Mixed f-divergence and inequalities for log concave functions *

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Abstract

Mixed f-divergences, a concept from information theory and statistics, measure the difference between multiple pairs of distributions. We introduce them for log concave functions and establish some of their properties. Among them are affine invariant vector entropy inequalities, like new Alexandrov-Fenchel type inequalities and an affine isoperimetric inequality for the vector form of the Kullback Leibler divergence for log concave functions.

Special cases of f-divergences are mixed L_λ-affine surface areas for log concave functions. For those, we establish various affine isoperimetric inequalities as well as a vector Blaschke Santaló type inequality.

1 Introduction

Affine invariant notions have had a transformative effect in convex geometry, e.g., [14, 25, 28, 45, 60]. One reason for this is that there are powerful inequalities associated to those notions. See, for instance, [14, 16, 23, 27, 28, 30, 55, 56]. Within the last few years, amazing connections have been discovered between some of these affine invariant notions and concepts from information theory, e.g., [13, 15, 18, 29, 31, 32, 43], leading to a totally new point of view and introducing a whole new set of tools in the area of convex geometry. In particular, it was observed in [53] that one of the most important affine invariant notions, the L_p-affine surface area for convex bodies [27, 50], is Rényi entropy from information theory and statistics. Rényi entropies are special cases of f-divergences. Consequently those were then introduced for convex bodies and their corresponding entropy inequalities have been established in [54]. An f-divergence (see below for the precise definition) is a function that measures the difference between (probability) densities. Aside from Rényi entropies, e.g., the relative entropy or Kullback-Leibler divergence [19] and the Bhattcharyya distance [5] are examples of f-divergences.

Much effort has been devoted lately to extend concepts and inequalities that hold for convex bodies to the corresponding ones for classes of functions. A natural analogue for a convex body is a log concave function. For those, functional analogues of important inequalities have been proved, such as the Blaschke Santaló inequality [2, 4, 11, 20] and the affine isoperimetric inequality [3]. In [6], f-divergences were introduced for log concave functions. This new concept yielded entropy inequalities which are stronger than

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the already existing ones, the reverse log-Sobolev and the reverse Poincare inequalities of [3].

Now we develop these ideas even further and introduce the mixed f-divergence for log concave functions. For convex bodies these were introduced and developed in [57]. Mixed f-divergence, which is important in applications, such as statistical hypothesis testing and classification, see e.g., [35, 40, 61], measures the difference between multiple pairs of (probability) distributions. Examples include, e.g., the Matusita's affinity [33, 34], the Toussaint's affinity [51], the information radius [48] and the average divergence [47]. Mixed f-divergence is an extension of the classical f-divergence and can be viewed as a vector form of classical f-divergence. For a vector $\vec{\varphi} = (\varphi_i)_{1 \leq i \leq n}$ consisting of log concave functions $\varphi_i : \mathbb{R}^d \to [0, \infty)$ and a vector $\vec{f} = (f_i)_{1 \leq i \leq n}$ consisting of concave or convex functions $f_i : (0, \infty) \to \mathbb{R}_+$, we define the mixed f-divergence for $\vec{\varphi} = (\varphi_1, \cdots, \varphi_n)$ by

$$D_{\vec{f}}(P_{\vec{\varphi}}, Q_{\vec{\varphi}}) \quad = \quad \int \prod_{i=1}^{n} \left[\varphi_i \ f_i \left(\frac{e^{\frac{\langle \nabla \varphi_i, x \rangle}{\varphi_i}}}{\varphi_i^2} \det \left[\mathrm{Hess} \left(-\log \varphi_i \right) \right] \right) \right]^{\frac{1}{n}} dx. \tag{1}$$

If all φ_i are the same and all f_i are the same, we recover the f-divergences of [6]. Like those, the new expressions are $SL(d)$ invariant. Here, $\nabla \varphi$ denotes the gradient and $\mathrm{Hess}(\varphi) = \left(\frac{\partial^2 \varphi}{\partial x_i \partial x_j} \right)_{1 \leq i,j \leq n}$ is the Hessian of φ.

One of the difficulties, to introduce this notion, was to find the right expression for the densities. A passage from functions to convex bodies and back, lets us achieve this goal and it can be seen that the expressions (1) appear naturally. This is demonstrated in [6].

The study of mixed f-divergences leads us to obtain new linear, respectively, affine invariant entropy inequalities, among them new Alexandrov-Fenchel type inequalities for log concave functions. Alexandrov-Fenchel inequality is a fundamental result in geometry. It is arguably one of the strongest inequality in this area as many important inequalities such as the Brunn-Minkowski inequality and Minkowski's first inequality follow from Alexandrov-Fenchel inequality (see, e.g., [13, 46]). Different generalization of Alexandrov-Fenchel inequalities for log concave functions can be found in e.g., [38]. Various vector entropy inequalities are consequences of this new Alexandrov-Fenchel inequality, for instance the following upper bound for the vector form of the f-divergence in terms of the classical f-divergences

$$\left[D_{\vec{f}}(P_{\vec{\varphi}}, Q_{\vec{\varphi}}) \right]^n \leq \prod_{k=1}^{n} D_{f_k}\left(P_{\varphi_k}, Q_{\varphi_k} \right),$$

and an affine isoperimetric inequality for the vector form of the relative entropy for normalized log concave functions,

$$D_{KL}\left(P_{\vec{\varphi}}, Q_{\vec{\varphi}} \right) \leq \log(2\pi)^n.$$

We refer to Theorem 4 and Corollary 7 for the detailed statements and the corresponding equality characterizations. While for the classical Alexandrov-Fenchel inequality for convex bodies the equality characterizations are not known in general, such equality characterizations can be established for these new Alexandrov-Fenchel inequalities for log concave functions. To do so, we use, among other things, the matrix version of the

Brunn-Minkowski inequality and recently established unique solutions of certain Monge Ampère differential equations [58].

Mixed L_λ-affine surface areas for a vector $\vec{\varphi}$ of log-concave functions, denoted by $as_\lambda(\vec{\varphi})$, are special cases of mixed f-divergences. This new definition corresponds, on the level of convex bodies, to the mixed L_p-affine surface areas (see, e.g., [26, 56, 59]), a generalization of L_p-affine surface areas. We refer to e.g., [16], [24], [25], [27], [36], [49], [50],[52]-[55] for more information on L_p-affine surface area for convex bodies. The L_p-affine surface areas for functions were introduced in [7].

We establish several affine isoperimetric inequalities for these quantities. Among them is a vector Blaschke Santaló type inequality for log concave functions with barycenter at 0,

$$as_\lambda(\vec{\varphi})as_\lambda(\vec{\varphi^\circ}) \ \leq \ (2\pi)^n.$$

Here, φ° is the dual function of φ, defined in (16) and $\lambda \in [0,1]$.

Please note that all the definitions and results hold, with obvious modifications, for s-concave functions as well. We refer to [6] for that.

Throughout the paper we will assume that the convex or concave functions $f : (0,\infty) \to \mathbb{R}$ and the log concave functions $\varphi : \mathbb{R}^d \to [0,\infty)$ have enough smoothness and integrability properties so that the expressions considered in the statements make sense, i.e., we will always assume that φ and $\varphi^\circ \in C^2 \cap L^1(\mathbb{R}^d, dx)$, where C^2 denotes the twice continuously differentiable functions, and that

$$\prod_{i=1}^{n} \left[\varphi_i f_i \left(\frac{e^{\frac{\langle \nabla \varphi_i, x\rangle}{\varphi_i}}}{\varphi_i^2} \det\left(\text{Hess}\left(-\ln \varphi_i\right)\right) \right) \right] \in L^1(\mathbb{R}^d, dx). \tag{2}$$

2 Mixed f-divergence

2.1 Background on mixed f-divergence

In information theory, probability theory and statistics, an f-divergence is a function that measures the difference between two (probability) distributions. This notion was introduced by Csiszár [9], and independently Morimoto [41] and Ali & Silvery [1].

Let (X,μ) be a finite measure space and let $P = p\mu$ and $Q = q\mu$ be (probability) measures on X that are absolutely continuous with respect to the measure μ. Let $f : (0,\infty) \to \mathbb{R}$ be a convex or a concave function. The $*$-adjoint function $f^* : (0,\infty) \to \mathbb{R}$ of f is defined by

$$f^*(t) = tf(1/t), \quad t \in (0,\infty). \tag{3}$$

It is obvious that $(f^*)^* = f$ and that f^* is again convex if f is convex, respectively concave if f is concave. Then the f-divergence $D_f(P,Q)$ of the measures P and Q is defined by

$$D_f(P,Q) = \int_X f\left(\frac{p}{q}\right) q d\mu. \tag{4}$$

It is a generalization of well known divergences, such as, the *variational distance*, the *Kullback-Leibler divergence* or *relative entropy*, the *Rényi divergence* and many more.

More on f-divergence can be found in e.g. [12, 21, 22, 42, 44, 54, 57].

For applications, such as statistical hypothesis test and classification, it is important to have extension of f-divergence from two (probability) measures to multiple (probability) measures, see e.g., [35, 40, 61].

For $1 \leq i \leq n$, let $P_i = p_i\mu$ and $Q_i = q_i\mu$ be probability measures on X that are absolutely continuous with respect to the measure μ. We also assume that the density functions p_i and q_i are nonzero almost everywhere with respect to μ. Denote by

$$\vec{\mathbf{P}} = (P_1, P_2, \cdots, P_n), \quad \vec{\mathbf{Q}} = (Q_1, Q_2, \cdots, Q_n).$$

We use $\vec{p}$ and $\vec{q}$ to denote the density vectors for $\vec{\mathbf{P}}$ and $\vec{\mathbf{Q}}$ respectively,

$$\frac{d\vec{\mathbf{P}}}{d\mu} = \vec{p} = (p_1, p_2, \cdots, p_n), \quad \frac{d\vec{\mathbf{Q}}}{d\mu} = \vec{q} = (q_1, q_2, \cdots, q_n).$$

For $1 \leq i \leq n$, let $f_i : (0, \infty) \to \mathbb{R}_+$ be either convex or concave functions. Denote by $\vec{f}$ the vector $\vec{f} = (f_1, f_2, \cdots, f_n)$ and the $*$-adjoint vector of $\vec{f}$ by $\vec{f^*} = (f_1^*, f_2^*, \cdots, f_n^*)$. The mixed f-divergence for $(\vec{f}, \vec{\mathbf{P}}, \vec{\mathbf{Q}})$ is defined in [57] as

$$D_{\vec{f}}(\vec{\mathbf{P}}, \vec{\mathbf{Q}}) = \int_X \prod_{i=1}^n \left[f_i \left(\frac{p_i}{q_i} \right) q_i \right]^{\frac{1}{n}} d\mu. \tag{5}$$

If $f_i = f$, $P_i = P$, and $Q_i = Q$, for all $1 \leq i \leq n$, then the mixed f-divergence becomes the classical f-divergence, defined in (4).

Similarly, the mixed f-divergence for $(\vec{f}, \vec{\mathbf{Q}}, \vec{\mathbf{P}})$ is

$$D_{\vec{f}}(\vec{\mathbf{Q}}, \vec{\mathbf{P}}) = \int_X \prod_{i=1}^n \left[f_i \left(\frac{q_i}{p_i} \right) p_i \right]^{\frac{1}{n}} d\mu. \tag{6}$$

It is obvious that $D_{\vec{f}}(\vec{\mathbf{P}}, \vec{\mathbf{Q}}) = D_{\vec{f^*}}(\vec{\mathbf{Q}}, \vec{\mathbf{P}})$. Therefore, it is enough to consider $D_{\vec{f}}(\vec{\mathbf{P}}, \vec{\mathbf{Q}})$, which we will do throughout the paper.

We now present some examples. For more examples and properties, see [57].

Examples.

1. For $1 \leq i \leq n$, let $f_i(t) = |t - 1|$. Then the mixed f-divergence becomes the *mixed total variation* of $\vec{P}$ and $\vec{Q}$, defined by Werner and Ye in [57],

$$D_{\vec{f}}(\vec{P}, \vec{Q}) = \int_X \prod_{i=1}^n |p_i - q_i|^{\frac{1}{n}} d\mu. \tag{7}$$

2. For $1 \leq i \leq n$, let $f_i(t) = \log t$. Then the mixed f-divergence is *mixed Kullback-Leibler divergence* or the *mixed relative entropy* of $\vec{P}$ and $\vec{Q}$ [57],

$$D_{KL}(\vec{P}, \vec{Q}) = D_{\vec{f}_+}(\vec{P}, \vec{Q}) = \int_X \prod_{i=1}^n \left[q_i \log \frac{p_i}{q_i} \right]_+^{\frac{1}{n}} d\mu \tag{8}$$

where for $a \in \mathbb{R}^n$, $a_+ = (\max\{a_1, 0\}, \max\{a_2, 0\}, \cdots, \max\{a_n, 0\})$. Recall that *Kullback-Leibler divergence* or *relative entropy* from P to Q is defined as (see, e.g., [8])

$$D_{KL}(P\|Q) = \int_X q \log \frac{p}{q} d\mu. \tag{9}$$

2.2 Mixed f-divergence for log concave functions

A function $\varphi : \mathbb{R}^d \to [0, \infty)$ is log concave, if it is of the form $\varphi(x) = e^{-\psi(x)}$, where $\psi : \mathbb{R}^d \to \mathbb{R}$ is a convex function. For $1 \le i \le n$, we put

$$q_{\varphi_i} = \varphi_i \quad \text{and} \quad p_{\varphi_i} = \varphi_i^{-1} e^{\frac{\langle \nabla \varphi_i, x \rangle}{\varphi_i}} \det \left[\text{Hess} \left(-\log \varphi_i \right) \right]. \tag{10}$$

We use the expressions (10) to define the mixed f-divergences for log concave functions. These quantities are the proper ones to use in order to define divergences for log concave functions. This was shown in [6].

Definition 1. *Let $f_i : (0, \infty) \to \mathbb{R}_+$ be convex and/or concave functions and let $\varphi_i : \mathbb{R}^d \to [0, \infty)$ be log concave functions. Then the mixed f-divergence for $\vec{\varphi} = (\varphi_1, \cdots, \varphi_n)$ is*

$$\begin{aligned} D_{\vec{f}}(P_{\vec{\varphi}}, Q_{\vec{\varphi}}) &= D_{\vec{f}}((P_{\varphi_1}, \cdots, P_{\varphi_n}), (Q_{\varphi_1}, \cdots, Q_{\varphi_n})) = \int \prod_{i=1}^n \left[f_i \left(\frac{p_{\varphi_i}}{q_{\varphi_i}} \right) q_{\varphi_i} \right]^{\frac{1}{n}} dx \\ &= \int \prod_{i=1}^n \left[\varphi_i \; f_i \left(\frac{e^{\frac{\langle \nabla \varphi_i, x \rangle}{\varphi_i}}}{\varphi_i^2} \; det \left[Hess \left(-\log \varphi_i \right) \right] \right) \right]^{\frac{1}{n}} dx. \end{aligned} \tag{11}$$

Remarks and Examples (i) Similarly to (11),

$$D_{\vec{f}}(Q_{\vec{\varphi}}, P_{\vec{\varphi}}) = D_{\vec{f}}((Q_{\varphi_1}, \cdots, Q_{\varphi_n}), (P_{\varphi_1}, \cdots, P_{\varphi_n})) = \int \prod_{i=1}^n \left[f_i \left(\frac{q_{\varphi_i}}{p_{\varphi_i}} \right) p_{\varphi_i} \right]^{\frac{1}{n}} dx$$

$$= \int \prod_{i=1}^n \left[\varphi_i^{-1} e^{\frac{\langle \nabla \varphi_i, x \rangle}{\varphi_i}} \det \left[-\text{Hess} \left(\log \varphi_i \right) \right] \; f_i \left(\frac{\varphi_i^2 \; e^{-\frac{\langle \nabla \varphi_i, x \rangle}{\varphi_i}}}{\det \left[\text{Hess} \left(-\log \varphi_i \right) \right]} \right) \right]^{\frac{1}{n}} dx.$$

(ii) If we let $f_i = f$ and $\varphi_i = \varphi$, $1 \le i \le n$, then we obtain the usual f-divergence for log concave functions, $D_f(P_\varphi, Q_\varphi)$, defined in [6],

$$D_f(P_\varphi, Q_\varphi) = \int \varphi \; f \left(\frac{e^{\frac{\langle \nabla \varphi, x \rangle}{\varphi}}}{\varphi^2} \det \left[\text{Hess} \left(-\log \varphi \right) \right] \right) dx. \tag{12}$$

(iii) If we write a log concave function as $\varphi = e^{-\psi}$, ψ convex, then (11) becomes

$$D_{\vec{f}}(P_{\vec{\varphi}}, Q_{\vec{\varphi}}) = \int \prod_{i=1}^n \left[e^{-\psi_i} \; f_i \left(e^{2\psi_i - \langle \nabla \psi_i, x \rangle} \det \left[\text{Hess} \psi_i \right] \right) \right]^{\frac{1}{n}} dx. \tag{13}$$

(iv) For $1 \leq i \leq n$, let A_i be a $(d \times d)$ positive definite matrix, $c_i > 0$ a constant and let $\varphi_i(x) = c_i e^{-\frac{1}{2}\langle A_i x, x\rangle}$. Then

$$D_{\vec{f}}(P_{\vec{\varphi}}, Q_{\vec{\varphi}}) = \frac{(2n\pi)^{\frac{n}{2}}}{\left(\det(\sum_{i=1}^n A_i)\right)^{\frac{1}{2}}} \prod_{i=1}^n \left[c_i \ f_i\left(\frac{\det(A_i)}{c_i^2}\right)\right]^{\frac{1}{n}}. \tag{14}$$

In particular, if $A_i = A$ for all i, where A is a $(d \times d)$ positive definite matrix, then (14) becomes

$$D_{\vec{f}}(P_{\vec{\varphi}}, Q_{\vec{\varphi}}) = \frac{(2\pi)^{n/2}}{\sqrt{\det(A)}} \prod_{i=1}^n \left[c_i \ f_i\left(\frac{\det(A)}{c_i^2}\right)\right]^{\frac{1}{n}}. \tag{15}$$

Proposition 2. *For $1 \leq i \leq n$, let $f_i : (0, \infty) \to \mathbb{R}_+$ be convex and/or concave functions and let $\varphi_i : \mathbb{R}^d \to [0, \infty)$ be log-concave functions. Then $D_{\vec{f}}(P_{\vec{\varphi}}, Q_{\vec{\varphi}})$ is invariant under self adjoint $SL(d)$ maps.*

Proof. Let $A : \mathbb{R}^d \to \mathbb{R}^d$ be a self adjoint, $SL(d)$ invariant linear map.

$$\begin{aligned}
&D_{\vec{f}}((P_{\varphi_1 \circ A}, \cdots, P_{\varphi_n \circ A}), (Q_{\varphi_1 \circ A}, \cdots, Q_{\varphi_n \circ A})) \\
&= \int \prod_{i=1}^n \left[\varphi_i(Ax) \ f_i\left(\frac{e^{\frac{\langle \nabla \varphi_i(Ax), x\rangle}{\varphi_i(Ax)}}}{(\varphi_i(Ax))^2} \ \det\left[\text{Hess}\left(-\log \varphi_i(Ax)\right)\right]\right)\right]^{\frac{1}{n}} dx \\
&= \frac{1}{|\det A|} \int \prod_{i=1}^n \left[\varphi_i \ f_i\left((\det A)^2 \frac{e^{\frac{\langle \nabla \varphi_i, x\rangle}{\varphi_i}}}{\varphi_i^2} \ \det\left[\text{Hess}\left(-\log \varphi_i\right)\right]\right)\right]^{\frac{1}{n}} dx \\
&= D_{\vec{f}}((P_{\varphi_1}, \cdots, P_{\varphi_n}), (Q_{\varphi_1}, \cdots, Q_{\varphi_n})).
\end{aligned}$$

□

Recall that for a function $\varphi : \mathbb{R}^d \to [0, \infty)$, the dual function φ° [2] is defined by

$$\varphi^\circ(y) = \inf_{x \in \mathbb{R}^d} \left[\frac{e^{-\langle x, y\rangle}}{\varphi(x)}\right].$$

If φ is a log concave function, i.e., $\varphi(x) = e^{-\psi(x)}$ with $\psi : \mathbb{R}^d \to \mathbb{R}$ convex, then this duality notion is connected with the Legendre transform $\psi^*(y) = \sup_{x \in \mathbb{R}^d} [\langle x, y\rangle - \psi(x)]$,

$$\varphi^\circ(y) = e^{-\psi^*(y)}. \tag{16}$$

For special forms of the log concave functions φ_i we have the following duality formula. This is the functional counterpart to the one proved in [59] for convex bodies and for special f.

Theorem 3. *For $1 \leq i \leq n$, let $f_i : (0, \infty) \to \mathbb{R}_+$ be convex and/or concave functions and let $\varphi_i = \lambda_i \varphi$, for some log concave function $\varphi : \mathbb{R}^d \to [0, \infty)$ and $\lambda_i > 0$. Then*

$$D_{\vec{f}}\left(P_{\vec{\varphi^\circ}}, Q_{\vec{\varphi^\circ}}\right) = D_{\vec{f^*}}(P_{\vec{\varphi}}, Q_{\vec{\varphi}}). \tag{17}$$

Proof. We write $\varphi = e^{-\psi}$, ψ convex, and let $\psi^*(y)$ be the Legendre transform of ψ. Please note that when ψ is a C^2 strictly convex function, then

$$\psi(x) + \psi^*(y) = \langle x, y\rangle \text{ if and only if } y = \nabla\psi(x) \text{ if and only if } x = \nabla\psi^*(y).$$

It follows that

$$\forall y \in \mathbb{R}^d,\ \psi(\nabla\psi^*(y)) = \langle y, \nabla\psi^*(y)\rangle - \psi^*(y) \tag{18}$$

and

$$\nabla\psi \circ \nabla\psi^* = \nabla\psi^* \circ \nabla\psi = \mathrm{I}d, \tag{19}$$

so that for any $x, y \in \mathbb{R}^d$,

$$\mathrm{Hess}\,\psi(\nabla\psi^*(y))\ \mathrm{Hess}\,\psi^*(y) = \mathrm{I}d = \mathrm{Hess}\,\psi^*(\nabla\psi(x))\ \mathrm{Hess}\,\psi(x). \tag{20}$$

Using equations (18), (19) and (20), the change of variable $x = \nabla\psi^*(y)$ gives

$$\begin{aligned}
&D_{\vec{f}^*}\left(P_{\vec{\varphi}}, Q_{\vec{\varphi}}\right)\\
&= \int \prod_{i=1}^n \left[\varphi_i\ f_i^*\left(\frac{e^{\frac{\langle\nabla\varphi_i, x\rangle}{\varphi_i}}}{\varphi_i^2}\ \det\left[\mathrm{Hess}\left(-\log\varphi_i\right)\right]\right)\right]^{\frac{1}{n}} dx\\
&= \int \prod_{i=1}^n \left[\frac{e^{\frac{\langle\nabla\varphi, x\rangle}{\varphi}}\ \det\left[\mathrm{Hess}\left(-\log\varphi\right)\right]}{\lambda_i\varphi} f_i\left(\frac{\lambda_i^2\varphi^2 e^{-\frac{\langle\nabla\varphi, x\rangle}{\varphi}}}{\det\left[\mathrm{Hess}\left(-\log\varphi\right)\right]}\right)\right]^{\frac{1}{n}} dx\\
&= \frac{1}{(\lambda_1\cdots\lambda_n)^{\frac{1}{n}}} \int \prod_{i=1}^n \left[\det\left(\mathrm{Hess}\psi(x)\right)\ e^{\psi(x)-\langle\nabla\psi, x\rangle} f_i\left(\frac{\lambda_i^2\ e^{-2\psi(x)+\langle\nabla\psi, x\rangle}}{\det\left(\mathrm{Hess}\psi(x)\right)}\right)\right]^{\frac{1}{n}} dx\\
&= \frac{1}{(\lambda_1\cdots\lambda_n)^{\frac{1}{n}}} \int \prod_{i=1}^n \left[\det\left(\mathrm{Hess}\psi(\nabla\psi^*(y))\right)\ e^{(\psi(\nabla\psi^*(y))-\langle y, \nabla(\psi^*(y))\rangle)}\right]^{\frac{1}{n}}\\
&\qquad \times \prod_{i=1}^n \left[f_i\left(\frac{\lambda_i^2\ e^{-2\psi(\nabla\psi^*(y))+\langle y, \nabla\psi^*(y)\rangle}}{\det\left(\mathrm{Hess}\psi(\nabla\psi^*(y))\right)}\right)\right]^{\frac{1}{n}} \det\left(\mathrm{Hess}\psi^*(y)\right) dy\\
&= \frac{1}{(\lambda_1\cdots\lambda_n)^{\frac{1}{n}}} \int \prod_{i=1}^n \left[e^{-\psi^*(y)}\ f_i\left(\lambda_i^2\ \det(\mathrm{Hess}\psi^*(y))\ e^{-\langle y, \nabla\psi^*(y)\rangle+2\psi^*(y)}\right)\right]^{\frac{1}{n}} dy\\
&= \frac{1}{(\lambda_1\cdots\lambda_n)^{\frac{1}{n}}} \int \prod_{i=1}^n \left[\varphi^\circ\ f_i\left(\lambda_i^2\ \det\left[\mathrm{Hess}\left(-\log\varphi^\circ\right)\right]\ \frac{e^{\frac{\langle\nabla\varphi^\circ, x\rangle}{\varphi^\circ}}}{(\varphi^\circ)^2}\right)\right]^{\frac{1}{n}} dx\\
&= D_{\vec{f}}\left(P_{\vec{\varphi^\circ}}, Q_{\vec{\varphi^\circ}}\right).
\end{aligned}$$

The last part follows from the fact that $(\lambda\varphi)^\circ = \frac{\varphi^\circ}{\lambda}$, for $\lambda \in \mathbb{R}$, $\lambda \neq 0$. ☐

Remark. If $f_i = f$ and $\lambda_i = 1$, i.e. $\varphi_i = \varphi$ for all $i = 1, \cdots, n$, then $D_f(P_{\varphi^\circ}, Q_{\varphi^\circ}) = D_{f^*}(P_\varphi, Q_\varphi)$. This was proved in [6].

The classical Alexandrov-Fenchel inequality for mixed volumes of convex bodies is one of the most important results in convex geometry. We refer to e.g., [46] for the details and prove now an Alexandrov-Fenchel type inequality for mixed f-divergences for log concave functions. The proof is similar to one given in [57]. We include it for completeness. We use the following notations.

For $1 \le m \le n-1$ and $k > n-m$, we put

$$\vec{f}_{m,k} = (f_1, f_2, \cdots, f_{n-m}, \underbrace{f_k, \cdots, f_k}_{m}),$$

$$P_{\vec{\varphi}_{m,k}} = (P_{\varphi_1}, \cdots, P_{\varphi_{n-m}}, \underbrace{P_{\varphi_k}, \cdots, P_{\varphi_k}}_{m}), \quad Q_{\vec{\varphi}_{m,k}} = (Q_{\varphi_1}, \cdots, Q_{\varphi_{n-m}}, \underbrace{Q_{\varphi_k}, \cdots, Q_{\varphi_k}}_{m}).$$

Following [17], we say that two functions f and g are *effectively proportional* if there are constants a and b, not both zero, such that $af = bg$. Functions $f_1, \ldots, f_m$ are effectively proportional if every pair $(f_i, f_j), 1 \le i, j \le m$ is effectively proportional. A null function is effectively proportional to any function.

Moreover, for $1 \le m \le n-1$, we let

$$h_0(x) = \prod_{i=1}^{n-m} \left[\varphi_i \ f_i \left(\frac{e^{\frac{\langle \nabla \varphi_i, x\rangle}{\varphi_i}}}{\varphi_i^2} \ \det\left[\text{Hess}\left(-\log \varphi_i\right)\right] \right) \right]^{\frac{1}{n}} \tag{21}$$

and for $j = 0, \cdots, m-1$,

$$h_{j+1}(x) = \left[\varphi_{n-j} \ f_{n-j} \left(\frac{e^{\frac{\langle \nabla \varphi_{n-j}, x\rangle}{\varphi_{n-j}}}}{\varphi_{n-j}^2} \ \det\left[\text{Hess}\left(-\log \varphi_{n-j}\right)\right] \right) \right]^{\frac{1}{n}}. \tag{22}$$

Then an Alexandrov-Fenchel type inequality holds for log concave functions, namely,

Theorem 4. *For $1 \le i \le n$, let $f_i : (0,\infty) \to \mathbb{R}_+$ be either all convex or all concave functions and let $\varphi_i : \mathbb{R}^d \to [0,\infty)$ be log-concave functions. Then, for $1 \le m \le n-1$,*

$$\left[D_{\vec{f}}\left(P_{\vec{\varphi}}, Q_{\vec{\varphi}}\right) \right]^m \le \prod_{k=n-m+1}^{n} D_{\vec{f}_{m,k}}\left(P_{\vec{\varphi}_{m,k}}, Q_{\vec{\varphi}_{m,k}}\right).$$

Equality holds if and only if one of the functions $h_0^{\frac{1}{m}} h_j$, $1 \le j \le m$, is null or all are effectively proportional.

If $m = n$, then

$$\left[D_{\vec{f}}\left(P_{\vec{\varphi}}, Q_{\vec{\varphi}}\right) \right]^n \le \prod_{k=1}^{n} D_{f_k}\left(P_{\varphi_k}, Q_{\varphi_k}\right).$$

Equality holds if and only if one of the functions h_j, $1 \le j \le n$, is null or all are effectively proportional.

Remark. In particular, equality holds in Theorem 4, if (i) all φ_i coincide and $f_i = \lambda_i f$ for some positive convex function f and $\lambda_i > 0$, $i = n-m+1, \cdots, n$, or (ii) $f_i = \lambda_i f$, for some positive convex function f, for some $\lambda_i > 0$, $\varphi_i = a_i \varphi$, for some positive, log concave function φ, for some a_i, $i = n-m+1, \cdots, n$ and f is homogeneous of degree $\alpha \in [0, 1)$.

Proof. We first treat the case $m = n$. By Hölder's inequality, e.g., [17],

$$\begin{aligned} D_{\vec{f}}(P_{\vec{\varphi}}, Q_{\vec{\varphi}}) &= \int \prod_{i=1}^{n} \left[\varphi_i \; f_i \left(\frac{e^{\frac{\langle \nabla \varphi_i, x\rangle}{\varphi_i}}}{\varphi_i^2} \det \left[\mathrm{Hess}\left(-\log \varphi_i\right)\right] \right)\right]^{\frac{1}{n}} dx \\ &\leq \prod_{i=1}^{n} \left[\int \varphi_i \; f_i \left(\frac{e^{\frac{\langle \nabla \varphi_i, x\rangle}{\varphi_i}}}{\varphi_i^2} \det \left(\mathrm{Hess}\left(-\log \varphi_i\right)\right) \right)\right]^{\frac{1}{n}} dx \\ &= \prod_{i=1}^{n} \left(D_{f_i}\left(P_{\varphi_i}, Q_{\varphi_i}\right)\right)^{\frac{1}{n}}. \end{aligned}$$

Let now $m \leq n-1$. Again, by Hölder's inequality,

$$\begin{aligned} \left[D_{\vec{f}}(P_{\vec{\varphi}}, Q_{\vec{\varphi}})\right]^m &= \left(\int \prod_{j=0}^{m-1} \left(h_0(x) h_{j+1}^m(x)\right)^{\frac{1}{m}} dx \right)^m \\ &\leq \prod_{j=0}^{m-1} \left(\int h_0(x) h_{j+1}^m(x) \right) dx \; = \prod_{k=n-m+1}^{n} D_{\vec{f}_{m,k}}\left(P_{\vec{\varphi}_{m,k}}, Q_{\vec{\varphi}_{m,k}}\right). \end{aligned}$$

In both cases, characterization of equality follows from the equality characterization in Hölder's inequality e.g., [17]. □

The following entropy inequality is a consequence of Theorem 4.

Theorem 5. *For $1 \leq i \leq n$, let $f_i : (0, \infty) \to \mathbb{R}_+$ be concave functions and let $\varphi_i : \mathbb{R}^d \to [0, \infty)$ be log-concave functions. Then*

$$\left[D_{\vec{f}}(P_{\vec{\varphi}}, Q_{\vec{\varphi}})\right]^n \leq \prod_{i=1}^{n} f_i \left(\frac{\int \varphi_i^\circ dx}{\int \varphi_i dx}\right) \left(\int \varphi_i dx \right). \tag{23}$$

Equality holds if and only if $\varphi_i(x) = c_i e^{-\frac{1}{2}\langle Ax, x\rangle}$ where c_i is a positive constant and A is a $(d \times d)$ positive definite matrix.

Proof. The inequality follows immediately from Theorem 4 for $m = n$ and Theorem 1 in [6], which says that for a concave function $f : (0, \infty) \to \mathbb{R}$ and a log-concave function $\varphi : \mathbb{R}^d \to [0, \infty)$, we have

$$D_f(P_\varphi, Q_\varphi) \leq f \left(\frac{\int \varphi^\circ dx}{\int \varphi dx}\right) \left(\int \varphi dx \right). \tag{24}$$

It was proved in [58], that equality holds in (24) if and only if $\varphi(x) = c e^{-\frac{1}{2}\langle Ax, x\rangle}$ where $c > 0$ is a constant and A is a $(d \times d)$ positive definite matrix.

We now treat the equality characterization. Using (15), it is easy to check that equality holds if $\varphi_i(x) = c_i e^{-\frac{1}{2}\langle Ax, x\rangle}$, $1 \leq i \leq n$. On the other hand, if equality holds in

(23), then in particular,

$$\prod_{i=1}^{n} D_{f_i}\left(P_{\varphi_i}, Q_{\varphi_i}\right) = \prod_{i=1}^{n} \; f_i\left(\frac{\int \varphi_i^\circ dx}{\int \varphi_i dx}\right)\; \left(\int \varphi_i dx\right).$$

Thus, equality holds in particular for all i in the entropy inequality (24), which, by the equality characterization of [58], means that for all i, $\varphi_i(x) = c_i e^{-\frac{1}{2}\langle A_i x, x\rangle}$, where c_i is a positive constant and A_i is a $(d \times d)$ positive definite matrix. Thus, also using (14), the equality condition leads to the following identity

$$\prod_{i=1}^{n} (\det(A_i))^{\frac{1}{n}} = \frac{\det\left(\sum_{i=1}^{n} A_i\right)}{n^n}. \tag{25}$$

The Brunn Minkowski inequality for matrices [8, 10, 39] says that for positive definite matrices A_i, $1 \le i \le n$, one has

$$\left(\det\left(\sum_{i=1}^{n} A_i\right)\right)^{\frac{1}{n}} \ge \sum_{i=1}^{n} (\det(A_i)^{\frac{1}{n}}, \tag{26}$$

with equality if and only if all $A_i = \lambda_i A$ for some positive definite matrix A and scalars $\lambda_i \ge 0$. It follows from the geometric arithmetic mean inequality that

$$\left(\sum_{i=1}^{n} (\det(A_i))^{\frac{1}{n}}\right)^{n} \ge n^n \prod_{i=1}^{n} (\det(A_i))^{\frac{1}{n}},$$

with equality if and only if $\det(A_i) = \det(A_j)$, for all i, j. With (26), we get altogether,

$$\prod_{i=1}^{n} (\det(A_i))^{\frac{1}{n}} \le \frac{1}{n^n} \;\left(\sum_{i=1}^{n} (\det(A_i))^{\frac{1}{n}}\right)^{n} \le \frac{1}{n^n} \;\left(\det\left(\sum_{i=1}^{n} A_i\right)\right). \tag{27}$$

By assumption, equality (25) holds. Therefore, we have equality in both, the geometric arithmetic mean inequality and the Brunn Minkowski inequality which means that for all i, $A_i = \lambda A$, for some $\lambda > 0$, for some positive definite matrix A. Hence we have that $\varphi_i(x) = c_i e^{-\langle Ax, x\rangle/2}$. □

If we let $f_i(t) = \log t$, $1 \le i \le n$, in Theorem 5, then we obtain the following corollaries. We use again the notation $a_+ = (\max\{a_1, 0\}, \max\{a_2, 0\}, \cdots, \max\{a_n, 0\})$, for $a \in \mathbb{R}^n$.

Corollary 6. *For $1 \le i \le n$, let $\varphi_i : \mathbb{R}^d \to [0, \infty)$ be log-concave functions. Then*

$$\left[D_{KL}\left(P_{\vec{\varphi}}, Q_{\vec{\varphi}}\right)\right]^n \le \prod_{i=1}^{n} \;\left[\log\left(\frac{\int \varphi_i^\circ dx}{\int \varphi_i dx}\right)\right]_+ \;\left(\int \varphi_i dx\right). \tag{28}$$

Equality holds if and only if $\varphi_i(x) = c_i e^{-\frac{1}{2}\langle Ax, x\rangle}$, where c_i is a positive constant and A is a $(d \times d)$ positive definite matrix.

Corollary 7. *For $1 \le i \le n$, let $\varphi_i : \mathbb{R}^d \to [0,\infty)$ be log-concave functions such that $\int x\varphi_i dx = 0$ for all i. Then*

$$[D_{KL}(P_{\vec{\varphi}}, Q_{\vec{\varphi}})]^n \le \prod_{i=1}^n \left[\log\left(\frac{(2\pi)^n}{\left(\int \varphi_i dx\right)^2}\right)\right]_+ \left(\int \varphi_i dx\right). \tag{29}$$

Equality holds if and only if $\varphi_i(x) = c_i e^{-\frac{1}{2}\langle Ax,x\rangle}$, where c_i is a positive constant and A is a $(d \times d)$ positive definite matrix.

Proof. The functional Blaschke Santaló inequality [2, 4, 11, 20] says that for a log concave function φ with barycenter at 0, i.e., $\int x\varphi dx = 0$, one has

$$\left(\int \varphi\, dx\right)\left(\int \varphi^\circ\, dx\right) \le (2\pi)^n,$$

with equality if and only if there exists a positive definite matrix A and $c > 0$ such that $\varphi(x) = ce^{-\langle Ax,x\rangle/2}$. We apply the functional Blaschke Santaló inequality on the right hand side of (28) to each φ_i and get inequality (29).

Using (15), it is easy to see that equality holds in (29) if $\varphi_i(x) = c_i e^{-\frac{1}{2}\langle Ax,x\rangle}$, where c_i is a positive constant and A is a $(d \times d)$ positive definite matrix. On the other hand, if equality holds in (29), then equality holds in particular for all i in the functional Blaschke Santaló inequality which means that for all i, $\varphi_i(x) = c_i e^{-\frac{1}{2}\langle A_i x,x\rangle}$, where c_i is a positive constant and A_i is a $(d \times d)$ positive definite matrix. Thus, as above in the proof of Theorem 5, the equality condition again leads to the identity

$$\prod_{i=1}^n (\det(A_i))^{\frac{1}{n}} = \frac{\det\left(\sum_{i=1}^n A_i\right)}{n^n}$$

and we conclude as above. □

3 The i-th mixed f-divergence for log-concave functions

Throughout this section, let $f_1, f_2 : (0,\infty) \to \mathbb{R}_+$ be either convex or concave functions. As above, let (X,μ) be a finite measure space and, for $l = 1, 2$, let $P_l = p_l\mu$ and $Q_l = q_l\mu$ be measures on X that are absolutely continuous with respect to the measure μ. Denote $\vec{f} = (f_1, f_2)$, $\vec{P} = (P_1, P_2)$ and $\vec{Q} = (Q_1, Q_2)$.

The i-th mixed f-divergence was introduced in [57]. We refer to [57] for properties and examples and only give the definition.

Definition 8. *Let $i \in \mathbb{R}$. The i-th mixed f-divergence for $(\vec{f}, \vec{\boldsymbol{P}}, \vec{\boldsymbol{Q}})$ is defined in [57] as*

$$D_{\vec{f}}(\vec{\boldsymbol{P}}, \vec{\boldsymbol{Q}}; i) = \int_X \left[f_1\left(\frac{p_1}{q_1}\right) q_1\right]^{\frac{i}{n}} \left[f_2\left(\frac{p_2}{q_2}\right) q_2\right]^{\frac{n-i}{n}} d\mu. \tag{30}$$

As before, for $l=1,2$, we let

$$q_{\varphi_l}=\varphi_l \quad \text{and} \quad p_{\varphi_l}=\varphi_l^{-1}e^{\frac{\langle\nabla\varphi_l,x\rangle}{\varphi_l}}\det\left[\text{Hess}\left(-\log\varphi_l\right)\right] \tag{31}$$

and use Definition 8 with $q_l=q_{\varphi_l}$ and $p_l=p_{\varphi_l}$, $l=1,2$, and get the *i-th mixed f-divergences* for log concave functions.

Definition 9. *Let $f_1,f_2:(0,\infty)\to\mathbb{R}_+$ be either convex or concave functions and let $\varphi_1,\varphi_2:\mathbb{R}^d\to[0,\infty)$ be log concave functions. Let $i\in\mathbb{R}$. Then the i-th mixed f-divergence of $\vec{\varphi}=(\varphi_1,\varphi_2)$ is*

$$D_{\vec{f}}((P_{\varphi_1},P_{\varphi_2}),(Q_{\varphi_1},Q_{\varphi_2});i)=\int\left[f_1\left(\frac{p_1}{q_1}\right)q_1\right]^{\frac{i}{n}}\left[f_2\left(\frac{p_2}{q_2}\right)q_2\right]^{\frac{n-i}{n}}dx \;=$$

$$\int\left[\varphi_1f_1\left(\frac{e^{\frac{\langle\nabla\varphi_1,x\rangle}{\varphi_1}}}{\varphi_1^2}det\left[Hess\left(-\log\varphi_1\right)\right]\right)\right]^{\frac{i}{n}}\left[\varphi_2f_2\left(\frac{e^{\frac{\langle\nabla\varphi_2,x\rangle}{\varphi_2}}}{\varphi_2^2}det\left[Hess\left(-\log\varphi_2\right)\right]\right)\right]^{\frac{n-i}{n}}dx.$$

If we let $q_l=q_{\varphi_l}$ and $p_l=p_{\varphi_l}$, $l=1,2$, then the following proposition is an immediate consequence of Proposition V.I of [57]. We also denote

$$P_{\vec{\varphi}}=(P_{\varphi_1},P_{\varphi_2}),\qquad Q_{\vec{\varphi}}=(Q_{\varphi_1},Q_{\varphi_2}).$$

Proposition 10. *Let $f_1,f_2:(0,\infty)\to\mathbb{R}_+$ be either convex or concave functions and let $\varphi_1,\varphi_2:\mathbb{R}^d\to[0,\infty)$ be log concave functions. If $j\le i\le k$ or $k\le i\le j$, then*

$$D_{\vec{f}}(P_{\vec{\varphi}}\;,Q_{\vec{\varphi}}\;;\;i)\;\le\;\left[D_{\vec{f}}(P_{\vec{\varphi}}\;,Q_{\vec{\varphi}}\;;\;j)\right]^{\frac{k-i}{k-j}}\;\times\;\left[D_{\vec{f}}(P_{\vec{\varphi}}\;,Q_{\vec{\varphi}}\;;\;k)\right]^{\frac{i-j}{k-j}}.$$

Equality holds trivially if $i=k$ or $i=j$. Otherwise, equality holds if and only if one of the functions $f_l\left(\frac{p_{\varphi_l}}{q_{\varphi_l}}\right)\,q_{\varphi_l}$, $l=1,2$ is null or are effectively proportional.

The next corollary follows immediately from Proposition 10 and (24).

Corollary 11. *Let $\varphi_1,\varphi_2:\mathbb{R}^d\to[0,\infty)$ be log concave functions and let $f_1,f_2:(0,\infty)\to\mathbb{R}_+$. If f_1,f_2 are concave and $0\le i\le n$, then*

$$\left[D_{\vec{f}}(P_{\vec{\varphi}}\;,Q_{\vec{\varphi}}\;;\;i)\right]^n$$
$$\le\left[f_1\left(\frac{\int\varphi_1^\circ dx}{\int\varphi_1 dx}\right)\;\left(\int\varphi_1dx\right)\right]^i\times\left[f_2\left(\frac{\int\varphi_2^\circ dx}{\int\varphi_2 dx}\right)\;\left(\int\varphi_2dx\right)\right]^{n-i}.$$

If (i) f_1 is convex, f_2 is concave and $i\ge n$, or (ii) f_1 is concave, f_2 is convex and $i\le0$, then the inequality is reversed.

Equality holds trivially if $i=0$ or $i=n$. Otherwise, equality holds if and only if $\varphi_l=c_le^{-\frac{1}{2}\langle Ax,x\rangle}$, $l=1,2$, where c_l is a positive constant and A is a $(d\times d)$ positive definite matrix.

Proof. We give the proof in the first case. The others are done similarly. Let $k = 0$ and $j = n$ (or $j = 0$ and $k = n$) in Proposition 10. By (24),

$$\begin{aligned}\left[D_{\vec{f}}\left(P_{\vec{\varphi}}\,, Q_{\vec{\varphi}}\,;\ i\right)\right]^n &\leq \left[D_{f_1}\left(P_{\varphi_1}\,, Q_{\varphi_1}\right)\right]^i \times \left[D_{f_2}\left(P_{\varphi_2}\,, Q_{\varphi_2}\right)\right]^{n-i} \\ &\leq \left[f_1\left(\frac{\int \varphi_1^\circ dx}{\int \varphi_1 dx}\right)\left(\int \varphi_1 dx\right)\right]^i \left[f_2\left(\frac{\int \varphi_2^\circ dx}{\int \varphi_2 dx}\right)\left(\int \varphi_2 dx\right)\right]^{n-i}.\end{aligned}$$

It is easy to see that equality holds if $\varphi_l = c_l e^{-\frac{1}{2}\langle Ax,x\rangle}$, $l = 1, 2$, where c_l is a positive constant and A is a $(d \times d)$ positive definite matrix. On the other hand, if equality holds in the inequality, then in particular, equality holds in (24), which means that $\varphi_l = c_l e^{-\frac{1}{2}\langle A_l x,x\rangle}$, $l = 1, 2$, where c_l are positive constants and A_l are $(d \times d)$ positive definite matrices. Thus, equality in the inequality leads to the following identity

$$\det\left(\frac{i}{n}A_1 + \left(1 - \frac{i}{n}\right)A_2\right) = \left(\det A_1\right)^{\frac{i}{n}} \left(\det A_2\right)^{1-\frac{i}{n}}.$$

We conclude again, by the Brunn Minkowski inequality for matrices [8, 10, 39], that $A_1 = A_2$. □

Remark. In particular, if we let $f_1(t) = f_2(t) = \log(t)$ in Corollary 11, then we obtain similar results for the *i-th mixed Kullback-Leibler divergence*, as in Corollary 6.

4 Applications to special functions: Mixed L_λ-affine surface area

Now we consider special functions f and obtain special cases of mixed f-divergences for log concave functions.

For $i = 1, \dots, n$, we let $f_i(t) = t^\lambda$, $-\infty < \lambda < \infty$, and we obtain the *mixed L_λ-affine surface area*, denoted by $as_\lambda(\vec{\varphi})$, for log concave functions φ_i,

$$as_\lambda(\vec{\varphi}) = \int \prod_{i=1}^n \left[\varphi_i \left(\frac{e^{\frac{\langle \nabla\varphi_i, x\rangle}{\varphi_i}}}{\varphi_i^2} \det\left[\mathrm{Hess}\left(-\log\varphi_i\right)\right]\right)^\lambda\right]^{\frac{1}{n}} dx, \tag{32}$$

or, writing $\varphi_i(x) = e^{-\psi_i(x)}$, ψ_i convex,

$$as_\lambda(\vec{\varphi}) = \int \prod_{i=1}^n \left[e^{(2\lambda-1)\psi_i(x) - \lambda\langle x, \nabla\psi_i(x)\rangle} \left(\det \mathrm{Hess}\ \psi_i(x)\right)^\lambda\right]^{\frac{1}{n}} dx. \tag{33}$$

In particular, $as_0(\vec{\varphi}) = \int (\varphi_1 \cdots \varphi_n)^{\frac{1}{n}} dx$. Please note that for any $\vec{\varphi}$, we have $as_\lambda(\vec{\varphi}) \geq 0$. Moreover, by Proposition 2, the $as_\lambda(\vec{\varphi})$ are invariant under self adjoint $SL(d)$ maps.

Remarks. (i) If we let $\varphi_i = \varphi$ for $i = 1, \cdots, n$, we recover the L_λ-*affine surface area*, $as_\lambda(\varphi)$, defined in [7] (see also [6]),

$$as_\lambda(\varphi) = \int \varphi \left(\frac{e^{\frac{\langle \nabla \varphi, x\rangle}{\varphi}}}{\varphi^2} \det\left[\text{Hess}\left(-\log \varphi\right)\right] \right)^\lambda dx. \tag{34}$$

(ii) For $1 \le i \le n$, let A_i be a $(d \times d)$ positive definite matrix, $c_i > 0$ a constant and let $\varphi_i(x) = c_i e^{-\frac{1}{2}\langle A_i x, x\rangle}$. Then,

$$as_\lambda(\vec{\varphi}) = \frac{(2n\pi)^{\frac{n}{2}}}{\left(\det(\sum_{i=1}^n A_i)\right)^{\frac{1}{2}}} \prod_{i=1}^n \left[c_i^{1-2\lambda} \ \left(\det(A_i)\right)^\lambda\right]^{\frac{1}{n}}. \tag{35}$$

We also give a definition for $as_\infty(\vec{\varphi})$ and $as_{-\infty}(\vec{\varphi})$, similarly as it was done for the L_λ-affine surface area [6] (see also [37]).

$$as_\infty(\vec{\varphi}) = \max_x \prod_{i=1}^n \left[\frac{e^{\frac{\langle \nabla \varphi_i, x\rangle}{\varphi_i}}}{\varphi_i^2} \det\left[\text{Hess}\left(-\log \varphi_i\right)\right] \right]^{\frac{1}{n}} \quad \text{and} \quad as_{-\infty}(\vec{\varphi}) = \frac{1}{as_\infty(\vec{\varphi})}. \tag{36}$$

The following two propositions are direct consequences of Theorem 3 and Theorem 4.

Proposition 12. *Let $\varphi_i : \mathbb{R}^d \to [0, \infty)$ be log-concave functions such that $\varphi_i = a_i \varphi$ for some log concave function $\varphi : \mathbb{R}^d \to [0, \infty)$ and $a_i > 0$, $i = 1, \cdots, n$. Then*

$$as_\lambda\left(\vec{\varphi}\right) = as_{1-\lambda}\left(\vec{\varphi^\circ}\right). \tag{37}$$

Proposition 12 is generalization of the duality $as_\lambda(\varphi) = as_{1-\lambda}(\varphi^\circ)$, proved in [7].

In the next proposition we use, for $k > n - m$, the notation

$$\begin{aligned} as_\lambda(\vec{\varphi}_{m,k}) \quad = \quad & \int \prod_{i=1}^{n-m} \left[\varphi_i \left(\frac{e^{\frac{\langle \nabla \varphi_i, x\rangle}{\varphi_i}}}{\varphi_i^2} \det\left[\text{Hess}\left(-\log \varphi_i\right)\right] \right)^\lambda \right]^{\frac{1}{n}} \\ & \left[\varphi_k \left(\frac{e^{\frac{\langle \nabla \varphi_k, x\rangle}{\varphi_k}}}{\varphi_k^2} \det\left[\text{Hess}\left(-\log \varphi_k\right)\right] \right)^\lambda \right]^{\frac{m}{n}} dx. \end{aligned}$$

Proposition 13. *For $1 \le i \le n$, let $\varphi_i : \mathbb{R}^d \to [0, \infty)$ be log-concave functions and let $-\infty < \lambda < \infty$. Then, if $1 \le m \le n - 1$,*

$$\left[as_\lambda\left(\vec{\varphi}\right)\right]^m \le \prod_{k=n-m+1}^n as_\lambda(\vec{\varphi}_{m,k}).$$

If $m = n$,

$$\left[as_\lambda(\vec{\varphi})\right]^n \le \prod_{k=1}^n as_\lambda(\varphi_k).$$

The equality characterization is the same as in Theorem 4.

Next, we prove affine isoperimetric inequalities for the mixed L_λ-affine surface area.

Proposition 14. *For $1 \le i \le n$, let $\varphi_i : \mathbb{R}^d \to [0,\infty)$ be log-concave functions such that φ_i has barycenter at 0. If $\lambda \in [0,1]$, then*

$$\left[\frac{as_\lambda(\vec{\varphi})}{as_\lambda(g,\cdots,g)}\right]^n \le \prod_{i=1}^n \left(\frac{\int \varphi_i}{\int g}\right)^{1-2\lambda}. \tag{38}$$

where $g(x) = e^{-\frac{\|x\|^2}{2}}$. Equality holds if and only if $\varphi_i = c_i e^{-\frac{1}{2}\langle Ax,x\rangle}$ where $c_i > 0$, $1 \le i \le n$, and A is a $(d \times d)$ positive definite matrix.

Proof. By Proposition 13,

$$\left[\frac{as_\lambda(\vec{\varphi})}{as_\lambda(g,\cdots,g)}\right]^n \le \prod_{i=1}^n \frac{as_\lambda(\varphi_i)}{as_\lambda(g)} \le \prod_{i=1}^n \left(\frac{\int \varphi_i}{\int g}\right)^{1-2\lambda}.$$

The last part follows from a corollary in [7], which says that for a log-concave function $\varphi : \mathbb{R}^d \to [0,\infty)$ with barycenter at 0,

$$\frac{as_\lambda(\varphi)}{as_\lambda(g)} \le \left(\frac{\int \varphi}{\int g}\right)^{1-2\lambda}. \tag{39}$$

It was proved in [7] that equality holds if and only if $\varphi(x) = ce^{-\frac{1}{2}\langle Ax,x\rangle}$ where $c > 0$ is a constant and A is a $(d \times d)$ positive definite matrix.

Using (35), it is easy to see that equality holds in (38) if $\varphi_i(x) = c_i e^{-\frac{1}{2}\langle Ax,x\rangle}$, where c_i is a positive constant and A is a $(d \times d)$ positive definite matrix. On the other hand, if equality holds in (38), then equality holds in particular, for all i, in the inequality (39) which means that for all i, $\varphi_i(x) = c_i e^{-\frac{1}{2}\langle A_i x,x\rangle}$, where c_i is a positive constant and A_i is a $(d \times d)$ positive definite matrix. Thus, as before, the equality condition again leads to the identity

$$\prod_{i=1}^n (\det(A_i))^{\frac{1}{n}} = \frac{\det\left(\sum_{i=1}^n A_i\right)}{n^n}$$

and we conclude as before.

□

We also have a Blaschke Santaló type inequality.

Proposition 15. *For $1 \le i \le n$, let $\varphi_i : \mathbb{R}^d \to [0,\infty)$ be log-concave functions such that φ_i has barycenter at 0. If $\lambda \in [0,1]$, then*

$$as_\lambda(\vec{\varphi}) as_\lambda(\vec{\varphi^\circ}) \le (2\pi)^n. \tag{40}$$

Equality holds if and only if $\varphi_i = c_i e^{-\frac{1}{2}\langle Ax,x\rangle}$ where $c_i > 0$, $1 \le i \le n$, and A is a $(d \times d)$ positive definite matrix.

Proof. By Proposition 13,

$$\left[as_\lambda(\vec{\varphi})as_\lambda(\vec{\varphi^\circ})\right]^n \le \prod_{i=1}^n as_\lambda(\varphi_i)as_\lambda(\varphi_i^\circ).$$

The following Blaschke Santaló type inequality was proved in [7],

$$as_\lambda(\varphi)as_\lambda(\varphi^\circ) \le (2\pi)^n, \tag{41}$$

where φ is a log-concave function with barycenter at 0. It was proved in [7] that equality holds if and only if $\varphi(x) = ce^{-\frac{1}{2}\langle Ax,x\rangle}$ where $c > 0$ is a constant and A is a $(d\times d)$ positive definite matrix. Thus, the statement of the theorem follows. By the duality formula (37) and (35), it is easy to see that equality holds in (40) if $\varphi_i(x) = c_i e^{-\frac{1}{2}\langle Ax,x\rangle}$. On the other hand, if equality holds in (40), then equality holds in particular, for all i, in the inequality (41) which means that for all i, $\varphi_i(x) = c_i e^{-\frac{1}{2}\langle A_i x,x\rangle}$, where c_i is a positive constant and A_i is a $(d \times d)$ positive definite matrix. Note that for $\varphi_i(x) = c_i e^{-\frac{1}{2}\langle A_i x,x\rangle}$, the dual function is $\varphi_i^\circ(x) = c_i^{-1} e^{-\frac{1}{2}\langle A_i^{-1}x,x\rangle}$. Thus, also using (35), the equality condition leads to the following identity

$$\left(\det(A_1 + \cdots + A_n)\ \det(A_1^{-1} + \cdots + A_n^{-1})\right)^{\frac{1}{2}} = n^n \tag{42}$$

Therefore, by (27), we must have for all i, $A_i = \lambda A$, for some $\lambda > 0$ and for some positive definite matrix A. Hence we have that $\varphi_i(x) = c_i e^{-\langle Ax,x\rangle/2}$. □

The next proposition gives a monotonicity behavior of the mixed L_λ-affine surface area. The proofs follow by Hölder's inequality (see also [6]).

Proposition 16. *Let $\alpha \neq \beta, \lambda \neq \beta$ be real numbers. Let $\varphi_1, \cdots, \varphi_n : \mathbb{R}^d \to [0,\infty)$ be log concave functions.*

(i) If $1 \le \frac{\alpha-\beta}{\lambda-\beta} < \infty$, then $as_\lambda(\vec{\varphi}) \le \left(as_\alpha(\vec{\varphi})\right)^{\frac{\lambda-\beta}{\alpha-\beta}} \left(as_\beta(\vec{\varphi})\right)^{\frac{\alpha-\lambda}{\alpha-\beta}}$.

(ii) If $1 \le \frac{\alpha}{\lambda} < \infty$, then $as_\lambda(\vec{\varphi}) \le (as_\alpha(\vec{\varphi}))^{\frac{\lambda}{\alpha}} \left(\int (\varphi_1 \cdots \varphi_2)^{\frac{1}{n}}\right)^{\frac{\alpha-\lambda}{\alpha}}$.

(iii) If $\beta \le \lambda$, then $as_\lambda(\vec{\varphi}) \le \left(as_\infty(\vec{\varphi})\right)^{\lambda-\beta}\ as_\beta(\vec{\varphi})$.

If $\frac{\alpha-\beta}{\lambda-\beta} = 1$ in (i), respectively $\frac{\alpha}{\lambda} = 1$ in (ii), then $\alpha = \lambda$ and equality holds trivially in (i) respectively (ii). Equality also holds if for $1 \le i \le n$, $\varphi_i(x) = c_i e^{-\frac{1}{2}\langle A_i x,x\rangle}$, where c_i is a positive constant and A_i is a $(d \times d)$ positive definite matrix.

It follows from Proposition 16 (ii) that for $0 < \lambda \le \alpha$,

$$0 \le \left(\frac{as_\lambda(\vec{\varphi})}{\int (\varphi_1 \cdots \varphi_n)^{\frac{1}{n}} dx}\right)^{\frac{1}{\lambda}} \le \left(\frac{as_\alpha(\vec{\varphi})}{\int (\varphi_1 \cdots \varphi_n)^{\frac{1}{n}} dx}\right)^{\frac{1}{\alpha}},$$

which means that for $\lambda > 0$ the function $\left(\frac{as_\lambda(\vec{\varphi})}{\int (\varphi_1 \cdots \varphi_n)^{\frac{1}{n}} dx}\right)^{\frac{1}{\lambda}}$ is bounded below by 0 and is increasing for $\lambda > 0$. Therefore, the limit

$$\Omega_{\vec{\varphi}} = \lim_{\lambda \downarrow 0} \left(\frac{as_\lambda(\vec{\varphi})}{\int (\varphi_1 \cdots \varphi_n)^{\frac{1}{n}}\ dx}\right)^{\frac{1}{\lambda}} \tag{43}$$

exists and the quantity $\Omega_{\vec{\varphi}}$ is invariant under self adjoint $SL(d)$ maps. This quantity was first introduced by Paouris and Werner in [43] for convex bodies, then by Caglar and Werner [6] for log concave functions using L_λ-affine surface area. It also follows from Proposition 16 (ii) that for $\lambda < 0$, the function $\lambda \to \left(\frac{as_\lambda(\vec{\varphi})}{\int(\varphi_1\cdots\varphi_n)^{\frac{1}{n}}dx}\right)^{\frac{1}{\lambda}}$ is increasing. Therefore, $\lim_{\lambda\uparrow 0}\left(\frac{as_\lambda(\vec{\varphi})}{\int(\varphi_1\cdots\varphi_n)^{\frac{1}{n}}dx}\right)^{\frac{1}{\lambda}}$ exists and, in fact, is equal to $\Omega_{\vec{\varphi}}$.

The quantity $\Omega_{\vec{\varphi}}$ is related to the relative entropy as follows.

Proposition 17. *Let $\varphi_i : \mathbb{R}^d \to [0,\infty)$ be log concave functions, $i = 1, \cdots, n$. Then*

$$\Omega_{\vec{\varphi}} = \exp\left[\frac{D_{KL}\left(P_{\prod_{i=1}^n \varphi_i^{\frac{1}{n}}}||Q_{\prod_{i=1}^n \varphi_i^{\frac{1}{n}}}\right)}{\int \prod_{i=1}^n \varphi_i^{\frac{1}{n}} dx} + \int \log\left(\frac{\prod_{i=1}^n \left(det\left[Hess\left(-\log\varphi_i\right)\right]\right)^{\frac{1}{n}}}{det\left[\frac{1}{n}\sum_{i=1}^n Hess\left(-\log\varphi_i\right)\right]}\right)d\mu\right],$$

where $d\mu = \frac{\prod_{i=1}^n \varphi_i^{\frac{1}{n}}dx}{\int\prod_{i=1}^n \varphi_i^{\frac{1}{n}}dx}$.

Proof. By definition and de l'Hôspital,

$$\begin{aligned}
\Omega_{\vec{\varphi}} &= \lim_{\lambda\downarrow 0}\left(\frac{as_\lambda(\vec{\varphi})}{\int\left(\varphi_1\cdots\varphi_n\right)^{\frac{1}{n}}dx}\right)^{\frac{1}{\lambda}} = \lim_{\lambda\downarrow 0}\exp\left(\frac{1}{\lambda}\log\left(\frac{as_\lambda(\vec{\varphi})}{\int\left(\varphi_1\cdots\varphi_n\right)^{\frac{1}{n}}dx}\right)\right) \\
&= \exp\left(\lim_{\lambda\downarrow 0}\frac{\int \frac{d}{d\lambda}\prod_{i=1}^n\left[\varphi_i\left(\frac{e^{\frac{\langle\nabla\varphi_i,x\rangle}{\varphi_i}}}{\varphi_i^2}\det\left[\text{Hess}\left(-\log\varphi_i\right)\right]\right)^{\lambda}\right]^{\frac{1}{n}}dx}{as_\lambda(\vec{\varphi})}\right) \\
&= \exp\left(\frac{\int\prod_{i=1}^n\varphi_i^{\frac{1}{n}}\;\log\left[\prod_{i=1}^n\left(\frac{e^{\frac{\langle\nabla\varphi_i,x\rangle}{\varphi_i}}}{\varphi_i^2}\det\left[\text{Hess}\left(-\log\varphi_i\right)\right]\right)^{\frac{1}{n}}\right]dx}{\int\left(\varphi_1\cdots\varphi_n\right)^{\frac{1}{n}}dx}\right).
\end{aligned}$$

Now we treat the exponent further. As $q_{\prod_{i=1}^n \varphi_i^{\frac{1}{n}}} = \prod_{i=1}^n \varphi_i^{\frac{1}{n}} = \prod_{i=1}^n q_{\varphi_i^{\frac{1}{n}}}$ and

$$p_{\prod_{i=1}^n \varphi_i^{\frac{1}{n}}} = \prod_{i=1}^n \frac{e^{\frac{1}{n}\frac{\langle\nabla\varphi_i,x\rangle}{\varphi_i}}}{\varphi_i^{\frac{1}{n}}}\det\left[\text{Hess}\left(-\log\prod_{i=1}^n\varphi_i^{\frac{1}{n}}\right)\right],$$

we get that

$$\int \prod_{i=1}^{n} \varphi_i^{\frac{1}{n}} \ \log \left[\prod_{i=1}^{n} \left(\frac{e^{\frac{\langle \nabla \varphi_i, x\rangle}{\varphi_i}}}{\varphi_i^2} \ \det\left[\text{Hess}\left(-\log \varphi_i\right)\right] \right)^{\frac{1}{n}} \right] dx$$

$$= \int q_{\prod_{i=1}^n \varphi_i^{\frac{1}{n}}} \log \left(\frac{p_{\prod_{i=1}^n \varphi_i^{\frac{1}{n}}}}{q_{\prod_{i=1}^n \varphi_i^{\frac{1}{n}}}} \ \frac{\prod_{i=1}^n \left(\det\left[\text{Hess}\left(-\log\varphi_i\right)\right]\right)^{\frac{1}{n}}}{\det\left[\text{Hess}\left(-\log \prod_{i=1}^n \varphi_i^{\frac{1}{n}}\right)\right]} \right) dx$$

$$= D_{KL}\left(P_{\prod_{i=1}^n \varphi_i^{\frac{1}{n}}} || Q_{\prod_{i=1}^n \varphi_i^{\frac{1}{n}}}\right) + \int \prod_{i=1}^{n} \varphi_i^{\frac{1}{n}} \ \log \left(\frac{\prod_{i=1}^n \left(\det\left[\text{Hess}\left(-\log\varphi_i\right)\right]\right)^{\frac{1}{n}}}{\det\left[\frac{1}{n}\sum_{i=1}^n \text{Hess}\left(-\log \varphi_i\right)\right]} \right).$$

□

Corollary 18. *Let $\varphi_i : \mathbb{R}^d \to [0,\infty)$ be log concave functions, $i = 1, \cdots, n$. Then*

$$\log \ \left(\Omega_{\vec{\varphi}}\right) \le \frac{D_{KL}\left(P_{\prod_{i=1}^n \varphi_i^{\frac{1}{n}}} || Q_{\prod_{i=1}^n \varphi_i^{\frac{1}{n}}}\right)}{\int \prod_{i=1}^n \varphi_i^{\frac{1}{n}} \, dx}.$$

If $n = 1$, equality holds trivially. Otherwise, equality holds if and only if for all i, Hess $(-\log \varphi_i) =$ Hess $(-\log \varphi)$, for some log concave φ.

Proof. For $i = 1, \ldots, n$, we put $H_i = \text{Hess}\left(-\log \varphi_i\right)$. Then, by Proposition 17,

$$\Omega_{\vec{\varphi}} = \exp \left[\frac{D_{KL}\left(P_{\prod_{i=1}^n \varphi_i^{\frac{1}{n}}} || Q_{\prod_{i=1}^n \varphi_i^{\frac{1}{n}}}\right)}{\int \prod_{i=1}^n \varphi_i^{\frac{1}{n}} \, dx} \right] \ \exp\left[\int \log \left(\frac{\prod_{i=1}^n \left(\det H_i\right)^{\frac{1}{n}}}{\det\left[\frac{1}{n}\sum_{i=1}^n H_i\right]} \right) d\mu \right].$$

It is easy to see that equality holds if $n = 1$. Otherwise, by (27),

$$\prod_{i=1}^{n} \left(\det(H_i)\right)^{\frac{1}{n}} \le \frac{1}{n^n} \ \left(\det\left(\sum_{i=1}^n H_i\right)\right) = \det\left(\frac{1}{n}\sum_{i=1}^n H_i\right),$$

with equality if and only if for all i, $H_i = \lambda H$, for some $\lambda > 0$ and $H = \text{Hess}\left(-\log \varphi\right)$, for some log concave φ. Therefore,

$$\Omega_{\vec{\varphi}} \le \exp \left[\frac{D_{KL}\left(P_{\prod_{i=1}^n \varphi_i^{\frac{1}{n}}} || Q_{\prod_{i=1}^n \varphi_i^{\frac{1}{n}}}\right)}{\int \prod_{i=1}^n \varphi_i^{\frac{1}{n}} \, dx} \right],$$

with equality if and only if for all i, $\text{Hess}\left(-\log\varphi_i\right) = \text{Hess}\left(-\log \varphi\right)$, for some log concave φ.

□

Corollary 19. *Let* $\varphi_i : \mathbb{R}^d \to [0,\infty)$ *be log concave functions,* $i = 1, \cdots, n$.

(i) $\Omega_{\vec{\varphi}} \le \left(\frac{as_\lambda(\vec{\varphi})}{\int (\varphi_1 \cdots \varphi_n)^{\frac{1}{n}} dx}\right)^{\frac{1}{\lambda}}$ *for all* $\lambda > 0$ *and* $\Omega_{\vec{\varphi}} \ge \left(\frac{as_\lambda(\vec{\varphi})}{\int (\varphi_1 \cdots \varphi_n)^{\frac{1}{n}} dx}\right)^{\frac{1}{\lambda}}$ *for all* $\lambda < 0$.

(ii) Let $\varphi_i = a_i \varphi$ *for some log concave function* $\varphi : \mathbb{R}^d \to [0,\infty)$ *and* $a_i > 0$. *Then*

$$\Omega_{\vec{\varphi}}\, \Omega_{\vec{\varphi}^\circ} \ \le\ 1.$$

(iii) Let $\varphi_i = a_i \varphi$ *for some log concave function* $\varphi : \mathbb{R}^d \to [0,\infty)$ *and* $a_i > 0$. *Then*

$$\Omega_{\vec{\varphi}} = \lim_{\alpha \to 1} \left(\frac{as_\alpha(\vec{\varphi}^\circ)}{\int (\varphi_1 \cdots \varphi_n)^{\frac{1}{n}} dx}\right)^{\frac{1}{1-\alpha}}.$$

Equality holds in (i) and (ii) if $\varphi_i = c_i e^{-\frac{1}{2}\langle Ax, x\rangle}$ *where* $c_i > 0$, $1 \le i \le n$, *and* A *is a* $(d \times d)$ *positive definite matrix.*

Proof. (i) is deduced immediately from the monotonicity behavior of the function $\lambda \to \left(\frac{as_\lambda(\vec{\varphi})}{\int (\varphi_1 \cdots \varphi_n)^{\frac{1}{n}} dx}\right)^{\frac{1}{\lambda}}$ and the definition of $\Omega_{\vec{\varphi}}$.

(ii) By (i) and Proposition 12,

$$\Omega_{\vec{\varphi}} \le \frac{as_1(\vec{\varphi})}{\int (\varphi_1 \cdots \varphi_n)^{\frac{1}{n}}\, dx} = \frac{as_1(\vec{\varphi})}{as_0(\vec{\varphi})} = \frac{as_0(\vec{\varphi}^\circ)}{as_1(\vec{\varphi}^\circ)}, \qquad \Omega_{\vec{\varphi}^\circ} \le \frac{as_1(\vec{\varphi}^\circ)}{as_0(\vec{\varphi}^\circ)}.$$

(iii) We use the duality formula (37). By definition

$$\begin{aligned} \Omega_{\vec{\varphi}^\circ} &= \lim_{\lambda \to 0} \left(\frac{as_\lambda(\vec{\varphi}^\circ)}{\int (\varphi_1^\circ \cdots \varphi_n^\circ)^{\frac{1}{n}}\, dx}\right)^{\frac{1}{\lambda}} = \lim_{\lambda \to 0} \left(\frac{as_{1-\lambda}(\vec{\varphi})}{\int (\varphi_1^\circ \cdots \varphi_n^\circ)^{\frac{1}{n}}\, dx}\right)^{\frac{1}{\lambda}} \\ &= \lim_{\alpha \to 1} \left(\frac{as_\alpha(\vec{\varphi})}{\int (\varphi_1^\circ \cdots \varphi_n^\circ)^{\frac{1}{n}}\, dx}\right)^{\frac{1}{1-\alpha}}. \end{aligned}$$

Therefore, $\Omega_{\vec{\varphi}} = \lim_{\alpha \to 1} \left(\frac{as_\alpha(\vec{\varphi}^\circ)}{\int (\varphi_1 \cdots \varphi_n)^{\frac{1}{n}} dx}\right)^{\frac{1}{1-\alpha}}$. □

We define the *i-th mixed* L_λ*-affine surface area* $as_{\lambda,i}(\vec{\varphi})$ of $\vec{\varphi} = (\varphi_1, \varphi_2)$ by

$$as_{\lambda,i}(\vec{\varphi}) =$$

$$\int \left[\varphi_1 \left(\frac{e^{\frac{\langle \nabla \varphi_1, x\rangle}{\varphi_1}}}{\varphi_1^2} \det\left[\text{Hess}\left(-\log \varphi_1\right)\right]\right)^\lambda\right]^{\frac{i}{n}} \left[\varphi_2 \left(\frac{e^{\frac{\langle \nabla \varphi_2, x\rangle}{\varphi_2}}}{\varphi_2^2} \det\left[\text{Hess}\left(-\log \varphi_2\right)\right]\right)^\lambda\right]^{\frac{n-i}{n}} dx.$$

Clearly, for all λ, $as_{\lambda,0}(\vec{\varphi}) = as_\lambda(\varphi_2)$ and $as_{\lambda,n}(\vec{\varphi}) = as_\lambda(\varphi_1)$. Moreover, $as_{0,n}(\vec{\varphi}) = \int \varphi_1 dx$ and $as_{1,n}(\vec{\varphi}) = \int \varphi_1^\circ dx$ (see, [6]). We also give a definition for $as_{\infty,i}(\vec{\varphi})$ and $as_{-\infty,i}(\vec{\varphi})$.

$$as_{\infty,i}(\vec{\varphi}) \;=\; \max_x \left[\frac{e^{\frac{\langle \nabla\varphi_1, x\rangle}{\varphi_1}}}{\varphi_1^2} \det\left[\mathrm{Hess}\left(-\log\varphi_1\right)\right]\right]^{\frac{i}{n}} \left[\frac{e^{\frac{\langle \nabla\varphi_2, x\rangle}{\varphi_2}}}{\varphi_2^2} \det\left[\mathrm{Hess}\left(-\log\varphi_2\right)\right]\right]^{\frac{n-i}{n}}.$$

$$as_{-\infty,i}(\vec{\varphi}) = \frac{1}{as_{\infty,i}(\vec{\varphi})}.$$

It is easy to see that these expressions are invariant under symmetric linear transformations with determinant 1.

Remarks. (i) It follows from Proposition 12 that $as_{1-\lambda,i}(\vec{\varphi}) = as_{\lambda,i}(\vec{\varphi^\circ})$ where $\vec{\varphi} = (\varphi_1, \varphi_2)$ such that $\varphi_1 = a\varphi_2$, $a > 0$.

(ii) Let $\varphi_l(x) = c_l e^{-\frac{1}{2}\langle A_l x, x\rangle}$, where c_l is a positive constant and A_l is a $(d\times d)$ positive definite matrix for $l = 1, 2$. Then,

$$as_{\lambda,i}(\vec{\varphi}) = \left(c_1^i\ c_2^{n-i}\right)^{\frac{1-2\lambda}{n}} \left((\det(A_1))^i(\det(A_2))^{n-i}\right)^{\frac{\lambda}{n}} \frac{(2n\pi)^{\frac{n}{2}}}{(\det(iA_1 + (n-i)A_2))^{\frac{1}{2}}}. \quad (44)$$

The next proposition is identical to Proposition 16 and the proof follows by Hölder's inequality.

Proposition 20. *Let $i \in \mathbb{R}$ and $\alpha \neq \beta, \lambda \neq \beta$ be real numbers. Let $\varphi_1, \varphi_2 : \mathbb{R}^d \to [0,\infty)$ be log concave functions.*

(i) If $1 \leq \frac{\alpha-\beta}{\lambda-\beta} < \infty$, then $as_{\lambda,i}(\vec{\varphi}) \leq \left(as_{\alpha,i}(\vec{\varphi})\right)^{\frac{\lambda-\beta}{\alpha-\beta}} \left(as_{\beta,i}(\vec{\varphi})\right)^{\frac{\alpha-\lambda}{\alpha-\beta}}$.

(ii) If $1 \leq \frac{\alpha}{\lambda} < \infty$, then $as_{\lambda,i}(\vec{\varphi}) \leq (as_{\alpha,i}(\vec{\varphi}))^{\frac{\lambda}{\alpha}} \left(\int \varphi_1^{\frac{i}{n}} \varphi_2^{\frac{n-i}{n}}\right)^{\frac{\alpha-\lambda}{\alpha}}$.

(iii) If $\beta \leq \lambda$, then $as_{\lambda,i}(\vec{\varphi}) \leq \left(as_{\infty,i}(\vec{\varphi})\right)^{\lambda-\beta} as_{\beta,i}(\vec{\varphi})$.

If $\frac{\alpha-\beta}{\lambda-\beta} = 1$ in (i), respectively $\frac{\alpha}{\lambda} = 1$ in (ii), then $\alpha = \lambda$ and equality holds trivially in (i) respectively (ii). Equality also holds if $\varphi_l(x) = c_l e^{-\frac{1}{2}\langle A_l x, x\rangle}$, where c_l is a positive constant and A_l is a $(d\times d)$ positive definite matrix for $l = 1, 2$.

The following proposition is a direct consequence of Proposition 10.

Proposition 21. *Let $\varphi_1, \varphi_2 : \mathbb{R}^d \to [0,\infty)$ be log concave functions. If $j \leq i \leq k$ or $k \leq i \leq j$, then*

$$as_{\lambda,i}(\vec{\varphi}) \;\leq\; \left[as_{\lambda,j}(\vec{\varphi})\right]^{\frac{k-i}{k-j}} \times \left[as_{\lambda,k}(\vec{\varphi})\right]^{\frac{i-j}{k-j}}.$$

Equality holds trivially if $i = k$ or $i = j$. Otherwise, equality holds if and only if one of the functions $\varphi_l \left(\frac{e^{\frac{\langle \nabla\varphi_l, x\rangle}{\varphi_l}}}{\varphi_l^2} det\left[Hess\left(-\log\varphi_l\right)\right]\right)^\lambda$, $l = 1, 2$, is null or they are effectively proportional.

In Proposition 21, if we let $j = 0$ and $k = n$, then for all λ and $0 \le i \le n$

$$[as_{\lambda,i}(\vec{\varphi})]^n \ \le \ [as_\lambda(\varphi_2)]^{n-i} \ [as_\lambda(\varphi_1)]^i . \tag{45}$$

If we let $i = 0$ and $j = n$, then for all λ and $k \le 0$

$$[as_{\lambda,k}(\vec{\varphi})]^n \ \ge \ [as_\lambda(\varphi_2)]^{n-k} \ [as_\lambda(\varphi_1)]^k . \tag{46}$$

From inequality (45) and an inequality of [7], already quoted here as inequality (41), one gets for functions with barycenter at 0,

$$\begin{aligned}[as_{\lambda,i}(\varphi_1,\varphi_2)]^n\,[as_{\lambda,i}(\varphi_1^\circ,\varphi_2^\circ)]^n \quad & \le \ [as_\lambda(\varphi_2)as_\lambda(\varphi_2^\circ)]^{n-i} \ [as_\lambda(\varphi_1)as_\lambda(\varphi_1^\circ)]^i \\ & \le (2\pi)^{n^2}\end{aligned}$$

holds true for all $\lambda \in [0,1]$ and $0 \le i \le n$. Hence, we have proved the following proposition which also follows directly from Proposition 15.

Proposition 22. *Let φ_1, φ_2 be log concave functions with barycenter at 0. If $\lambda \in [0,1]$ and $0 \le i \le n$, then*

$$as_{\lambda,i}(\vec{\varphi})as_{\lambda,i}(\vec{\varphi^\circ}) \ \le (2\pi)^n. \tag{47}$$

Equality holds if and only if $\varphi_l = c_l e^{-\frac{1}{2}\langle Ax,x\rangle}$ where $c_l > 0$, $l = 1,2$, and A is a $(d \times d)$ positive definite matrix.

Proof. The inequality follows from above. Using (44) and the duality formula $as_{1-\lambda,i}(\vec{\varphi}) = as_{\lambda,i}(\vec{\varphi^\circ})$, it is easy to see that equality holds in (47) if $\varphi_l = c_l e^{-\frac{1}{2}\langle Ax,x\rangle}$ where $c_l > 0$ and A is a $(d \times d)$ positive definite matrix. On the other hand, if equality holds in (47) then equality holds in particular, for $l = 1,2$, in the inequality (41) which means that, $\varphi_l(x) = c_l e^{-\frac{1}{2}\langle A_l x,x\rangle}$, where c_l is a positive constant and A_l is a $(d \times d)$ positive definite matrix. Note that for $\varphi_l(x) = c_l e^{-\frac{1}{2}\langle A_l x,x\rangle}$, the dual function is $\varphi_l^\circ(x) = c_l^{-1} e^{-\frac{1}{2}\langle A_l^{-1}x,x\rangle}$. Thus, also using (44), the equality condition leads to the following identity

$$\big(\det(iA_1 + (n-i)A_2)\ \det(iA_1^{-1} + (n-i)A_2^{-1})\big)^{\frac{1}{2}} = n^n \tag{48}$$

Therefore, by (27), we must have $A_1 = A_2$. Hence we have that $\varphi_l(x) = c_l e^{-\langle Ax,x\rangle/2}$. □

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