

Divergence for s -concave and log concave functions *

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Abstract

We prove new entropy inequalities for log concave and s -concave functions that strengthen and generalize recently established reverse log Sobolev and Poincaré inequalities for such functions. This leads naturally to the concept of f -divergence and, in particular, relative entropy for s -concave and log concave functions. We establish their basic properties, among them the affine invariant valuation property. Applications are given in the theory of convex bodies.

1 Introduction

There is a general approach to extend invariants and inequalities of convex bodies to the corresponding invariants and inequalities for functions. Among the best known affine isoperimetric inequalities is the Blaschke Santaló inequality [10, 49, 59]. The corresponding inequalities for log concave functions were proved by Ball [7] and Artstein, Klartag and Milman [5] (see also [17, 31]). A stronger inequality than the Blaschke Santaló inequality is the affine isoperimetric inequality for convex bodies [10, 15, 59]. The equivalent of this inequality for log concave functions was established in [6]: For every log-concave function $\varphi : \mathbb{R}^n \rightarrow [0, \infty)$ with enough smoothness and integrability properties and such that $\int \varphi dx = 1$,

$$\int_{\text{supp}(\varphi)} \varphi \ln \left(\det (\text{Hess} (-\ln \varphi)) \right) dx \leq 2 [\text{Ent}(\varphi) - \text{Ent}(g)], \quad (1)$$

where g is the Gaussian, $\text{supp}(\varphi)$ is the support, $\text{Hess}(\varphi) = \left(\frac{\partial^2 \varphi}{\partial x_i \partial x_j} \right)_{1 \leq i, j \leq n}$ is the Hessian and $\text{Ent}(\varphi) = \int_{\text{supp}(\varphi)} \varphi \ln \varphi dx$ is the entropy of φ . Thus, the affine isoperimetric inequality corresponds to a reverse log Sobolev inequality for entropy. Equality holds in (1) if and only if $\varphi(x) = Ce^{-\langle Ax, x \rangle}$, where C is a positive constant and A is an $n \times n$ positive definite matrix. This characterization of equality in inequality (1) was achieved in [12].

Here, we strengthen and generalize inequality (1).

Inequality (1) is yet another instance of the rapidly developing, fascinating connection between convex geometric analysis and information theory. Further examples can be found in e.g., [16, 28, 42, 44, 46, 47, 54]. In particular, it has been observed [67] that a

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fundamental notion of affine convex geometry, the L_p -affine surface area can be viewed as Rényi entropy from information theory, thus establishing a link between information theory and the powerful L_p -Brunn-Minkowski theory [41] of affine convex geometry. Due to a number of highly influential works (see, e.g., [19]- [23], [29], [30], [34]- [48], [58], [62]- [67], [69], [72]), this theory is now a central part of modern convex geometry. Rényi entropies are special cases of f -divergences whose definition is given in Section 2. Such divergences and their related inequalities are important tools in information theory, statistics, probability theory and machine learning [8, 13, 18, 24, 32, 33, 53, 55, 71]. Consequently, it is desirable to have such divergences available also in the theory of convex bodies and this was achieved in [68].

In this paper, we further develop that direction. We introduce f -divergences for functions and establish some of their basic properties, among them the affine invariance property and the valuation property. Valuations were the critical ingredient in Dehn's solution of Hilbert's third problem and, in the last decade, have seen rapid growth as is demonstrated by e.g., [1]-[3], [9], [22], [34]- [40], [60].

We prove the following entropy inequality for log concave functions, i.e. functions of the form $\varphi = e^{-\psi}$ with $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ convex. This inequality is stronger than inequality (1). Its proof uses methods different from the ones used in [6].

Theorem 1. *Let $f : (0, \infty) \rightarrow \mathbb{R}$ be a convex function. Let $\varphi : \mathbb{R}^n \rightarrow [0, \infty)$ be a log-concave function. Then*

$$\begin{aligned} \int_{\text{supp}(\varphi)} \varphi f \left(e^{\langle \frac{\nabla \varphi}{\varphi}, x \rangle} \varphi^{-2} (\det (\text{Hess} (-\ln \varphi))) \right) dx \\ \geq f \left(\frac{\int \varphi^\circ dx}{\int \varphi dx} \right) \left(\int_{\text{supp}(\varphi)} \varphi dx \right). \end{aligned} \quad (2)$$

If f is concave, the inequality is reversed. If f is linear, equality holds in (2). Equality also holds if $\varphi(x) = Ce^{-\langle Ax, x \rangle}$, where C is a positive constant and A is an $n \times n$ positive definite matrix.

Here, $\nabla \varphi$ denotes the gradient of φ and $\varphi^\circ = \inf_{y \in \mathbb{R}^n} \left[\frac{e^{-\langle x, y \rangle}}{\varphi(y)} \right]$ [5] is the dual function of φ . We will demonstrate that the left hand side of the inequality (2) is **the** natural definition of f -divergence $D_f(\varphi)$ for a log concave function φ , so that inequality (2) can be rewritten as

$$D_f(\varphi) \geq f \left(\frac{\int \varphi^\circ dx}{\int \varphi dx} \right) \left(\int_{\text{supp}(\varphi)} \varphi dx \right). \quad (3)$$

This is shown in Section 3. Inequality (3) also holds for s -concave functions. We prove this in Theorem 8.

If we let $f(t) = \ln t$ in Theorem 1, we obtain the following corollary.

Corollary 2. *Let $\varphi : \mathbb{R}^n \rightarrow [0, \infty)$ be a log-concave function. Then*

$$\int_{\text{supp}(\varphi)} \varphi \ln \left(\det (\text{Hess} (-\ln \varphi)) \right) dx \leq 2 \text{Ent}(\varphi) + \|\varphi\|_{L^1} \ln \left[e^n \left(\int \varphi \right) \left(\int \varphi^\circ \right) \right], \quad (4)$$

with equality if $\varphi(x) = Ce^{-\langle Ax, x \rangle}$, where C is a positive constant and A is an $n \times n$ positive definite matrix.

We show in Section 3 that inequality (4) involves the *relative entropy* or *Kullback Leibler divergence* $D_{KL}(\varphi)$ (see Section 2 for the definition) of the function φ and thus inequality (4) is equivalent to

$$D_{KL}(\varphi) \leq \left(\int_{\text{supp}(\varphi)} \varphi dx \right) \ln \left(\frac{\int \varphi^\circ dx}{\int \varphi dx} \right).$$

Moreover, as it is shown in Section 3, the inequality of Corollary 2 is stronger than inequality (1).

It is important to note the affine invariant nature of the expressions (2), (4) and of (5) below. Both, the respective left-hand sides and the right-hand sides, are invariant under volume-preserving linear transformations.

The key ingredient to prove Theorem 1 is (a special case) of the following duality relation for log concave functions $\varphi : \mathbb{R}^n \rightarrow [0, \infty)$ and their duals φ° .

Theorem 3. *Let $\varphi : \mathbb{R}^n \rightarrow [0, \infty)$ be a log-concave function. For a convex or concave function $f : (0, \infty) \rightarrow \mathbb{R}$, let $f^*(t) = tf(1/t)$. Then*

$$D_f(\varphi^\circ) = D_{f^*}(\varphi). \tag{5}$$

We present several applications. In Section 4, we consider f -divergence for special functions f , which, on the level of convex bodies, correspond to L_p -affine surface areas. We refer to [41, 51, 63] for the definition and to e.g., [23], [37], [39], [40], [50], [61], [62], [66]-[70] for more information on L_p -affine surface area for convex bodies. The L_p -affine surface areas for functions were already introduced in [12]. Here, we establish several affine isoperimetric inequalities for these quantities. They are the functional counterparts of known inequalities for convex bodies. Another application is given in Section 5, where we apply our results about log concave functions to convex bodies. Finally, in Section 6 we obtain a reverse Poincaré inequality that is stronger than the one proved in [6].

Throughout the paper we will assume that the convex or concave functions $f : (0, \infty) \rightarrow \mathbb{R}$ and the s -concave and log concave functions $\varphi : \mathbb{R}^n \rightarrow [0, \infty)$ have enough smoothness and integrability properties so that the expressions considered in the statements make sense, i.e., we will always assume that $\varphi^\circ \in L^1(\text{supp}(\varphi), dx)$, the Lebesgue integrable functions on the support of φ , that

$$\varphi \in C^2(\text{supp}(\varphi)) \cap L^1(\mathbb{R}^n, dx), \tag{6}$$

where $C^2(\text{supp}(\varphi))$ denotes the twice continuously differentiable functions on their support, and that

$$\varphi f \left(\frac{e^{\langle \nabla \varphi, x \rangle}}{\varphi^2} \det(\text{Hess}(-\ln \varphi)) \right) \in L^1(\text{supp}(\varphi), dx). \tag{7}$$

See also Remark (iv) after Definition 9.

2 f -divergence for s -concave functions.

2.1 Background on f -divergence.

In information theory, probability theory and statistics, an f -divergence is a function that measures the difference between two (probability) distributions. This notion was introduced by Csiszár [14], and independently Morimoto [52] and Ali & Silvery [4].

Let (X, μ) be a measure space and let $P = p\mu$ and $Q = q\mu$ be (probability) measures on X that are absolutely continuous with respect to the measure μ . Let $f : (0, \infty) \rightarrow \mathbb{R}$ be a convex or a concave function. The $*$ -adjoint function $f^* : (0, \infty) \rightarrow \mathbb{R}$ of f is defined by

$$f^*(t) = tf(1/t), \quad t \in (0, \infty). \quad (8)$$

It is obvious that $(f^*)^* = f$ and that f^* is again convex if f is convex, respectively concave if f is concave. Then the f -divergence $D_f(P, Q)$ of the measures P and Q is defined by

$$\begin{aligned} D_f(P, Q) &= \int_{\{pq>0\}} f\left(\frac{p}{q}\right) qd\mu + f(0) Q(\{x \in X : p(x) = 0\}) \\ &+ f^*(0) P(\{x \in X : q(x) = 0\}), \end{aligned} \quad (9)$$

provided the expressions exist. Here

$$f(0) = \lim_{t \downarrow 0} f(t) \quad \text{and} \quad f^*(0) = \lim_{t \downarrow 0} f^*(t). \quad (10)$$

We make the convention that $0 \cdot \infty = 0$.

Please note that

$$D_f(P, Q) = D_{f^*}(Q, P). \quad (11)$$

With (10) and as

$$f^*(0) P(\{x \in X : q(x) = 0\}) = \int_{\{q=0\}} f^*\left(\frac{q}{p}\right) pd\mu = \int_{\{q=0\}} f\left(\frac{p}{q}\right) qd\mu,$$

we can write in short

$$D_f(P, Q) = \int_X f\left(\frac{p}{q}\right) qd\mu. \quad (12)$$

Examples of f -divergences are as follows.

1. For $f(t) = t \ln t$ (with $*$ -adjoint function $f^*(t) = -\ln t$), the f -divergence is *Kullback-Leibler divergence* or *relative entropy* from P to Q (see [13])

$$D_{KL}(P\|Q) = \int_X p \ln \frac{p}{q} d\mu. \quad (13)$$

2. For the convex or concave functions $f(t) = t^\alpha$ we obtain the *Hellinger integrals* (e.g. [33])

$$H_\alpha(P, Q) = \int_X p^\alpha q^{1-\alpha} d\mu. \quad (14)$$

Those are related to the Rényi divergence of order α , $\alpha \neq 1$, introduced by Rényi [56] (for $\alpha > 0$) as

$$D_\alpha(P\|Q) = \frac{1}{\alpha - 1} \ln \left(\int_X p^\alpha q^{1-\alpha} d\mu \right) = \frac{1}{\alpha - 1} \ln (H_\alpha(P, Q)). \quad (15)$$

The case $\alpha = 1$ is the relative entropy $D_{KL}(P\|Q)$.

More on f -divergence can be found in e.g. [18, 32, 33, 53, 55, 68, 71].

2.2 f -divergence for s -concave functions.

Let $s \in \mathbb{R}$, $s \neq 0$. Let $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}_+$. Following Borell [11], we say that φ is s -concave if for every $\lambda \in [0, 1]$ and all x and y such that $\varphi(x) > 0$ and $\varphi(y) > 0$,

$$\varphi((1 - \lambda)x + \lambda y) \geq ((1 - \lambda)\varphi(x)^s + \lambda\varphi(y)^s)^{1/s}.$$

Note that s can be negative. Now we want to define f -divergence for s -concave functions. To do that, let

$$P_\varphi^{(s)} = \frac{\det \left[\frac{-\text{Hess}(\varphi)}{\varphi} + (1 - s) \frac{\nabla\varphi \otimes \nabla\varphi}{\varphi^2} \right]}{\varphi \left(1 - s \frac{\langle \nabla\varphi, x \rangle}{\varphi} \right)^{n + \frac{1}{s}}}, \quad Q_\varphi^{(s)} = \varphi \left(1 - s \frac{\langle \nabla\varphi, x \rangle}{\varphi} \right).$$

Recall that we assume that the functions satisfy the conditions (6) and (7).

Definition 4. Let $f : (0, \infty) \rightarrow \mathbb{R}$ be a convex or concave function. Let $s \in \mathbb{R}$ and let $\varphi : \mathbb{R}^n \rightarrow [0, \infty)$ be an s -concave function. Then the f -divergence $D_f^{(s)}(P_\varphi^{(s)}, Q_\varphi^{(s)})$ of φ is

$$D_f^{(s)}(P_\varphi^{(s)}, Q_\varphi^{(s)}) = \int_{\text{supp}(\varphi)} \varphi f \left(\frac{\det \left[\frac{-\text{Hess}(\varphi)}{\varphi} + (1 - s) \frac{\nabla\varphi \otimes \nabla\varphi}{\varphi^2} \right]}{\varphi^2 \left(1 - s \frac{\langle \nabla\varphi, x \rangle}{\varphi} \right)^{n + \frac{1}{s} + 1}} \right) \left(1 - s \frac{\langle \nabla\varphi, x \rangle}{\varphi} \right) dx.$$

We will sometimes write in short $D_f^{(s)}(\varphi)$ for $D_f^{(s)}(P_\varphi^{(s)}, Q_\varphi^{(s)})$.

Please note also that for $s \neq 1$ expression of the definition can be rewritten as

$$D_f^{(s)}(\varphi) = D_f^{(s)}(P_\varphi^{(s)}, Q_\varphi^{(s)}) = \int_{\text{supp}(\varphi)} \varphi f \left(\frac{\det \left[\left(\text{Hess}(-\ln \varphi) + s \frac{\nabla\varphi \otimes \nabla\varphi}{\varphi^2} \right) \right]}{\varphi^2 \left(1 - s \frac{\langle \nabla\varphi, x \rangle}{\varphi} \right)^{n + \frac{1}{s} + 1}} \right) \left(1 - s \frac{\langle \nabla\varphi, x \rangle}{\varphi} \right) dx. \quad (16)$$

Remark. A similar expression holds for $D_f^{(s)}(Q_\varphi^{(s)}, P_\varphi^{(s)})$, namely

$$D_f^{(s)}(Q_\varphi^{(s)}, P_\varphi^{(s)}) = \int_{\text{supp}(\varphi)} f \left(\frac{\varphi^2 \left(1 - s \frac{\langle \nabla\varphi, x \rangle}{\varphi} \right)^{n + \frac{1}{s} + 1}}{\det \left[\left(\text{Hess}(-\ln \varphi) + s \frac{\nabla\varphi \otimes \nabla\varphi}{\varphi^2} \right) \right]} \right) \frac{\det \left[\left(\text{Hess}(-\ln \varphi) + s \frac{\nabla\varphi \otimes \nabla\varphi}{\varphi^2} \right) \right]}{\varphi \left(1 - s \frac{\langle \nabla\varphi, x \rangle}{\varphi} \right)^{n + \frac{1}{s}}}. \quad (17)$$

By (11), $D_f^{(s)}(Q_\varphi^{(s)}, P_\varphi^{(s)}) = D_{f^*}^{(s)}(P_\varphi^{(s)}, Q_\varphi^{(s)}) = D_{f^*}^{(s)}(\varphi)$. Therefore it is enough to only consider $D_f^{(s)}(P_\varphi^{(s)}, Q_\varphi^{(s)})$. We will do this throughout the paper.

The motivation for this definition of f -divergence for s -concave functions comes from convex geometry. In [68], f -divergence for a convex body K in \mathbb{R}^n was introduced. We refer to [68] for more information and special cases and give here only the definition.

For $x \in \partial K$, the boundary of a sufficiently smooth convex body K , let $N_K(x)$ denote the outer unit normal to ∂K in x and let $\kappa_K(x)$ be the Gauss curvature in x . μ_K is the usual surface area measure on ∂K . We put

$$p_K(x) = \frac{\kappa_K(x)}{\langle x, N_K(x) \rangle^n}, \quad q_K(x) = \langle x, N_K(x) \rangle \quad (18)$$

and

$$P_K = p_K \mu_K \quad \text{and} \quad Q_K = q_K \mu_K. \quad (19)$$

Then P_K and Q_K are measures on ∂K that are absolutely continuous with respect to μ_K . Let $f : (0, \infty) \rightarrow \mathbb{R}$ be a convex or concave function. In fact, Q_K and P_K are (up to the factor n) the *cone measures* (e.g. [54]) of K and its polar

$$K^\circ = \{y : \langle x, y \rangle \leq 1 \ \forall x \in K\}, \quad (20)$$

the latter provided K has sufficiently smooth boundary.

The f -divergence of K with respect to the measures P_K and Q_K was defined in [68] as

$$D_f(P_K, Q_K) = \int_{\partial K} f\left(\frac{p_K}{q_K}\right) q_K d\mu_K = \int_{\partial K} f\left(\frac{\kappa_K(x)}{\langle x, N_K(x) \rangle^{n+1}}\right) \langle x, N_K(x) \rangle d\mu_K. \quad (21)$$

For $s > 0$ such that $\frac{1}{s} \in \mathbb{N}$, we associate with an s -concave function φ a convex body $K_s(\varphi)$ [5] (see also [6]) in $\mathbb{R}^n \times \mathbb{R}^{\frac{1}{s}}$,

$$K_s(\varphi) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^{\frac{1}{s}} : \sqrt{1/s} x \in \overline{\text{supp}(\varphi)}, \|y\| \leq \varphi^s(\sqrt{1/s} x)\}. \quad (22)$$

The following proposition relates the definitions of f -divergence for the convex bodies and s -concave functions.

Proposition 5. *Let $s > 0$ be such that $\frac{1}{s} \in \mathbb{N}$. Let $f : (0, \infty) \rightarrow \mathbb{R}$ be a convex or concave function. Let $\varphi : \mathbb{R}^n \rightarrow [0, \infty)$ be an s -concave function. Then*

$$D_f^{(s)}(P_\varphi^{(s)}, Q_\varphi^{(s)}) = \frac{D_f(P_{K_s(\varphi)}, Q_{K_s(\varphi)})}{s^{\frac{n}{2}} \text{vol}_{\frac{1}{s}-1}^{\frac{1}{s}}(S^{\frac{1}{s}-1})}, \quad (23)$$

where $S^{\frac{1}{s}-1}$ is the $(\frac{1}{s} - 1)$ -dimensional Euclidean sphere.

Proof. It was shown in [6] that for $z \in \partial(K_s(\varphi))$

$$\langle z, N_{K_s(\varphi)}(z) \rangle = \frac{\varphi^s - \langle \nabla(\varphi^s), x \rangle}{(1 + \|\nabla(\varphi^s)\|^2)^{\frac{1}{2}}} \quad (24)$$

and

$$\frac{\kappa_{K_s(\varphi)}(z)}{\langle z, N_{K_s(\varphi)}(z) \rangle^{n+\frac{1}{s}+1}} = \frac{\det\left(\frac{-\text{Hess } \varphi}{\varphi} + (1-s)\frac{\nabla\varphi\otimes\nabla\varphi}{\varphi^2}\right)}{\varphi^2\left(1-\sqrt{s}\frac{\langle\nabla\varphi, x\rangle}{\varphi}\right)^{n+\frac{1}{s}+1}}, \quad (25)$$

where φ is evaluated at $\sqrt{1/s}x = (\sqrt{1/s}x_1, \dots, \sqrt{1/s}x_n) \in \mathbb{R}^n$. We denote the collection of all points $(x_1, \dots, x_{n+\frac{1}{s}}) \in \partial K_s(\varphi)$ such that $(\sqrt{1/s}x_1, \dots, \sqrt{1/s}x_n) \in \text{int}(\text{supp}(\varphi))$ by $\tilde{\partial}K_s(\varphi)$. Since there is no contribution to the integral of $D_f(P_{K_s(\varphi)}, Q_{K_s(\varphi)})$ from $\partial K_s(\varphi) \setminus \tilde{\partial}K_s(\varphi)$ (since the Gauss curvature vanishes on the part with full dimension, if it exists), we get with (24) and (25),

$$\begin{aligned} D_f(P_{K_s(\varphi)}, Q_{K_s(\varphi)}) &= \int_{\partial K_s(\varphi)} f\left(\frac{\kappa_{K_s(\varphi)}(z)}{\langle z, N_{K_s(\varphi)}(z) \rangle^{n+1}}\right) \langle z, N_{K_s(\varphi)}(z) \rangle d\mu_{K_s(\varphi)} \\ &= \int_{\tilde{\partial}K_s(\varphi)} f\left(\frac{\det\left(\frac{-\text{Hess } \varphi}{\varphi} + (1-s)\frac{\nabla\varphi\otimes\nabla\varphi}{\varphi^2}\right)}{\varphi^2\left(1-\sqrt{s}\frac{\langle\nabla\varphi, x\rangle}{\varphi}\right)^{n+s+1}}\right) \frac{(\varphi^s - \langle\nabla(\varphi^s), x\rangle)}{(1 + \|\nabla(\varphi^s)\|^2)^{\frac{1}{2}}} d\mu_{K_s(\varphi)} \\ &= 2 \int_{\mathbb{R}^{n+\frac{1}{s}-1}} f\left(\frac{\det\left(\frac{-\text{Hess}\varphi}{\varphi} + (1-s)\frac{\nabla\varphi\otimes\nabla\varphi}{\varphi^2}\right)}{\varphi^2\left(1-\sqrt{s}\frac{\langle\nabla\varphi, x\rangle}{\varphi}\right)^{n+\frac{1}{s}+1}}\right) \frac{\varphi^s - \langle\nabla(\varphi^s), x\rangle}{\varphi^{-s}|x_{n+\frac{1}{s}}|} dx_1 \dots dx_{n+\frac{1}{s}-1} \end{aligned}$$

where φ is evaluated at $\sqrt{1/s}x = (\sqrt{1/s}x_1, \dots, \sqrt{1/s}x_n)$. The last equality follows as the boundary of $K_s(\varphi)$ consists of two, ‘‘positive’’ and ‘‘negative’’, parts. As in [6],

$$\int_{\mathbb{R}^{\frac{1}{s}-1}} \frac{dx_{n+1} \dots dx_{n+\frac{1}{s}-1}}{|x_{n+\frac{1}{s}}|} = \frac{\varphi^{1-2s}(\sqrt{1/s}x)}{2s} \text{vol}_{\frac{1}{s}}\left(B_2^{\frac{1}{s}}\right).$$

Hence,

$$\begin{aligned} D_f(P_{K_s(\varphi)}, Q_{K_s(\varphi)}) &= \\ c_s \int_{\{x: \sqrt{1/s}x \in \text{supp}(\varphi)\}} \varphi f\left(\frac{\det\left(\frac{-\text{Hess } \varphi}{\varphi} + (1-s)\frac{\nabla\varphi\otimes\nabla\varphi}{\varphi^2}\right)}{\varphi^2\left(1-\sqrt{s}\frac{\langle\nabla\varphi, x\rangle}{\varphi}\right)^{n+\frac{1}{s}+1}}\right) \left(1-\sqrt{s}\frac{\langle\nabla\varphi, x\rangle}{\varphi}\right) dx, \end{aligned}$$

where φ is evaluated at $\sqrt{1/s}x = (\sqrt{1/s}x_1, \dots, \sqrt{1/s}x_n)$ and where $c_s = \frac{1}{s} \text{vol}_{\frac{1}{s}}\left(B_2^{\frac{1}{s}}\right) = \text{vol}_{\frac{1}{s}-1}\left(S^{\frac{1}{s}-1}\right)$. With the change of variable $\sqrt{1/s}x = y$,

$$\begin{aligned} D_f(P_{K_s(\varphi)}, Q_{K_s(\varphi)}) &= \\ c_s s^{\frac{n}{2}} \int_{\text{supp}(\varphi)} \varphi f\left(\frac{\det\left[-\left(\frac{\text{Hess}(\varphi)}{\varphi} - (1-s)\frac{\nabla\varphi\otimes\nabla\varphi}{\varphi^2}\right)\right]}{\varphi^2\left(1-s\frac{\langle\nabla\varphi, y\rangle}{\varphi}\right)^{n+\frac{1}{s}+1}}\right) \left(1-s\frac{\langle\nabla\varphi, y\rangle}{\varphi}\right) dy = \\ c_s s^{\frac{n}{2}} D_f^{(s)}\left(P_\varphi^{(s)}, Q_\varphi^{(s)}\right). \end{aligned}$$

□

Now we describe some properties of the f -divergence for s -concave functions. By (11), it is enough to do this for $D_f^{(s)}(P_\varphi^{(s)}, Q_\varphi^{(s)})$ only.

Lemma 6. *Let $\varphi : \mathbb{R}^n \rightarrow [0, \infty)$ be an s -concave function and let $f : (0, \infty) \rightarrow \mathbb{R}$ be a convex or concave function. Then $D_f^{(s)}(\varphi) = D_f^{(s)}(P_\varphi^{(s)}, Q_\varphi^{(s)})$ is invariant under self adjoint $SL(n)$ invariant linear maps and it is a valuation: If $\max(\varphi_1, \varphi_2)$ is s -concave, then*

$$D_f^{(s)}(\varphi_1) + D_f^{(s)}(\varphi_2) = D_f^{(s)}(\max(\varphi_1, \varphi_2)) + D_f^{(s)}(\min(\varphi_1, \varphi_2)).$$

Proof. Let $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a self adjoint, $SL(n)$ invariant linear map. By Definition 4,

$$\begin{aligned} D_f^{(s)}(P_{\varphi \circ A}^{(s)}, Q_{\varphi \circ A}^{(s)}) &= \int_{\text{supp}(\varphi \circ A)} \varphi(Ax) \left(1 - s \frac{\langle \nabla(\varphi(Ax)), x \rangle}{\varphi(Ax)} \right) \\ &\quad f \left(\frac{\det \left(\frac{-\varphi(Ax) \text{Hess}(\varphi(Ax)) + (1-s)(\nabla(\varphi(Ax)) \otimes \nabla(\varphi(Ax)))}{(\varphi(Ax))^2} \right)}{(\varphi(Ax))^2 \left(1 - s \frac{\langle \nabla(\varphi(Ax)), x \rangle}{\varphi(Ax)} \right)^{n+\frac{1}{s}+1}} \right) dx \\ &= \frac{1}{|\det A|} \int_{\text{supp}(\varphi)} \varphi \left(1 - s \frac{\langle \nabla \varphi, y \rangle}{\varphi} \right) f \left((\det A)^2 \frac{\det \left[\frac{-\text{Hess}(\varphi)}{\varphi} + (1-s) \frac{\nabla \varphi \otimes \nabla \varphi}{\varphi^2} \right]}{\varphi^2 \left(1 - s \frac{\langle \nabla \varphi, y \rangle}{\varphi} \right)^{n+\frac{1}{s}+1}} \right) dy \\ &= D_f^{(s)}(P_\varphi^{(s)}, Q_\varphi^{(s)}). \end{aligned}$$

Next, we establish the valuation property. There, A^C denotes the complement of a set $A \subset \mathbb{R}^n$.

$$\begin{aligned} D_f^{(s)}(P_{\varphi_1}^{(s)}, Q_{\varphi_1}^{(s)}) + D_f^{(s)}(P_{\varphi_2}^{(s)}, Q_{\varphi_2}^{(s)}) &= \\ &\int_{\text{supp}(\varphi_1) \cap \text{supp}(\varphi_2)} \varphi_1 f \left(\frac{\det \left[\frac{-\text{Hess}(\varphi_1)}{\varphi_1} + (1-s) \frac{\nabla \varphi_1 \otimes \nabla \varphi_1}{\varphi_1^2} \right]}{\varphi_1^2 \left(1 - s \frac{\langle \nabla \varphi_1, x \rangle}{\varphi_1} \right)^{n+\frac{1}{s}+1}} \right) \left(1 - s \frac{\langle \nabla \varphi_1, x \rangle}{\varphi_1} \right) dx + \\ &\int_{\text{supp}(\varphi_1) \cap (\text{supp}(\varphi_2))^C} \varphi_1 f \left(\frac{\det \left[\frac{-\text{Hess}(\varphi_1)}{\varphi_1} + (1-s) \frac{\nabla \varphi_1 \otimes \nabla \varphi_1}{\varphi_1^2} \right]}{\varphi_1^2 \left(1 - s \frac{\langle \nabla \varphi_1, x \rangle}{\varphi_1} \right)^{n+\frac{1}{s}+1}} \right) \left(1 - s \frac{\langle \nabla \varphi_1, x \rangle}{\varphi_1} \right) dx + \\ &\int_{\text{supp}(\varphi_1) \cap \text{supp}(\varphi_2)} \varphi_2 f \left(\frac{\det \left[\frac{-\text{Hess}(\varphi_2)}{\varphi_2} + (1-s) \frac{\nabla \varphi_2 \otimes \nabla \varphi_2}{\varphi_2^2} \right]}{\varphi_2^2 \left(1 - s \frac{\langle \nabla \varphi_2, x \rangle}{\varphi_2} \right)^{n+\frac{1}{s}+1}} \right) \left(1 - s \frac{\langle \nabla \varphi_2, x \rangle}{\varphi_2} \right) dx + \\ &\int_{\text{supp}(\varphi_2) \cap (\text{supp}(\varphi_1))^C} \varphi_2 f \left(\frac{\det \left[\frac{-\text{Hess}(\varphi_2)}{\varphi_2} + (1-s) \frac{\nabla \varphi_2 \otimes \nabla \varphi_2}{\varphi_2^2} \right]}{\varphi_2^2 \left(1 - s \frac{\langle \nabla \varphi_2, x \rangle}{\varphi_2} \right)^{n+\frac{1}{s}+1}} \right) \left(1 - s \frac{\langle \nabla \varphi_2, x \rangle}{\varphi_2} \right) dx \\ &= D_f^{(s)}(P_{\max(\varphi_1, \varphi_2)}^{(s)}, Q_{\max(\varphi_1, \varphi_2)}^{(s)}) + D_f^{(s)}(P_{\min(\varphi_1, \varphi_2)}^{(s)}, Q_{\min(\varphi_1, \varphi_2)}^{(s)}), \end{aligned}$$

provided that $\max(\varphi_1, \varphi_2)$ is s -concave. \square

Let $s \in \mathbb{R}$, $s \neq 0$ and let $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be an s -concave function. Let $\text{supp}(\varphi) = \{x : \varphi(x) > 0\}$ be the support of φ . Then $\text{supp}(\varphi)$ is convex. We will assume throughout the rest of this section that $\text{supp}(\varphi)$ is open and bounded, that φ is C^2 on $\text{supp}(\varphi)$ and that $\lim_{x \rightarrow \partial \text{supp}(\varphi)} \varphi^s(x) = 0$. We define a function ψ on $\text{supp}(\varphi)$ (see also [57]) by

$$\psi(x) = \frac{1 - \varphi^s(x)}{s}, \quad x \in \text{supp}(\varphi). \quad (26)$$

As $\varphi > 0$ on $\text{supp}(\varphi)$, ψ is well defined, ψ is convex on $\text{supp}(\varphi)$, $\psi < \frac{1}{s}$, if $s > 0$ and $\psi > \frac{1}{s}$, if $s < 0$. We will use the following duality definition from [12]. First, let $(\text{supp}(\varphi))^* = \{y : \sup_{x \in \text{supp}(\varphi)} \langle x, y \rangle < 1\}$. Note that we can assume without loss of generality that $0 \in \text{supp}(\varphi)$. If $0 \notin \text{supp}(\varphi)$, pick $z \in \text{supp}(\varphi)$ and consider $(\text{supp}(\varphi) - z)^* + z$. Then $(\text{supp}(\varphi))^*$ is convex, open, bounded and $0 \in (\text{supp}(\varphi))^*$. On the set $\frac{(\text{supp}(\varphi))^*}{s}$ we define

$$\psi_{(s)}^*(y) = \sup_{x \in \text{supp}(\varphi)} \frac{\langle x, y \rangle - \psi(x)}{1 - s\psi(x)}, \quad y \in \frac{(\text{supp}(\varphi))^*}{s}. \quad (27)$$

Then $\psi_{(s)}^*$ is convex, and, as for $s > 0$, $\langle x, y \rangle < \frac{1}{s}$ for $x \in \text{supp}(\varphi)$ and $y \in (\text{supp}(\varphi))^*$, we have that $\psi_{(s)}^* < \frac{1}{s}$, if $s > 0$ and, similarly, that $\psi_{(s)}^* > \frac{1}{s}$, if $s < 0$. Observe also that for $s \rightarrow 0$ we obtain the Legendre transform $\mathcal{L}\psi(y) = \sup_x [\langle x, y \rangle - \psi(x)]$. We denote

$$\varphi_{(s)}^*(x) = \left(1 - s\psi_{(s)}^*(x)\right)^{1/s}$$

the function corresponding to $\psi_{(s)}^*$. $\varphi_{(s)}^*$ is well defined, s -concave and, putting $\varphi_{(s)}^* \equiv 0$ outside $\frac{(\text{supp}(\varphi))^*}{s}$, coincides for $s > 0$ with $\mathcal{L}_s(\varphi)(y) = \inf_{\text{supp}(\varphi)} \frac{(1 - s\langle x, y \rangle)_+^{1/s}}{\varphi(x)}$ from [5].

The supremum in (27) is attained at x such that

$$y = \frac{1 - s\langle x, y \rangle}{1 - s\psi(x)} \nabla \psi(x) \text{ which means } y = (1 - s\psi_{(s)}^*(y)) \nabla \psi(x).$$

Moreover,

$$\frac{1}{1 - s\psi_{(s)}^*(y)} = \frac{1 - s\psi(x)}{1 - s\langle x, y \rangle} = 1 + s(\langle \nabla \psi(x), x \rangle - \psi(x)), \quad (28)$$

and the relation between y and x is

$$y = \frac{\nabla \psi(x)}{1 + s(\langle \nabla \psi(x), x \rangle - \psi(x))} = T_\psi(x). \quad (29)$$

It was noted in [12] that the Jacobian is given by

$$dy = |\det dT_\psi(x)| dx = \frac{1 - s\psi(x)}{(1 + s(\langle \nabla \psi(x), x \rangle - \psi(x)))^{n+1}} \det \text{Hess } \psi(x) dx.$$

It was also noted in [12] that the duality $(\psi_{(s)}^*)_{(s)}^* = \psi$ holds and that therefore,

$$\det (dT_\psi(x)) \det \left(dT_{\psi_{(s)}^*}(y) \right) = 1. \quad (30)$$

Now, the next theorem provides a duality formula for an s -concave function φ and $\varphi_{(s)}^*$. It is a generalization of a duality formula proved for special f in [12].

Theorem 7. Let $f : (0, \infty) \rightarrow \mathbb{R}$ be a convex or concave function. Let $\varphi : \mathbb{R}^n \rightarrow [0, \infty)$ be an s -concave function such that C_φ is open and bounded, φ is differentiable on C_φ and that $\lim_{x \rightarrow \partial \text{supp}(\varphi)} \varphi^s(x) = 0$. Then

$$D_f^{(s)}(P_{\varphi_{(s)}^*}, Q_{\varphi_{(s)}^*}) = D_{f^*}^{(s)}(P_\varphi^{(s)}, Q_\varphi^{(s)}). \quad (31)$$

Remark. In particular, if $f \equiv 1$, or, equivalently, $f^* = Id$, formula (31) becomes

$$(1 + sn) \int \varphi_{(s)}^* dx = \int_{\text{supp}(\varphi)} \frac{\det \left[\frac{-\text{Hess}(\varphi)}{\varphi} + (1-s) \frac{\nabla \varphi \otimes \nabla \varphi}{\varphi^2} \right]}{\varphi \left(1 - s \frac{\langle \nabla \varphi, x \rangle}{\varphi} \right)^{n + \frac{1}{s}}}. \quad (32)$$

Proof. By Definition 4, the change of variable (29) and (30)

$$\begin{aligned} D_{f^*}^{(s)}(P_\varphi^{(s)}, Q_\varphi^{(s)}) &= \int_{\mathbb{R}^n} f^* \left(\frac{\det \text{Hess } \psi(x)}{(1 - s\psi(x))^{\frac{1}{s}-1} |1 - s\psi(x) + s\langle x, \nabla \psi(x) \rangle|^{\frac{1}{s} + n + 1}} \right) \\ &\quad \times (1 - s\psi(x))^{\frac{1}{s}-1} |1 - s(\psi(x) - \langle x, \nabla \psi(x) \rangle)| dx \\ &= \int_{\mathbb{R}^n} f^* \left(\frac{\det dT_\psi(x)}{(1 - s\psi(x))^{\frac{1}{s}} |1 - s\psi(x) + s\langle x, \nabla \psi(x) \rangle|^{\frac{1}{s}}} \right) \\ &\quad \times (1 - s\psi(x))^{\frac{1}{s}-1} (1 - s\psi_{(s)}^*(y))^{-1} dx \\ &= \int_{\mathbb{R}^n} f \left(\frac{(1 - s\psi_{(s)}^*(y))^{1-\frac{1}{s}} \det \text{Hess } \psi_{(s)}^*(y)}{|1 - s\psi_{(s)}^*(y) + s\langle y, \nabla \psi_{(s)}^*(y) \rangle|^{\frac{1}{s} + n + 1}} \right) \\ &\quad \times (1 - s\psi_{(s)}^*(y))^{\frac{1}{s}-1} |1 - s\psi_{(s)}^*(y) + s\langle y, \nabla \psi_{(s)}^*(y) \rangle| dy \\ &= D_f^{(s)}(P_{\varphi_{(s)}^*}^{(s)}, Q_{\varphi_{(s)}^*}^{(s)}). \end{aligned}$$

□

The proof of the following entropy inequality for s -concave functions is immediate with Jensen's inequality and identity (32).

Theorem 8. Let $f : (0, \infty) \rightarrow \mathbb{R}$ be a convex function. Let $\varphi : \mathbb{R}^n \rightarrow (0, \infty)$ be an s -concave function such that C_φ is open and bounded, φ is differentiable on C_φ and that $\lim_{x \rightarrow \partial \text{supp}(\varphi)} \varphi^s(x) = 0$. Then

$$D_f^{(s)}(P_\varphi^{(s)}, Q_\varphi^{(s)}) \geq (1 + ns) \left(\int_{\text{supp}(\varphi)} \varphi dx \right) f \left(\frac{\int \varphi_{(s)}^* dx}{\int \varphi dx} \right).$$

If f is concave, the inequality is reversed.

3 f -divergence for log concave functions.

A function $\varphi : \mathbb{R}^n \rightarrow [0, \infty)$ is log concave, if it is of the form $\varphi(x) = e^{-\psi(x)}$, where $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ is a convex function. A log-concave function φ can be approximated by the sequence of k -concave functions $\{\varphi_k\}_{k=1}^{\infty}$

$$\varphi_k = (1 + k \ln \varphi)_+^{\frac{1}{k}}, \quad k \in \mathbb{N} \quad (33)$$

where for $a \in \mathbb{R}$, $a_+ = \max\{a, 0\}$. This motivates our definition for f -divergence for log concave functions. We put

$$Q_\varphi = \varphi \quad \text{and} \quad P_\varphi = \varphi^{-1} e^{\frac{\langle \nabla \varphi, x \rangle}{\varphi}} \det [\text{Hess}(-\ln \varphi)] \quad (34)$$

and define now the f -divergences for log concave functions.

Definition 9. Let $f : (0, \infty) \rightarrow \mathbb{R}$ be a convex or concave function and let $\varphi : \mathbb{R}^n \rightarrow [0, \infty)$ be a log concave function. Then the f -divergence $D_f(P_\varphi, Q_\varphi)$ of φ is

$$D_f(P_\varphi, Q_\varphi) = \int_{\text{supp}(\varphi)} \varphi f \left(\frac{e^{\frac{\langle \nabla \varphi, x \rangle}{\varphi}}}{\varphi^2} \det [\text{Hess}(-\ln \varphi)] \right) dx. \quad (35)$$

Again, we will sometimes write in short $D_f(\varphi)$ for $D_f(P_\varphi, Q_\varphi)$.

Remarks and Examples. (i) Similarly to (35),

$$D_f(Q_\varphi, P_\varphi) = \int_{\text{supp}(\varphi)} \varphi^{-1} e^{\frac{\langle \nabla \varphi, x \rangle}{\varphi}} \det [-\text{Hess}(\ln \varphi)] f \left(\frac{\varphi^2}{e^{\frac{\langle \nabla \varphi, x \rangle}{\varphi}} \det [\text{Hess}(-\ln \varphi)]} \right) dx.$$

As by (11), $D_f(Q_\varphi, P_\varphi) = D_{f^*}(P_\varphi, Q_\varphi)$, it is enough to consider $D_f(P_\varphi, Q_\varphi)$.

(ii) If we write a log concave function as $\varphi = e^{-\psi}$, ψ convex, then (35) (and similarly $D_f(Q_\varphi, P_\varphi)$) can be written as

$$D_f(P_\varphi, Q_\varphi) = \int_{\text{supp}(\psi)} e^{-\psi} f \left(e^{2\psi - \langle \nabla \psi, x \rangle} \det [\text{Hess} \psi] \right) dx. \quad (36)$$

(iii) Let A be a positive definite, symmetric matrix, $C > 0$ a constant and $\varphi(x) = C e^{-\langle Ax, x \rangle}$. Then

$$D_f(P_\varphi, Q_\varphi) = f \left(\frac{2^n \det(A)}{C^2} \right) \frac{C \pi^{n/2}}{\sqrt{\det(A)}}. \quad (37)$$

(iv) Let a be a non-zero vector in \mathbb{R}^n , $C > 0$ a constant and let $\varphi(x) = C e^{-\langle a, x \rangle}$. Then

$$D_f(P_\varphi, Q_\varphi) = \frac{C f(0)}{\prod_{i=1}^n a_i} \left(\int_{\mathbb{R}} e^{-x} dx \right)^n,$$

which is infinity, unless $f(0) = 0$. Therefore, we require that $\varphi = e^{-\psi}$ is such that ψ is strictly convex.

If φ is an s_0 -concave function, then φ is s -concave for all $s \leq s_0$. In particular, φ is log concave. Thus $D_f(P_\varphi, Q_\varphi)$ is defined for φ and $D_f(P_\varphi, Q_\varphi) = D_f^{(0)}(P_\varphi^{(0)}, Q_\varphi^{(0)})$.

On the other hand, as it was remarked in (33), every log concave function can be approximated by s -concave functions. The next Proposition shows that Definition 9 is compatible with Definition 4 for s -concave functions.

Proposition 10. *Let $f : (0, \infty) \rightarrow \mathbb{R}$ be a convex or concave function. Let $s > 0$ and for a log concave function $\varphi : \mathbb{R}^n \rightarrow [0, \infty)$ put $\varphi_s = (1 + s \ln \varphi)_+^{\frac{1}{s}}$. Then*

$$\lim_{s \rightarrow 0} D_f^{(s)} \left(P_{\varphi_s}^{(s)}, Q_{\varphi_s}^{(s)} \right) = D_f(P_\varphi, Q_\varphi).$$

Proof. Let $s > 0$ and let $\varphi_s = (1 + s \ln \varphi)_+^{\frac{1}{s}}$. Then

$$D_f^{(s)} \left(P_{\varphi_s}^{(s)}, Q_{\varphi_s}^{(s)} \right) = \int (1 + s \ln \varphi)_+^{\frac{1}{s}} \left[1 - \frac{s \langle \nabla \varphi, x \rangle}{\varphi (1 + s \ln \varphi)_+} \right] f \left(\frac{\det \left[\frac{\text{Hess}(-\ln \varphi)}{(1 + s \ln \varphi)_+^2} + \frac{s \nabla(\ln \varphi) \otimes \nabla(\ln \varphi)}{(1 + s \ln \varphi)_+^2} - \frac{s \nabla \varphi \otimes \nabla \varphi}{\varphi^2 (1 + s \ln \varphi)_+^2} \right]}{(1 + s \ln \varphi)_+^{\frac{2}{s}} \left(1 - \frac{s \langle \nabla \varphi, x \rangle}{\varphi (1 + s \ln \varphi)_+} \right)^{n + \frac{1}{s} + 1}} \right) dx.$$

Therefore

$$\lim_{s \rightarrow 0} D_f^{(s)} \left(P_{\varphi_s}^{(s)}, Q_{\varphi_s}^{(s)} \right) = D_f(P_\varphi, Q_\varphi).$$

Note that we can interchange integration and limit because conditions (6) and (7) hold. Compare also [6]. \square

Similar to Lemma 6, f -divergences for log concave functions are affine invariant valuations. Also, the proof is similar to the one of Lemma 6 and we omit it.

Corollary 11. *Let $f : (0, \infty) \rightarrow \mathbb{R}$ be a convex or concave function and let $\varphi : \mathbb{R}^n \rightarrow [0, \infty)$ be a log-concave function. Then $D_f(P_\varphi, Q_\varphi)$ is invariant under self adjoint $SL(n)$ maps and it is a valuation.*

Recall that for $\varphi : \mathbb{R}^n \rightarrow [0, \infty)$, the dual function φ° [5] is defined by

$$\varphi^\circ(x) = \inf_{y \in \mathbb{R}^n} \left[\frac{e^{-\langle x, y \rangle}}{\varphi(y)} \right].$$

This definition is connected with the Legendre transform $\mathcal{L}\varphi(y) = \sup_{x \in \mathbb{R}^n} [\langle x, y \rangle - \varphi(x)]$, namely for $\varphi = e^{-\psi}$,

$$\varphi^\circ = e^{-\mathcal{L}(-\ln \varphi)} = e^{-\mathcal{L}(\psi)}. \quad (38)$$

Remark. Please observe that Proposition 5 justifies to call Q_φ and P_φ the *cone measures of the log-concave function φ and its polar φ°* .

The next Theorem 3, already mentioned in the introduction, gives a duality relation for a log concave function and its polar. We will see in Section 5 that it is the functional analogue of a duality formula for convex bodies.

Theorem 3. Let $f : (0, \infty) \rightarrow \mathbb{R}$ be a convex or concave function. Let $\varphi : \mathbb{R}^n \rightarrow [0, \infty)$ be a log-concave function. Then

$$D_f(P_{\varphi^\circ}, Q_{\varphi^\circ}) = D_{f^*}(P_\varphi, Q_\varphi). \quad (39)$$

Remark. In particular, for $f \equiv 1$, or, equivalently, $f^* = Id$, formula (39) becomes

$$\int_{\text{supp}(\varphi)} \varphi^\circ dx = \int_{\text{supp}(\varphi)} \varphi^{-1} (\det (\text{Hess} (-\ln \varphi))) e^{\frac{\langle \nabla \varphi, x \rangle}{\varphi}}. \quad (40)$$

Proof. We give a direct proof. But please observe that the proof also follows from Theorem 7, if we let $s \rightarrow 0$, together with (38) and Proposition 10.

We write $\varphi = e^{-\psi}$, ψ convex, and let $\mathcal{L}\psi(y)$ be the Legendre transform of ψ . Please note that when ψ is a C^2 strictly convex function, then

$$\psi(x) + \mathcal{L}\psi(y) = \langle x, y \rangle \text{ if and only if } y = \nabla\psi(x) \text{ if and only if } x = \nabla\mathcal{L}\psi(y).$$

It follows that

$$\forall y \in \mathbb{R}^n, \psi(\nabla\mathcal{L}\psi(y)) = \langle y, \nabla\mathcal{L}\psi(y) \rangle - \mathcal{L}\psi(y) \quad (41)$$

and

$$\nabla\psi \circ \nabla\mathcal{L}\psi = \nabla\mathcal{L}\psi \circ \nabla\psi = Id, \quad (42)$$

so that for any $x, y \in \mathbb{R}^n$,

$$\text{Hess} \psi(\nabla\mathcal{L}\psi(y)) \text{Hess} \mathcal{L}\psi(y) = Id = \text{Hess} \mathcal{L}\psi(\nabla\psi(x)) \text{Hess} \psi(x). \quad (43)$$

Using equations (41), (42) and (43), the change of variable $x = \nabla\mathcal{L}(\psi(y))$ gives

$$\begin{aligned} D_{f^*}(P_\varphi, Q_\varphi) &= \int_{\mathbb{R}^n} \varphi f^* \left(\det [\text{Hess} (-\ln \varphi)] \frac{e^{\frac{\langle \nabla \varphi, x \rangle}{\varphi}}}{\varphi^2} \right) dx \\ &= \int_{\mathbb{R}^n} \det (\text{Hess} \psi(x)) e^{\psi(x) - \langle \nabla \psi, x \rangle} f \left(\frac{e^{-2\psi(x) + \langle \nabla \psi, x \rangle}}{\det (\text{Hess} \psi(x))} \right) dx \\ &= \int_{\mathbb{R}^n} \det (\text{Hess} \psi(\nabla\mathcal{L}\psi(y))) e^{\psi(\nabla\mathcal{L}\psi(y)) - \langle y, \nabla(\mathcal{L}\psi(y)) \rangle} \\ &\quad \times f \left(\frac{e^{-2\psi(\nabla\mathcal{L}\psi(y)) + \langle y, \nabla\mathcal{L}\psi(y) \rangle}}{\det (\text{Hess} \psi(\nabla\mathcal{L}\psi(y)))} \right) \det (\text{Hess} \mathcal{L}\psi(y)) dy \\ &= \int_{\mathbb{R}^n} e^{-\mathcal{L}\psi(y)} f \left(\det (\text{Hess} \mathcal{L}\psi(y)) e^{-\langle y, \nabla\mathcal{L}\psi(y) \rangle + 2\mathcal{L}\psi(y)} \right) dy \\ &= \int_{\mathbb{R}^n} \varphi^\circ f \left(\det [\text{Hess} (-\ln \varphi^\circ)] \frac{e^{\frac{\langle \nabla \varphi^\circ, x \rangle}{\varphi^\circ}}}{(\varphi^\circ)^2} \right) \\ &= D_f(P_{\varphi^\circ}, Q_{\varphi^\circ}). \end{aligned}$$

□

A consequence of Theorem 3 is the following entropy inequality for log concave functions. This is Theorem 1 of the introduction.

Theorem 1. *Let $f : (0, \infty) \rightarrow \mathbb{R}$ be a convex function and let $\varphi : \mathbb{R}^n \rightarrow [0, \infty)$ be a log-concave function. Then*

$$D_f(P_\varphi, Q_\varphi) \geq f\left(\frac{\int \varphi^\circ dx}{\int \varphi dx}\right) \left(\int_{\text{supp}(\varphi)} \varphi dx\right).$$

If f is concave, the inequality is reversed. If f is linear, equality holds. Equality also holds if $\varphi(x) = Ce^{-\langle Ax, x \rangle}$, where C is a positive constant and A is a $n \times n$ positive definite matrix.

Proof. The inequality follows immediately from Jensen's inequality and identity (40). Or, if we approximate φ by $\varphi_s = (1 + s \ln \varphi)_+^{\frac{1}{s}}$, the inequality follows from Theorem 8 letting $s \rightarrow 0$.

It is easy to check that equality holds if f is linear and that equality holds for $\varphi(x) = Ce^{-\langle Ax, x \rangle}$ by (37). In fact, one can assume that A is positive definite and symmetric. \square

If we let $f(t) = \ln t$ in Theorem 1, we obtain the following corollary which is a reformulation of Corollary 2 of the introduction.

Corollary 12. *Let $\varphi : \mathbb{R}^n \rightarrow [0, \infty)$ be a log-concave function. Then*

$$D(P_\varphi \| Q_\varphi) \leq \left(\int_{\text{supp}(\varphi)} \varphi dx\right) \ln \left(\frac{\int \varphi^\circ dx}{\int \varphi dx}\right). \quad (44)$$

Equality holds if $\varphi(x) = Ce^{-\langle Ax, x \rangle}$, where C is a positive constant and A is a $n \times n$ positive definite matrix.

Remarks.

(i) Inequality (44) is stronger than (1). Indeed, as

$$D(P_\varphi \| Q_\varphi) = -2 \int \varphi \ln \varphi dx - n \int \varphi dx + \int \varphi \ln \left(\det(\text{Hess}(-\ln \varphi))\right) dx,$$

inequality (44) is equivalent to

$$\begin{aligned} & \int_{\text{supp}(\varphi)} \varphi \ln \left(\det(\text{Hess}(-\ln \varphi))\right) dx \\ & \leq 2 \text{Ent}(\varphi) + \|\varphi\|_{L^1} \ln \left[e^n \left(\int_{\text{supp}(\varphi)} \varphi dx\right) \left(\int_{\text{supp}(\varphi)} \varphi^\circ dx\right) \right]. \end{aligned}$$

Now we apply the functional form of the Blaschke Santaló inequality [5, 7, 17, 31]. We assume without loss of generality that $\int \varphi dx = 1$. Observe that we can also assume

without loss of generality that $\int x\varphi(x)dx = 0$. If $\int x\varphi(x)dx = x_0$, replace φ by $\tilde{\varphi}(x) = \varphi(x + x_0)$. We then get,

$$\int_{\text{supp}(\varphi)} \varphi \ln \left(\det (\text{Hess} (-\ln \varphi)) \right) dx \leq 2 \text{Ent}(\varphi) + \ln (2\pi e)^n ,$$

which is inequality (1).

(ii) The characterization of equality in (1) now follows by the equality characterization of the Blaschke Santaló inequality. Indeed, from the arguments in (i), if there is equality in (1), there is also equality in the functional Blaschke-Santaló inequality. This implies that the function has the form $\varphi(x) = Ce^{-\langle Ax, x \rangle}$ by [5].

Let us state another corollary to Theorem 1. Its proof follows immediately from Theorem 1 and the functional Blaschke Santaló inequality and its equality characterization.

Corollary 13. *Let $\varphi : \mathbb{R}^n \rightarrow [0, \infty)$ be a log-concave function that has center of mass at 0. Let $f : (0, \infty) \rightarrow \mathbb{R}$ be a convex, decreasing function. Then*

$$D_f(P_\varphi, Q_\varphi) \geq f \left(\frac{(2\pi)^n}{(\int \varphi dx)^2} \right) \left(\int_{\text{supp}(\varphi)} \varphi dx \right).$$

If f is a concave, increasing function, the inequality is reversed.

Equality holds in both cases if and only if $\varphi(x) = ce^{-\langle Ax, x \rangle}$, where c is a positive constant and A is an $n \times n$ positive definite matrix.

4 Applications to special functions.

Now we consider special cases of f -divergences for log concave functions. Please recall that in subsection 2.1, the α -Rényi entropies (15) were introduced as special f -divergences. Examples of such Rényi entropies are, for log concave functions $\varphi : \mathbb{R}^n \rightarrow [0, \infty)$, for $f(t) = t^\lambda$, $-\infty < \lambda < \infty$, the L_λ -affine surface areas $as_\lambda(\varphi)$ of φ , introduced in [12],

$$as_\lambda(\varphi) = \int_{\text{supp}(\varphi)} \varphi \left(\frac{e^{\frac{\langle \nabla \varphi, x \rangle}{\varphi}}}{\varphi^2} \det [\text{Hess} (-\ln \varphi)] \right)^\lambda dx, \quad (45)$$

or, writing $\varphi(x) = e^{-\psi(x)}$, ψ convex,

$$as_\lambda(\varphi) = as_\lambda(e^{-\psi}) = \int_{\mathbb{R}^n} e^{(2\lambda-1)\psi(x) - \lambda \langle x, \nabla \psi(x) \rangle} (\det \text{Hess } \psi(x))^\lambda dx. \quad (46)$$

Especially, $as_0(\varphi) = \int_{\text{supp}(\varphi)} \varphi dx$ and, by (40), $as_1(\varphi) = \int_{\text{supp}(\varphi)} \varphi^\circ dx$. Please note also that for any log concave function φ we have that $as_\lambda(\varphi) \geq 0$. Moreover, by Corollary 11, the $as_\lambda(\varphi)$ are affine invariant valuations.

We first want to give a definition for $as_\infty(\varphi)$ and $as_{-\infty}(\varphi)$, similarly as it was done for convex bodies [51]. To that end, for $\lambda > 0$, let $\tilde{as}_\lambda(\varphi) = (as_\lambda(\varphi))^{\frac{1}{\lambda}}$ and denote

$$h = (\varphi)^{\frac{1}{\lambda}} \frac{e^{\frac{\langle \nabla \varphi, x \rangle}{\varphi}}}{\varphi^2} \det [\text{Hess} (-\ln \varphi)].$$

Denote by $\|f\|_\lambda = (\int f^\lambda dx)^{\frac{1}{\lambda}}$ the L_λ norm of a function f . Then, for $\lambda \rightarrow \infty$,

$$\tilde{as}_\lambda(\varphi) = \left(\int_{\text{supp}(\varphi)} h^\lambda dx \right)^{\frac{1}{\lambda}} = \|h\|_\lambda \rightarrow \|h\|_\infty = \max_{x \in \text{supp}(\varphi)} \frac{e^{\frac{\langle \nabla \varphi, x \rangle}{\varphi}}}{\varphi^2} \det [\text{Hess}(-\ln \varphi)].$$

Therefore, it is natural to put

$$as_\infty(\varphi) = \max_{x \in \text{supp}(\varphi)} \frac{e^{\frac{\langle \nabla \varphi, x \rangle}{\varphi}}}{\varphi^2} \det [\text{Hess}(-\ln \varphi)]. \quad (47)$$

Similarly, for $\lambda \rightarrow -\infty$,

$$\begin{aligned} \tilde{as}_\lambda(\varphi) &\rightarrow \frac{1}{\max_{x \in \text{supp}(\varphi)} \frac{e^{\frac{\langle \nabla \varphi, x \rangle}{\varphi}}}{\varphi^2} \det [\text{Hess}(-\ln \varphi)]} \\ &= \min_{x \in \text{supp}(\varphi)} \frac{\varphi^2}{e^{\frac{\langle \nabla \varphi, x \rangle}{\varphi}} \det [\text{Hess}(-\ln \varphi)]} = as_{-\infty}(\varphi). \end{aligned} \quad (48)$$

Hence we have that

$$as_{-\infty}(\varphi) = \frac{1}{as_\infty(\varphi)}.$$

It is also easy to see that these expressions are invariant under symmetric affine transformations with determinant 1.

The next theorem gives the analogue, for log concave functions, of a monotonicity behavior of the L_λ -affine surface area that was established for convex bodies in [41, 69]. The case $\beta = 0$ and $\alpha = 1$ was already proved in [12].

Proposition 14. *Let $\alpha \neq \beta, \lambda \neq \beta$ be real numbers. Let $\varphi : \mathbb{R}^n \rightarrow [0, \infty)$ be a log concave function.*

(i) *If $1 \leq \frac{\alpha-\beta}{\lambda-\beta} < \infty$, then $as_\lambda(\varphi) \leq (as_\alpha(\varphi))^{\frac{\lambda-\beta}{\alpha-\beta}} (as_\beta(\varphi))^{\frac{\alpha-\lambda}{\alpha-\beta}}$.*

(ii) *If $1 \leq \frac{\alpha}{\lambda} < \infty$, then $as_\lambda(\varphi) \leq (as_\alpha(\varphi))^{\frac{\lambda}{\alpha}} (\int \varphi)^{\frac{\alpha-\lambda}{\alpha}}$.*

(iii) *If $\beta \leq \lambda$, then $as_\lambda(\varphi) \leq (as_\infty(\varphi))^{\lambda-\beta} as_\beta(\varphi)$.*

If $\frac{\alpha-\beta}{\lambda-\beta} = 1$ in (i), respectively $\frac{\alpha}{\lambda} = 1$ in (ii), then $\alpha = \lambda$ and equality holds trivially in (i) respectively (ii). Equality also holds if $\varphi(x) = Ce^{-\langle Ax, x \rangle}$.

Proof. The proofs follow by Hölder's inequality, which, in (i), enforces the condition $\frac{\alpha-\beta}{\lambda-\beta} > 1$. The case(ii) is a special case of (i) for $\beta = 0$. We show (i). The others follow

similarly.

$$\begin{aligned}
as_\lambda(\varphi) &= \int_{\text{supp}(\varphi)} \varphi \left(\frac{e^{\frac{\langle \nabla \varphi, x \rangle}{\varphi}}}{\varphi^2} \det [\text{Hess}(-\ln \varphi)] \right)^\lambda dx \\
&= \int \left[\varphi \left(\frac{e^{\frac{\langle \nabla \varphi, x \rangle}{\varphi}}}{\varphi^2} \det [\text{Hess}(-\ln \varphi)] \right)^\alpha \right]^{\frac{\lambda-\beta}{\alpha-\beta}} \\
&\quad \times \left[\varphi \left(\frac{e^{\frac{\langle \nabla \varphi, x \rangle}{\varphi}}}{\varphi^2} \det [\text{Hess}(-\ln \varphi)] \right)^\beta \right]^{\frac{\alpha-\lambda}{\alpha-\beta}} dx \\
&\leq (as_\alpha(\varphi))^{\frac{\lambda-\beta}{\alpha-\beta}} (as_\beta(\varphi))^{\frac{\alpha-\lambda}{\alpha-\beta}}.
\end{aligned}$$

□

It follows from Proposition 14 (ii) that for $0 < \lambda \leq \alpha$,

$$0 \leq \left(\frac{as_\lambda(\varphi)}{\int \varphi dx} \right)^{\frac{1}{\lambda}} \leq \left(\frac{as_\alpha(\varphi)}{\int \varphi dx} \right)^{\frac{1}{\alpha}},$$

which means that for $\lambda > 0$ the function $\lambda \rightarrow \left(\frac{as_\lambda(\varphi)}{\int \varphi dx} \right)^{\frac{1}{\lambda}}$ is bounded below by 0 and is increasing for $\lambda > 0$. Therefore, the limit

$$\Omega_\varphi = \lim_{\lambda \downarrow 0} \left(\frac{as_\lambda(\varphi)}{\int \varphi dx} \right)^{\frac{1}{\lambda}} \quad (49)$$

exists and the quantity Ω_φ is an affine invariant. It is the analogue for log concave functions of an affine invariant introduced by Paouris and Werner in [54] for convex bodies. The quantity Ω_φ is related to the relative entropy as follows.

Proposition 15. *Let $\varphi : \mathbb{R}^n \rightarrow [0, \infty)$ be a log concave function. Then*

$$\Omega_\varphi = \lim_{\lambda \downarrow 0} \left(\frac{as_\lambda(\varphi)}{\int \varphi dx} \right)^{\frac{1}{\lambda}} = \lim_{\lambda \uparrow 0} \left(\frac{as_\lambda(\varphi)}{\int \varphi dx} \right)^{\frac{1}{\lambda}} = \exp \left(\frac{D(P_\varphi \| Q_\varphi)}{\int \varphi dx} \right).$$

Proof. By definition and de l'Hôpital,

$$\begin{aligned}
\Omega_\varphi &= \lim_{\lambda \downarrow 0} \left(\frac{as_\lambda(\varphi)}{\int \varphi} \right)^{\frac{1}{\lambda}} = \lim_{\lambda \downarrow 0} \exp \left(\frac{1}{\lambda} \ln \left(\frac{as_\lambda(\varphi)}{\int \varphi dx} \right) \right) \\
&= \exp \left(\lim_{\lambda \downarrow 0} \frac{\int \frac{d}{d\lambda} \left[\varphi \left(\frac{e^{\frac{\langle \nabla \varphi, x \rangle}{\varphi}}}{\varphi^2} \det [\text{Hess}(-\ln \varphi)] \right)^\lambda \right] dx}{as_\lambda(\varphi)} \right) \\
&= \exp \left(\frac{\int \varphi \ln \left(\frac{e^{\frac{\langle \nabla \varphi, x \rangle}{\varphi}}}{\varphi^2} \det [\text{Hess}(-\ln \varphi)] dx \right)}{\int \varphi dx} \right) \\
&= \exp \left(\frac{D(P_\varphi || Q_\varphi)}{\int \varphi dx} \right).
\end{aligned}$$

It also follows from Proposition 14 (ii) that for $\lambda < 0$, the function $\lambda \rightarrow \left(\frac{as_\lambda(\varphi)}{\int \varphi} dx \right)^{\frac{1}{\lambda}}$ is increasing. We compute $\lim_{\lambda \uparrow 0} \left(\frac{as_\lambda(\varphi)}{\int \varphi^\circ} \right)^{\frac{1}{\lambda}}$ as above. \square

Corollary 16. *Let $\varphi : \mathbb{R}^n \rightarrow [0, \infty)$ be a log concave function.*

- (i) $\Omega_\varphi \leq \left(\frac{as_\lambda(\varphi)}{\int \varphi dx} \right)^{\frac{1}{\lambda}}$ for all $\lambda > 0$ and $\Omega_\varphi \geq \left(\frac{as_\lambda(\varphi)}{\int \varphi dx} \right)^{\frac{1}{\lambda}}$ for all $\lambda < 0$.
- (ii) $\Omega_\varphi \Omega_{\varphi^\circ} \leq 1$.
- (iii) $\Omega_\varphi = \lim_{\alpha \rightarrow 1} \left(\frac{as_\alpha(\varphi^\circ)}{\int \varphi dx} \right)^{\frac{1}{1-\alpha}}$.

Equality holds in (i) and (ii) if $\varphi = Ce^{-\langle Ax, x \rangle}$.

Proof. (i) is deduced immediately from the monotonicity behavior of the function $\lambda \rightarrow \left(\frac{as_\lambda(\varphi)}{\int \varphi} dx \right)^{\frac{1}{\lambda}}$ and the definition of Ω_φ .

(ii) By (i), $\Omega_\varphi \leq \frac{as_1(\varphi)}{\int \varphi dx} = \frac{\int \varphi^\circ dx}{\int \varphi dx}$ and $\Omega_{\varphi^\circ} \leq \frac{as_1(\varphi^\circ)}{\int \varphi^\circ dx} = \frac{\int \varphi dx}{\int \varphi^\circ dx}$. Here, we have also used the bipolar property $(\varphi^\circ)^\circ = \varphi$. Thus (ii) follows.

(iii) We use the duality formula $as_\lambda(\varphi) = as_{1-\lambda}(\varphi^\circ)$ which was first proved in [12]. Note that it can also be obtained as a special case of Theorem 3 for $f(t) = t^\lambda$. By definition

$$\Omega_{\varphi^\circ} = \lim_{\lambda \rightarrow 0} \left(\frac{as_\lambda(\varphi^\circ)}{\int \varphi^\circ} \right)^{\frac{1}{\lambda}} = \lim_{\lambda \rightarrow 0} \left(\frac{as_{1-\lambda}(\varphi)}{\int \varphi^\circ} \right)^{\frac{1}{\lambda}} = \lim_{\alpha \rightarrow 1} \left(\frac{as_\alpha(\varphi)}{\int \varphi^\circ} \right)^{\frac{1}{1-\alpha}}.$$

Therefore, $\Omega_\varphi = \lim_{\alpha \rightarrow 1} \left(\frac{as_\alpha(\varphi^\circ)}{\int \varphi dx} \right)^{\frac{1}{1-\alpha}}$. \square

5 Application to convex bodies.

Let us now consider the case of 2-homogeneous functions ψ , that is $\psi(\lambda x) = \lambda^2 \psi(x)$ for any $\lambda \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$. Such functions ψ are necessarily (and this is obviously sufficient) of the form $\psi(x) = \|x\|_K^2/2$ for a certain convex body K with 0 in its interior, where we have denoted by $\|\cdot\|_K$ the gauge function of K ,

$$\|x\|_K = \min\{\lambda \geq 0 : \lambda x \in K\} = \max_{y \in K^\circ} \langle x, y \rangle = h_{K^\circ}(x).$$

Differentiating with respect to λ at $\lambda = 1$, we get

$$\langle x, \nabla \psi(x) \rangle = 2\psi(x). \quad (50)$$

Now we apply this function to the identities and inequalities which we have obtained for f -divergences for log concave functions. It was already observed in [12] that the L_λ -affine surface area for log concave functions is a generalization of L_λ -affine surface area for convex bodies. Indeed, it was noted there that if one applies the log concave function $\varphi_K = \exp\left(-\frac{\|\cdot\|_K^2}{2}\right)$ to Definition 45, then one obtains L_λ -affine surface area for convex bodies. Please note also that

$$\int e^{-\frac{\|x\|_K^2}{2}} dx = \frac{(2\pi)^{\frac{n}{2}} |K|}{|B_2^n|} \quad \text{and} \quad \int e^{-\frac{\|x\|_{K^\circ}^2}{2}} dx = \frac{(2\pi)^{\frac{n}{2}} |K^\circ|}{|B_2^n|}. \quad (51)$$

Recall that B_2^n denotes the n -dimensional Euclidean unit ball, and for a convex body K in \mathbb{R}^n , K° is the polar of K (20) and $|K|$ is its volume.

The following is a generalization of a result in [12] but it is proved in exactly the same way. We include the proof for completeness.

Theorem 17. *Let K be a convex body in \mathbb{R}^n with 0 in its interior. Let $\varphi_K = \exp\left(-\frac{\|\cdot\|_K^2}{2}\right)$ and $f : (0, \infty) \rightarrow \mathbb{R}$ be a convex or concave function. Then*

$$D_f(P_{\varphi_K}, Q_{\varphi_K}) = \frac{(2\pi)^{\frac{n}{2}}}{n|B_2^n|} D_f(P_K, Q_K).$$

Here, P_K and Q_K are as in (19) and, for φ_K , P_{φ_K} and Q_{φ_K} are as in (34).

Proof. We will use formula (36) for $\psi = \frac{\|\cdot\|_K^2}{2}$ and integrate in polar coordinates with respect to the normalized cone measure $\frac{P_K}{n|K|}$ (19) of K . Thus, if we write $x = rz$, with $z \in \partial K$, $dx = r^{n-1} dr dP_K(z)$. We also use that the map $x \mapsto \det \text{Hess } \psi(x)$ is 0-homogeneous. Therefore, with (50),

$$\begin{aligned} D_f(P_{\varphi_K}, Q_{\varphi_K}) &= \int_0^{+\infty} r^{n-1} e^{-\frac{r^2}{2}} dr \int_{\partial K} f(\det \text{Hess } \psi(z)) dP_K(z) \\ &= \frac{(2\pi)^{\frac{n}{2}}}{n|B_2^n|} \int_{\partial K} f(\det \text{Hess } \psi(z)) dP_K(z). \end{aligned}$$

It was proved in [12] that for all $z \in \partial K$,

$$\det(\text{Hess}_z \psi) = \frac{\kappa_K(z)}{\langle z, N_K(z) \rangle^{n+1}}. \quad (52)$$

Observe that for the $G_K(z)$ introduced in this lemma, $\|G_K(z)\|_{K^\circ} = \langle z, N_K(z) \rangle$. Thus

$$\begin{aligned} D_f(P_{\varphi_K}, Q_{\varphi_K}) &= \frac{(2\pi)^{\frac{n}{2}}}{n|B_2^n|} \int_{\partial K} f\left(\frac{\kappa(x)}{\langle x, N_K(x) \rangle^{n+1}}\right) \langle x, N_K(x) \rangle d\mu_K(x) \\ &= \frac{(2\pi)^{\frac{n}{2}}}{n|B_2^n|} D_f(P_K, Q_K). \end{aligned}$$

□

Now we apply Theorem 1 to $\varphi = \exp\left(-\frac{\|\cdot\|_K^2}{2}\right)$ and we obtain the following inequalities. Those were already proved, with different methods, in [68]. In fact, it was shown there that equality holds if and only if K is an ellipsoid.

Corollary 18. *Let K be a convex body in \mathbb{R}^n with the origin in its interior. Let $f : (0, \infty) \rightarrow \mathbb{R}$ be a convex function. Then*

$$D_f(P_K, Q_K) \geq n |K| f\left(\frac{|K^\circ|}{|K|}\right). \quad (53)$$

If f is concave, the inequality is reversed. Equality holds if K is an ellipsoid.

Proof. Let $\varphi = \exp\left(-\frac{\|\cdot\|_K^2}{2}\right)$ and let f be convex. By Theorem 1, together with (51),

$$\int_{\text{supp}(\varphi)} e^{-\frac{\|x\|_K^2}{2}} f\left(\left[\det\left(\text{Hess}\left(\frac{\|x\|_K^2}{2}\right)\right)\right]\right) dx \geq \frac{(2\pi)^{\frac{n}{2}} |K|}{|B_2^n|} f\left(\frac{|K^\circ|}{|K|}\right).$$

Now we use again (52) and, as above, make the change of variable, $x = rz$, $z \in \partial K$. Then this becomes

$$\int_{\partial K} f\left(\frac{\kappa(x)}{\langle x, N_K(x) \rangle^{n+1}}\right) \langle x, N_K(x) \rangle d\mu_K(x) \geq n |K| f\left(\frac{|K^\circ|}{|K|}\right).$$

For f concave, the direction in the inequality changes. □

In this way, by applying them to the particular log concave function $\exp\left(-\frac{\|\cdot\|_K^2}{2}\right)$, many of the known inequalities for convex bodies can be deduced from the corresponding ones for log concave functions. Examples include:

(i) For $-\infty \leq p \leq \infty$, $p \neq -n$, the function $f(t) = t^{\frac{p}{n+p}}$ is concave for $p \geq 0$ and convex for $p \leq 0$. Then, as a consequence of Theorem 1 applied to $\varphi = \exp\left(-\frac{\|\cdot\|_K^2}{2}\right)$, and the Blaschke Santaló inequality [10, 49, 59] we obtain the L_p -affine isoperimetric inequalities of [41, 69]. If we apply Proposition 14 to $\varphi = \exp\left(-\frac{\|\cdot\|_K^2}{2}\right)$, we obtain the monotonicity behavior of the L_p -affine surface area for convex bodies proved in [41, 69].

(ii) If we apply Proposition 15 and Corollary 16 to $\varphi = \exp\left(-\frac{\|\cdot\|_K^2}{2}\right)$, then we obtain entropy inequalities first proved in [54] for convex bodies. E.g, Corollary 16, together with the Blaschke Santaló inequality gives the following isoperimetric inequality of [54]

$$\Omega_{K^\circ} \leq \Omega\left(\frac{B_2^n}{|B_2^n|^{\frac{1}{n}}}\right)^\circ = |B_2^n|^{2n}.$$

Finally, the duality formula (39) applied to $\varphi = \exp\left(-\frac{\|\cdot\|_K^2}{2}\right)$ corresponds to a duality formula for convex bodies and is a generalization of previously established duality formulas [27, 69] for convex bodies, $as_p(K) = as_{\frac{n-2}{p}}(K^\circ)$. See also [37]. We skip the proof.

Proposition 19. *Let K be a convex body in \mathbb{R}^n and let $f : (0, \infty) \rightarrow \mathbb{R}$ be a convex or concave function. Then*

$$D_f(P_{K^\circ}, Q_{K^\circ}) = D_{f^*}(P_K, Q_K).$$

6 Linearization.

In this section we linearize the inequalities of Corollary 13 around its equality case. We treat only one inequality. The other one is done in the same way. We rewrite the inequality in terms of a convex function $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\varphi = e^{-\psi}$ and get

$$\int_{\mathbb{R}^n} e^{-\psi} f\left(e^{2\psi - \langle \nabla \psi, x \rangle} \det(\text{Hess}(\psi))\right) dx \geq f\left(\frac{(2\pi)^n}{\left(\int_{\mathbb{R}^n} e^{-\psi} dx\right)^2}\right) \left(\int_{\mathbb{R}^n} e^{-\psi} dx\right). \quad (54)$$

Corollary 13 requires that φ has center of mass at the origin. This is the case if ψ is even. We then linearize around the equality case $\psi(x) = \|x\|^2/2$ and obtain the following functional inequalities. See also [6], [25], [26]. The proof, which we include for completeness, follows [6]. Throughout, $\|\cdot\|_{HS}$ denotes the Hilbert Schmidt norm and $\Delta\eta = \text{tr}(\text{Hess } \eta)$ is the Laplacian of η . γ_n is the normalized Gaussian measure on \mathbb{R}^n and $\text{Var}_{\gamma_n}(\eta) = \int_{\mathbb{R}^n} \eta^2 d\gamma_n - \left(\int_{\mathbb{R}^n} \eta d\gamma_n\right)^2$ is the variance of η .

Corollary 20. *Let $\eta \in C^2(\mathbb{R}^n) \cap L^2(\mathbb{R}^n, \gamma_n)$ be even. Then*

$$(i) \quad \frac{1}{2} \int_{\mathbb{R}^n} (\Delta\eta - \langle \nabla\eta, x \rangle)^2 d\gamma_n \leq \int_{\mathbb{R}^n} \|\text{Hess } \eta\|_{HS}^2 d\gamma_n.$$

$$(ii) \quad \int_{\mathbb{R}^n} \|\nabla\eta\|^2 d\gamma_n - \frac{1}{4} \int_{\mathbb{R}^n} (\Delta\eta - \langle \nabla\eta, x \rangle)^2 d\gamma_n \leq \text{Var}_{\gamma_n}(\eta) \leq \int_{\mathbb{R}^n} \|\nabla\eta\|^2 d\gamma_n - \frac{1}{2} \int_{\mathbb{R}^n} (\Delta\eta - \langle \nabla\eta, x \rangle)^2 d\gamma_n + \frac{1}{2} \int_{\mathbb{R}^n} \|\text{Hess}\eta\|_{HS}^2 d\gamma_n.$$

Remark. The left hand side of Corollary 20 (ii) together with Corollary 20 (i) gives the following reverse Poincaré inequality obtained in [6] (see also [25], [26])

$$\int_{\mathbb{R}^n} \left[\|\nabla\eta\|^2 - \frac{\|\text{Hess } \eta\|_{HS}^2}{2} \right] d\gamma_n \leq \text{Var}_{\gamma_n}(\eta).$$

Proof. We first prove the corollary for functions with bounded support. Thus, let η be an even, twice continuously differentiable function with bounded support and let $\psi(x) = \|x\|^2/2 + \varepsilon\eta(x)$. Note that for sufficiently small ε the function ψ is convex and that ψ is even as η is even. Therefore we can plug ψ into inequality (54) and develop in powers of ε . We evaluate first the right hand expression of (54).

$$\begin{aligned} f & \left(\frac{(2\pi)^n}{\left(\int_{\mathbb{R}^n} e^{-\|x\|^2/2-\varepsilon\eta} dx\right)^2} \right) \left(\int_{\mathbb{R}^n} e^{-\|x\|^2/2-\varepsilon\eta} dx \right) \\ &= (2\pi)^{n/2} f \left(1 + 2\varepsilon \int_{\mathbb{R}^n} \eta d\gamma_n - \varepsilon^2 \int_{\mathbb{R}^n} \eta^2 d\gamma_n + 3\varepsilon^2 \left(\int_{\mathbb{R}^n} \eta d\gamma_n \right)^2 \right) \\ & \quad \left(1 - \varepsilon \int_{\mathbb{R}^n} \eta d\gamma_n + \frac{\varepsilon^2}{2} \int_{\mathbb{R}^n} \eta^2 d\gamma_n \right) + O(\varepsilon^3). \end{aligned}$$

As $f(1+t) = f(1) + f'(1)t + \frac{f''(1)}{2}t^2 + O(t^3)$, we get that the right hand side of (54) equals

$$\begin{aligned} & (2\pi)^{n/2} \left(f(1) + \varepsilon [2f'(1) - f(1)] \int_{\mathbb{R}^n} \eta d\gamma_n + \varepsilon^2 \left[\int_{\mathbb{R}^n} \eta^2 d\gamma_n \left(\frac{f(1)}{2} - f'(1) \right) + \right. \right. \\ & \left. \left. \left(\int_{\mathbb{R}^n} \eta d\gamma_n \right)^2 [f'(1) + 2f''(1)] \right] \right) + O(\varepsilon^3). \end{aligned}$$

We evaluate now the left hand expression of (54). Since $\text{Hess } \psi = I + \varepsilon \text{ Hess } \eta$, we obtain for the left hand side

$$\int_{\mathbb{R}^n} e^{-\|x\|^2/2-\varepsilon\eta} f \left(e^{\varepsilon(2\eta - \langle \nabla \eta, x \rangle)} \det(I + \varepsilon \text{ Hess } \eta) \right) dx.$$

By Taylor's theorem this equals

$$\begin{aligned} & \int_{\mathbb{R}^n} e^{-\|x\|^2/2} \left(1 - \varepsilon\eta + \frac{\varepsilon^2}{2}\eta^2 \right) \cdot f \left(\left(1 + \varepsilon(2\eta - \langle \nabla \eta, x \rangle) + \frac{\varepsilon^2}{2}(2\eta - \langle \nabla \eta, x \rangle)^2 \right) \right. \\ & \quad \left. \cdot \det(I + \varepsilon \text{ Hess } \eta) \right) dx + O(\varepsilon^3). \end{aligned}$$

For a matrix $A = (a_{i,j})_{i,j=1,\dots,n}$, let $D(A) = \sum_{i=1}^n \sum_{j \neq i}^n [a_{i,i}a_{j,j} - a_{i,j}^2]$. Note that each 2×2 minor is counted twice. Then $\det(I + \varepsilon \text{ Hess } \eta) = 1 + \varepsilon \Delta \eta + \frac{\varepsilon^2}{2} D(\text{Hess } \eta) + O(\varepsilon^3)$.

Therefore the left hand side of (54) equals

$$\begin{aligned}
& (2\pi)^{n/2} \left[\int_{\mathbb{R}^n} \left(1 - \varepsilon\eta + \frac{\varepsilon^2}{2}\eta^2 \right) f \left(1 + \varepsilon(2\eta + \Delta\eta - \langle \nabla\eta, x \rangle) \right) \right. \\
& \left. + \varepsilon^2 \left(\frac{(2\eta - \langle \nabla\eta, x \rangle)^2}{2} + \Delta\eta(2\eta - \langle \nabla\eta, x \rangle) + \frac{D(\text{Hess } \eta)}{2} \right) d\gamma_n \right] + O(\varepsilon^3) = \\
& (2\pi)^{n/2} \left(f(1) + \varepsilon \left[-f(1) \int_{\mathbb{R}^n} \eta d\gamma_n + f'(1) \left(\int_{\mathbb{R}^n} (2\eta - \langle \nabla\eta, x \rangle + \Delta\eta) d\gamma_n \right) \right] \right) \\
& + \varepsilon^2 \left[f(1) \int_{\mathbb{R}^n} \frac{\eta^2}{2} d\gamma_n - f'(1) \left(\int_{\mathbb{R}^n} \eta(2\eta - \langle \nabla\eta, x \rangle + \Delta\eta) d\gamma_n \right) + \right. \\
& \left. f'(1) \left(\int_{\mathbb{R}^n} \left(\frac{(2\eta - \langle \nabla\eta, x \rangle)^2}{2} + \Delta\eta(2\eta - \langle \nabla\eta, x \rangle) + \frac{D(\text{Hess } \eta)}{2} \right) d\gamma_n \right) \right. \\
& \left. + \frac{f''(1)}{2} \left(\int_{\mathbb{R}^n} (2\eta - \langle \nabla\eta, x \rangle + \Delta\eta)^2 d\gamma_n \right) \right] + O(\varepsilon^3).
\end{aligned}$$

Now observe that $\int_{\mathbb{R}^n} (\Delta\eta - \langle \nabla\eta, x \rangle) d\gamma_n = 0$. Also, as the coefficients of order zero and of order ε are the same on the left hand side and the right hand side, we discard them. We divide both sides by ε^2 and take the limit for $\varepsilon \rightarrow 0$. Thus, the inequality (54) is equivalent to

$$\begin{aligned}
& \left(f'(1) + 2f''(1) \right) \left[\left(\int_{\mathbb{R}^n} \eta d\gamma_n \right)^2 d\gamma_n - \int_{\mathbb{R}^n} \eta^2 d\gamma_n \right] \leq \\
& \left(f'(1) + 2f''(1) \right) \left[\int_{\mathbb{R}^n} \eta \Delta\eta d\gamma_n - \int_{\mathbb{R}^n} \eta \langle \nabla\eta, x \rangle d\gamma_n \right] + \frac{f'(1)}{2} \int_{\mathbb{R}^n} D(\text{Hess } \eta) d\gamma_n \\
& + \frac{f''(1)}{2} \int_{\mathbb{R}^n} (\Delta\eta)^2 d\gamma_n + \left(f'(1) + f''(1) \right) \left[\int_{\mathbb{R}^n} \frac{\langle \nabla\eta, x \rangle^2}{2} d\gamma_n - \int_{\mathbb{R}^n} \Delta\eta \langle \nabla\eta, x \rangle d\gamma_n \right].
\end{aligned}$$

Integration by parts yields $\int_{\mathbb{R}^n} \eta \langle \nabla\eta, x \rangle d\gamma_n = \frac{1}{2} \int_{\mathbb{R}^n} \eta^2(x) (\|x\|^2 - n) d\gamma_n$ and

$$\int_{\mathbb{R}^n} \eta \Delta\eta d\gamma_n = - \int_{\mathbb{R}^n} \|\nabla\eta\|^2 d\gamma_n + \frac{1}{2} \int_{\mathbb{R}^n} \eta^2(x) (\|x\|^2 - n) d\gamma_n.$$

We put $a = f'(1)$ and $b = f''(1)$. Note that $a \leq 0$ and $b \geq 0$, as f is convex and decreasing. Thus, the inequality becomes

$$\begin{aligned}
& (a + 2b) \left(\text{Var}_{\gamma_n}(\eta) - \int_{\mathbb{R}^n} \|\nabla\eta\|^2 d\gamma_n \right) \\
& \geq \frac{a}{2} \int_{\mathbb{R}^n} \|\text{Hess } \eta\|_{HS}^2 d\gamma_n - \frac{a+b}{2} \int_{\mathbb{R}^n} (\Delta\eta - \langle \nabla\eta, x \rangle)^2 d\gamma_n.
\end{aligned}$$

Hence we have shown that the inequality holds for all twice continuously differentiable functions η with bounded support. Now we extend it to all twice continuously differentiable functions $\eta \in L^2(\mathbb{R}^n, \gamma_n)$ satisfying the necessary integrability conditions, by a standard approximation argument, as follows.

Let χ_k be a twice continuously differentiable function bounded between zero and one such that $\chi_k(x) = 1$ for all $\|x\| \leq k$ and $\chi_k(x) = 0$ for all $\|x\| > k + 1$. Then, for all

$k \in \mathbb{N}$

$$\begin{aligned} & (a + 2b) \left(\text{Var}_{\gamma_n}(\eta \cdot \chi_k) - \int_{\mathbb{R}^n} \|\nabla(\eta \cdot \chi_k)\|^2 d\gamma_n \right) \\ & \geq \frac{a}{2} \int_{\mathbb{R}^n} \|\text{Hess}(\eta \cdot \chi_k)\|_{HS}^2 d\gamma_n - \frac{a+b}{2} \int_{\mathbb{R}^n} \left(\Delta(\eta \cdot \chi_k) - \langle \nabla(\eta \cdot \chi_k), x \rangle \right)^2 d\gamma_n, \end{aligned}$$

which is equivalent to

$$\begin{aligned} & 2b \left[\left(\int_{\mathbb{R}^n} (\eta \cdot \chi_k) d\gamma_n \right)^2 + \int_{\mathbb{R}^n} \|\nabla(\eta \cdot \chi_k)\|^2 d\gamma_n \right] \\ & - a \left[\int_{\mathbb{R}^n} (\eta \cdot \chi_k)^2 d\gamma_n + \int_{\mathbb{R}^n} \frac{1}{2} \left(\Delta(\eta \cdot \chi_k) - \langle \nabla(\eta \cdot \chi_k), x \rangle \right)^2 d\gamma_n \right] \\ & \leq (-a) \left[\left(\int_{\mathbb{R}^n} (\eta \cdot \chi_k) d\gamma_n \right)^2 + \int_{\mathbb{R}^n} \|\nabla(\eta \cdot \chi_k)\|^2 d\gamma_n + \int_{\mathbb{R}^n} \frac{\|\text{Hess} \eta\|_{HS}^2}{2} d\gamma_n \right] \\ & + b \left[\int_{\mathbb{R}^n} 2(\eta \cdot \chi_k)^2 d\gamma_n + \int_{\mathbb{R}^n} \frac{1}{2} \left(\Delta(\eta \cdot \chi_k) - \langle \nabla(\eta \cdot \chi_k), x \rangle \right)^2 d\gamma_n \right]. \end{aligned}$$

Now we pass to the limit $k \rightarrow \infty$ on both sides and obtain

$$\begin{aligned} & 2b \left[\liminf_{k \rightarrow \infty} \left(\int_{\mathbb{R}^n} (\eta \cdot \chi_k) d\gamma_n \right)^2 + \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^n} \|\nabla(\eta \cdot \chi_k)\|^2 d\gamma_n \right] \\ & - a \left[\liminf_{k \rightarrow \infty} \int_{\mathbb{R}^n} (\eta \cdot \chi_k)^2 d\gamma_n + \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^n} \frac{1}{2} \left(\Delta(\eta \cdot \chi_k) - \langle \nabla(\eta \cdot \chi_k), x \rangle \right)^2 d\gamma_n \right] \\ & \leq (-a) \left[\limsup_{k \rightarrow \infty} \left(\int_{\mathbb{R}^n} (\eta \cdot \chi_k) d\gamma_n \right)^2 + \limsup_{k \rightarrow \infty} \int_{\mathbb{R}^n} \|\nabla(\eta \cdot \chi_k)\|^2 d\gamma_n \right] + \\ & b \left[\limsup_{k \rightarrow \infty} \int_{\mathbb{R}^n} 2(\eta \cdot \chi_k)^2 d\gamma_n + \limsup_{k \rightarrow \infty} \int_{\mathbb{R}^n} \frac{1}{2} \left(\Delta(\eta \cdot \chi_k) - \langle \nabla(\eta \cdot \chi_k), x \rangle \right)^2 d\gamma_n \right] \\ & - a \left[\limsup_{k \rightarrow \infty} \int_{\mathbb{R}^n} \frac{\|\text{Hess} \eta\|_{HS}^2}{2} d\gamma_n \right]. \end{aligned}$$

Fatou's lemma and the dominated convergence theorem yield

$$\begin{aligned} & \frac{a}{2} \int_{\mathbb{R}^n} \|\text{Hess} \eta\|_{HS}^2 d\gamma_n - \frac{a+b}{2} \int_{\mathbb{R}^n} (\Delta\eta - \langle \nabla\eta, x \rangle)^2 d\gamma_n \\ & \leq (a + 2b) \left(\text{Var}_{\gamma_n}(\eta) - \int_{\mathbb{R}^n} \|\nabla\eta\|^2 d\gamma_n \right). \end{aligned}$$

Finally, we consider the cases $a + 2b = 0$, $a + 2b > 0$ and $a + 2b < 0$ and optimize in each case. \square

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