X be a random variable uniformly distributed on the Hilbert–Schmidt sphere in $\mathcal{M}_d$. It is easy to check, using an elementary ε-net argument, that the expectation of $\|X\|_\infty$ is bounded by $C_0 d^{-1/2}$ for some absolute constant C_0. Using the (pointwise) inequality $\|X\|_q \leq \|X\|_2^{2/q}\|X\|_\infty^{1-2/q}$ and Hölder's inequality, we get

$$M = \mathbb{E}\|X\|_q \leq \big(\mathbb{E}\|X\|_\infty\big)^{1-2/q} \leq \big(C_0 d^{-1/2}\big)^{1-2/q} = C_0^{1-2/q} d^{1/q-1/2}.$$

If we are interested in good values of numerical constants, the best possible choice is $C_0 = 2$ – the same "2" as in the Wigner's semicircle law. The needed generality and precision can be extracted – at least in the real case – from [6]; see also [10], [3] (Theorem 2.11) or [23] (Appendix F) for related calculations.

9. Derandomization. Milman's proof of Dvoretzky's theorem relies on concentration of measure via Lévy's lemma, and is highly nonconstructive. Some effort has been put recently in finding explicit subspaces satisfying the conclusion of the theorem. Of course this must depend on the choice of the initial norm $\|\cdot\|$. The prominent example is the case of the ℓ_1 norm on $\mathbb{R}^n$,

[1] It is shown in [5] that $m \sim d^{1+2/q}$ is the optimal (i.e., the largest) dimension for which (approximate) proportionality of norms does hold. Now, if we have had $M \gg d^{1/q-1/2}$, Dvoretzky's theorem would have yielded a nearly Euclidean subspace of dimension $m \gg d^{1+2/q}$ (just repeat the argument from the preceding paragraph with $\gg$ instead of $\sim$), which contradicts the optimality assertion.

which is relevant to (classical) theoretical computer science. In this case the dimension of the subspace is proportional to n. Although no explicit construction of such a subspace exists yet, recent results are promising (see [14, 9, 15] and references therein). We might hope to adapt such techniques to obtain constructive counterexamples to the additivity conjectures. However, to date the best result in this direction seems to be [8], with an explicit example which works for all $p > 2$.

10. Shrinking under random projections and related remarks. The Hayden–Winter construction requires only an *upper* estimate on $\|x\|_q$ for all x in a (random) subspace of $\mathcal{M}_d$ (cf. (3), (5)). This observation leads to counterexamples to additivity conjecture based on a phenomenon that is conceptually simpler (even if less known) than Dvoretzky's theorem. One way to express it is as follows: *if, in the notation of Dvoretzky's theorem, $(M/b)^2 n =: m_0 \le m \le n$, then the one sided estimate $\|x\| \le C\sqrt{m/n}\; b\,|x|$ holds for all x in a typical m-dimensional subspace.*

While the choice $m \sim d^{1+1/p}$ (in the construction of a random channel) results in an extremal violation of multiplicativity, the above remark shows that similar calculations for (e.g.) all $m \ge d^{1+1/p}$ lead to estimates of order m/d^2 on all the norms $\|\Phi\|_{1\to p}$, $\|\bar{\Phi}\|_{1\to p}$ and $\|\Phi \otimes \bar{\Phi}\|_{1\to p}$, and so a violation occurs as long as m/d^2 is small enough, i.e., smaller than a certain numerical constant $c > 0$. However, it should be noted that the restrictions $cd^2 > m \ge d^{1+1/p}$ still imply that $d \to \infty$ as $p \to 1$.

It may be more geometrically compelling to express the phenomenon referred to above in its dual form. First, the dual reformulation of Dvoretzky's theorem states that if K is a symmetric body in the n-dimensional Euclidean space, then there is (relatively large) m_0 such that a typical orthogonal m_0-dimensional projection of K is approximately a Euclidean ball. (Determining the threshold m_0 involves considering the norm, for which K is the unit ball, and then calculating parameters M and b for the *dual* norm.) The relaxed version states that if $m_0 \le m \le n$, then the diameter of a typical m-dimensional projection of K does not exceed $C\sqrt{m/n}$ times the diameter of K. References for these remarks are, e.g., [18] and [19] (section 2.3.1), but the phenomenon can in fact be traced back (at least) to [7] or [16], among others.

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Institut Camille Jordan, Université Claude Bernard Lyon 1, 43 boulevard du 11 novembre 1918, 69622 Villeurbanne CEDEX, France
E-mail address: aubrun@math.univ-lyon1.fr

Equipe d'Analyse Fonctionnelle, Institut de Mathématiques de Jussieu, Université Pierre et Marie Curie-Paris 6, 4 place Jussieu 75252 Paris, France *and* Department of Mathematics, Case Western Reserve University, Cleveland, Ohio 44106, USA
E-mail address: szarek@math.jussieu.fr

Department of Mathematics, Case Western Reserve University, Cleveland, Ohio 44106, USA *and* Université de Lille 1, UFR de Mathématique, 59655 Villeneuve d'Ascq, France
E-mail address: elisabeth.werner@case.edu