Probabilités/Probability Theory

Minima of sequences of Gaussian random variables
Minima des suites des variables aléatoires Gaussiennes

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Abstract. For a given sequence of real numbers $a_1, \dots, a_n$ we denote the k-th smallest one by $k\text{-}\min_{1\le i\le n} a_i$. We show that there exist two absolute positive constants c and C such that for every sequence of positive real numbers $x_1, \dots, x_n$ and every $k \le n$ one has

$$c \max_{1\le j\le k} \frac{k+1-j}{\sum_{i=j}^n 1/x_i} \le \mathbb{E}\, k\text{-}\min_{1\le i\le n} |x_i g_i| \le C \ln(k+1) \max_{1\le j\le k} \frac{k+1-j}{\sum_{i=j}^n 1/x_i},$$

where $g_i \in N(0,1)$, $i = 1, \dots, n$, are independent Gaussian random variables. Moreover, if $k = 1$ then the left hand side estimate does not require independence of the g_i's. Similar estimates hold for $\mathbb{E}\, k\text{-}\min_{1\le i\le n} |x_i g_i|^p$ as well.

Résumé. Pour une suite $a_1, \dots, a_n$ des nombres réels on denote le k-ième plus petit membre par $k\text{-}\min_{1\le i\le n} a_i$. On demontre qu'il existe deux constants positives c et C telle que pour toute suite $x_1, \dots, x_n$ des nombres réels et pour tout $k \le n$ on a

$$c \max_{1\le j\le k} \frac{k+1-j}{\sum_{i=j}^n 1/x_i} \le \mathbb{E}\, k\text{-}\min_{1\le i\le n} |x_i g_i| \le C \ln(k+1) \max_{1\le j\le k} \frac{k+1-j}{\sum_{i=j}^n 1/x_i}.$$

$g_i \in N(0,1)$, $i = 1, \dots, n$, sont des variables aléatoires Gaussiennes indépendentes. En plus, si $k = 1$, on n'a pas besoin de l'indépendence des g_i's pour obtenir l'inégalité du gauche. On demontre également les inégalités correspondentes pour $\mathbb{E}\, k\text{-}\min_{1\le i\le n} |x_i g_i|^p$.

For a given sequence of real numbers $(a_i)_{i=1}^n$ we denote its non-decreasing rearrangement by $(k\text{-}\min_{1\le i\le n} a_i)_{k=1}^n$, thus $1\text{-}\min_{1\le i\le n} a_i = \min_{1\le i\le n} a_i$, $2\text{-}\min_{1\le i\le n} a_i$ is the next smallest, etc.

Given $A \subset \mathbb{N}$ we denote its cardinality by $|A|$. We say that $(A_j)_{j=1}^k$ is a partition of $\{1, 2, \dots, n\}$ if $\emptyset \ne A_j \subset \{1, 2, \dots, n\}$, $j \le k$, $\cup_{j\le k} A_j = \{1, 2, \dots, n\}$, and $A_i \cap A_j = \emptyset$ for $i \ne j$. The canonical

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Euclidean norm and the canonical inner product on $\mathbb{R}^n$ we denote by $|\cdot|$ and $\langle \cdot, \cdot \rangle$. By $1/t$ we mean ∞ if $t = 0$ and 0 if $t = \infty$.

In this note we present two theorems. The first one investigates the behavior of the expectation of the minimum of symmetric Gaussian random variables.

Theorem 1 *Let $p > 0$. Let $(x_i)_{i=1}^n$ be a sequence of real numbers. Let $g_i \in N(0,1)$, $i \le n$, be Gaussian random variables. Then*

$$\tfrac{1}{1+p} \left(\tfrac{\pi}{2}\right)^{p/2} \left(\sum_{i=1}^n |x_i|^{-1} \right)^{-p} \le \mathbb{E} \min_{1 \le i \le n} |x_i g_i|^p .$$

Moreover, if the g_i's are independent then

$$\mathbb{E} \min_{1 \le i \le n} |x_i g_i|^p \le \Gamma(1+p) \left(\tfrac{\pi}{2}\right)^{p/2} \left(\sum_{i=1}^n |x_i|^{-1} \right)^{-p} .$$

An immediate consequence of this theorem is the following Corollary.

Corollary 2 *Let $p > 0$. Let $(x_i)_{i=1}^n$ be a sequence of real numbers. Let $f_i \in N(0,1)$, $i \le n$, be Gaussian random variables and $g_i \in N(0,1)$, $i \le n$, be independent Gaussian random variables. Then*

$$\mathbb{E} \min_{1 \le i \le n} |x_i g_i|^p \le \Gamma(2+p)\, \mathbb{E} \min_{1 \le i \le n} |x_i f_i|^p .$$

Remark. This inequality is connected to the Mallat-Zeitouni problem ([2]). In fact, to prove a particular case of Conjecture 1 from [2] it is enough to prove our Corollary for $p = 2$ and with factor 1 instead of $\Gamma(2+p)$ ([3]). Thus we provide the solution of this case up to constant 6.

Next theorem deals with the moments of k-min of independent symmetric Gaussian variables.

Theorem 3 *Let $p > 0$. Let $2 \le k \le n$. Let $0 < x_1 \le x_2 \le ... \le x_n$. Let $g_i \in N(0,1)$, $i \le n$, be independent Gaussian random variables. Then*

$$c_p \max_{1 \le j \le k} \frac{k+1-j}{\sum_{i=j}^n 1/x_i} \le \left(\mathbb{E}\ k\text{-}\min_{1 \le i \le n} |x_i g_i|^p \right)^{1/p} \le C(p,k) \max_{1 \le j \le k} \frac{k+1-j}{\sum_{i=j}^n 1/x_i},$$

where $c_p = \frac{1}{2e} \sqrt{\frac{\pi}{2}} \left(1 - \frac{1}{4\sqrt{\pi}}\right)^{1/p}$ and $C(p,k) = 4\sqrt{\pi} \max\{p, \ln(k+1)\}$.

Remark. Theorem 3 shows that we may evaluate sums of the form $\sum_{k \in I} \mathbb{E}\ k\text{-}\min_{1 \le i \le n} |x_i g_i|^p$, where $I \subset \{1, 2, ..., n\}$ is any subset of integers. Related inequalities, though in a different context, were developed initially in [1].

Theorems 1 and 3 are consequences of the following Lemmas, which are of independent interest.

Lemma 4 *Let $0 < x_1 \le x_2 \le ... \le x_n$. Let $g_i \in N(0,1)$, $i \le n$, be Gaussian random variables. Let $a = \sqrt{2/\pi} \sum_{i=1}^n 1/x_i$. Then for every $t > 0$*

$$\mathbb{P} \left\{ \omega \,\Big|\, \min_{1 \le i \le n} |x_i g_i(\omega)| \le t \right\} \le at.$$

Moreover, if the g_i's are independent then for every $t > 0$

$$\mathbb{P} \left\{ \omega \,\Big|\, \min_{1 \le i \le n} |x_i g_i(\omega)| > t \right\} \le e^{-at}.$$

Lemma 5 *Let $1 \le k \le n$. Let $0 < x_1 \le x_2 \le ... \le x_n$. Let $g_i \in N(0,1)$, $i \le n$, be independent Gaussian random variables. Let*

$$a = \frac{e}{k} \sqrt{\frac{2}{\pi}} \sum_{i=1}^{n} \frac{1}{x_i}.$$

Then for every $0 < t < 1/a$ one has

$$\mathbb{P}\left\{\omega \,\middle|\, k\text{-}\min_{1\le i\le n} |x_i g_i(\omega)| \le t\right\} \le \frac{1}{\sqrt{2\pi k}} \,\frac{(at)^k}{1-at} \tag{1}$$

In the rest of this note we provide proofs of Theorems 1 and 3. Proofs of all lemmas will be shown in the forthcoming paper.

Proof of Theorem 1. Let us note that if $x_i = 0$ for some i then the expectation is 0 and the Theorem is trivial. Therefore, without loss of generality, we assume that $x_i > 0$ for every i.

Denote

$$A = \left(\sqrt{\tfrac{2}{\pi}} \sum_{k=1}^{n} 1/x_k\right)^{-p}.$$

Then, by the first estimate in Lemma 4, we have

$$\mathbb{E} \min_{1\le i\le n} |x_i g_i|^p = \int_0^\infty \mathbb{P}\left\{\omega \,\middle|\, \min_{1\le i\le n} |x_i g_i(\omega)| > t^{1/p}\right\} dt \ge \int_0^A \left(1 - t^{1/p}\, A^{-1/p}\right) dt = \frac{A}{1+p},$$

which proves the first estimate.

Now assume that the g_i's are independent and use the second estimate of Lemma 4. We obtain

$$\mathbb{E} \min_{1\le i\le n} |x_i g_i|^p = \int_0^\infty \mathbb{P}\left\{\omega \,\middle|\, \min_{1\le i\le n} |x_i g_i(\omega)| > t^{1/p}\right\} dt \le \int_0^\infty \exp\left(-t^{1/p}\, A^{-1/p}\right) dt = A\, p\, \Gamma(p),$$

which implies the desired result. □

To prove Theorem 3 we need also the following combinatorial lemma.

Lemma 6 *Let $1 \le k \le n$. Let $(a_i)_{i=1}^n$, be a nonincreasing sequence of positive real numbers. Then there exists a partition $(A_l)_{l\le k}$ of $\{1, 2, ..., n\}$ such that*

$$\min_{1\le l\le k} \sum_{i\in A_l} a_i \ge a := \tfrac{1}{2} \min_{1\le j\le k} \frac{1}{k+1-j} \sum_{i=j}^{n} a_i.$$

Remark. In fact one can show that the A_l's can be taken as intervals, i.e. $A_l = \{i \mid n_{l-1} < i \le n_l\}$, $l \le k$, for some sequence $0 = n_0 < 1 \le n_1 < n_2 < \cdots < n_k = n$.

Proof of Theorem 3. First we show the lower estimate. Since for every sequence $(a_i)_{i=1}^n$ and every $r < k$ one has

$$k\text{-}\min(a_i)_{i=1}^n \ge (k-r)\text{-}\min(a_i)_{i=r+1}^n,$$

it is enough to show that for every k we have

$$c_p\, k \left(\sum_{i=1}^{n} 1/x_i\right)^{-1} \le \left(\mathbb{E}\; k\text{-}\min_{1\le i\le n} |x_i g_i|^p\right)^{1/p}. \tag{2}$$

Let a be as in Lemma 5 and $t = (2a)^{-p}$. Then, by Lemma 5 and since $k \ge 2$, we have

$$\mathbb{P}\left\{\omega \,\middle|\, k\text{-}\min_{1\le i\le n} |x_i g_i(\omega)|^p \ge t\right\} \ge 1 - \frac{1}{\sqrt{2\pi k}}\, \frac{(at^{1/p})^k}{1-at^{1/p}} \ge 1 - \frac{1}{4\sqrt{\pi}}.$$

Therefore (2) follows from the standard estimate

$$\mathbb{E}\ k\text{-}\min_{1\le i\le n} |x_i g_i|^p \ge t^p\ \mathbb{P}\left\{\omega\ \middle|\ k\text{-}\min_{1\le i\le n} |x_i g_i(\omega)| \ge t\right\}.$$

Now we prove the upper bound. Let $(A_j)_{j\le k}$ be the partition given by Lemma 6 for sequence $a_i = 1/x_i$, $i \le k$. The number q, $q \ge 1$, will be specified later. It is easy to see that $k\text{-}\min_{1\le i\le n} |x_i g_i|^p \le \max_{j\le k} \left\{\min_{i\in A_j} |x_i g_i|^p\right\}_{j\le k}$. Therefore, using Theorem 1, we get

$$\left(\mathbb{E}\ k\text{-}\min_{1\le i\le n} |x_i g_i|^p\right)^{1/p} \le \left(\mathbb{E}\ \left(\sum_{j\le k}\left(\min_{i\in A_j}|x_i g_i|^p\right)^q\right)^{1/q}\right)^{1/p} \le \left(\mathbb{E}\ \sum_{j\le k}\min_{i\in A_j}|x_i g_i|^{pq}\right)^{1/(pq)}$$

$$\le \sqrt{\tfrac{\pi}{2}}\left(\Gamma(1+pq)\sum_{j\le k}\left(\sum_{i\in A_j} 1/x_i\right)^{-pq}\right)^{1/(pq)} \le \sqrt{\tfrac{\pi}{2}}\ \left(k\ \Gamma(1+pq)\right)^{1/(pq)} \max_{j\le k}\left(\sum_{i\in A_j} 1/x_i\right)^{-1}.$$

Applying Lemma 6, we obtain

$$\left(\mathbb{E}\ k\text{-}\min_{1\le i\le n} |x_i g_i|^p\right)^{1/p} \le \sqrt{2\pi}\ \left(k\ \Gamma(1+pq)\right)^{1/(pq)} \max_{1\le j\le k} \frac{k+1-j}{\sum_{i=j}^n 1/x_i}.$$

To complete the proof we choose $q = \frac{\ln(k+1)}{p}$ if $p \le \ln(k+1)$, $q = 1$ otherwise, and apply Stirling's formula. $\Box$

Remark. Finally we would like to note that our results can be extended to the case of general distributions satisfying certain conditions. Namely, fix $\alpha > 0$, $\beta > 0$ and say that a random variable ξ satisfies an (α, β)-condition if for every $t > 0$ one has

$$\mathbb{P}\ (|\xi| \le t) \le \alpha t \qquad \text{and} \qquad \mathbb{P}\ (|\xi| > t) \le e^{-\beta t}.$$

Then Theorems 1, 3 and Lemmas 4, 5 hold for g_i's satisfying an (α, β)-condition (even not identically distributed), with constants depending on α, β. More precisely, in the estimates of Theorem 1, $(\pi/2)^{p/2}$ should be substituted by α^{-p} and β^{-p} correspondingly; in Theorem 3, $\sqrt{\pi/2}$ should be substituted by $1/\alpha$ and, in the upper estimate, $4\sqrt{\pi}$ by $4\sqrt{2}/\beta$; in Lemma 5 and in the first estimate of Lemma 4, $\sqrt{2/\pi}$ should be substituted by α; in the second estimate of Lemma 4, $\sqrt{2/\pi}$ should be substituted by β.

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