Uniform estimates for order statistics and Orlicz functions *

Yehoram Gordon †‡ Alexander Litvak † Carsten Schütt†
Elisabeth Werner† §

Abstract

We establish uniform estimates for order statistics: Given a sequence of independent identically distributed random variables $\xi_1, \ldots, \xi_n$ with log-concave distribution and scalars $x_1, \ldots, x_n$, for every $k \le n$ we provide estimates for $\mathbb{E}\ k\text{-}\min_{1\le i\le n} |x_i\xi_i|$ and $\mathbb{E}\,\text{k-max}_{1\le i\le n} |x_i\xi_i|$ in terms of the value k and the appropriate Orlicz norm $\|(1/x_1, \ldots, 1/x_n)\|_M$, associated with the distribution function of the random variable $|\xi_1|$. For example, if ξ_1 is the standard $N(0,1)$ Gaussian random variable, then the corresponding Orlicz function is $M(s) = \sqrt{\frac{2}{\pi}} \int_0^s e^{-\frac{1}{2t^2}}\, dt$. We would like to emphasize that our estimates do not depend on the length n of the sequence.

1 Introduction

In this paper we establish uniform estimates for order statistics. The k-th order statistic of a statistical sample of size n is equal to its k-th smallest value, or equivalently its $(n-k+1)$-th largest value. Order statistics are among the most fundamental tools in non-parametric statistics and inference and consequently there is extensive literature on order statistics. We only cite [1, 3, 5, 7, 11, 39] and references therein.

Order statistics are more resilient to faulty sensor reading than max, min or average and thus they find applications when methods are needed to study configurations that take on a ranked order. To name only a few: Wireless networks, signal processing, image processing, compressed sensing, data reconstruction, learning theory and data mining. A sample of works done in this area are [2, 4, 6, 8, 9, 10, 12, 13, 17, 31, 33].

Order statistics on random sequences appear naturally in Banach space theory, in computations of various random parameters associated with the geometry of convex bodies

*Keywords: expectations, normal distribution, order statistics, Orlicz function, Orlicz norm, sequence spaces, sequences of random variables. 2000 Mathematical Subject Classification: 60E15, 62G30, 60G15, 60B11, 46E40, 46B45.

†Partially supported by AIM, Palo Alto

‡Supported in part by “France-Israel Cooperation Grant #3-1350” and by the “Fund for the Promotion of Research at the Technion”.

§Partially supported by a NSF Grant, by a FRG Grant, and by a BSF Grant

in high dimensions, in random matrix theory (computing the distribution of eigenvalues), and in approximation theory (see e.g. [15, 16, 17, 19, 22, 32, 36, 37, 38]). This list of course does not include the enormous quantity of published works which deal with evaluations and applications of *max* and *min* associated with various random parameters, e.g., smallest and largest eigenvalues of random matrices, as these are the extreme values in the scale of order statistics.

For these important special cases of order statistics, the minimum and maximum value of a sample, very precise estimates were obtained in [18, 20, 21]. The new approach started there was to give estimates of the minimum and maximum value of the sample

$$\mathbb{E} \min_{1\le i\le n} |x_i\xi_i| \quad \text{and} \quad \mathbb{E} \max_{1\le i\le n} |x_i\xi_i|, \tag{1}$$

in terms of *Orlicz norms* (see the definition below). The expressions for the estimate in case (1) are relatively simple. For instance, it was shown in [18] that

$$c_1\|x\|_M \le \mathbb{E} \max_{1\le i\le n} |x_i\xi_i(\omega)| \le c_2\|x\|_M,$$

and in [20, 21] that

$$c_3 \left(\sum_{i=1}^n \frac{1}{|x_i|}\right)^{-1} \le \mathbb{E} \min_{1\le i\le n} |x_i\xi_i| \le c_4 \left(\sum_{i=1}^n \frac{1}{|x_i|}\right)^{-1},$$

where c_1, c_2, c_3, and c_4 are absolute positive constants and $\|\cdot\|_M$ is an Orlicz norm, depending on the distribution of ξ_1 only. In fact, in [18] much more general case was considered (see also [25] and [34]).

Here we study the values

$$\mathbb{E}\ k\text{-}\min_{1\le i\le n} |x_i\xi_i| \quad \text{and} \quad \mathbb{E}\ \text{k-max}_{1\le i\le n} |x_i\xi_i|, \tag{2}$$

for general order statistics for i.i.d. (independent identically distributed) random variables $\xi_1,\dots,\xi_n$ and scalars $x_1,\dots,x_n$, where for a given sequence of real numbers $a_1,\dots,a_n$ we denote the k-th smallest one by $k\text{-}\min_{1\le i\le n} a_i$. In particular, $1\text{-}\min_{1\le i\le n} a_i = \min_{1\le i\le n} a_i$ and $n\text{-}\min_{1\le i\le n} a_i = \max_{1\le i\le n} a_i$. In the same way we denote the k-th biggest number by $k\text{-}\max_{1\le i\le n} a_i$. Thus, $k\text{-}\max_{1\le i\le n} a_i = (n-k+1)\text{-}\min_{1\le i\le n} a_i$. In fact, in the theory of order statistics the standard notation for $k\text{-}\min$ is $a_{k:n}$. In this paper such a notation could be misleading and we prefer to use $k\text{-}\min$.

Now the expressions get more involved than in case (1). In view of possible applications we strive to keep them as simple as possible – at the expense of the constants involved. We show that if ξ_1 has a log-concave distribution then for $1\le k\le n/2$

$$c_1 \max_{1\le j\le k} \|\,(1/x_i)_{i=j}^n\,\|^{-1}_{\frac{2e}{k-j+1}N} \le \mathbb{E}\ k\text{-}\min_{1\le i\le n} |x_i\xi_i| \le c_2 \max_{1\le j\le k} \|\,(1/x_i)_{i=j}^n\,\|^{-1}_{\frac{2e}{k-j+1}N},$$

and for $1 \leq k \leq cn$

$$
\begin{aligned}
& c_3 \left(\max_{0 \leq \ell \leq ck-1} \| (1/x_i)_{i=1}^{k+\ell} \|^{-1}_{\frac{2e}{\ell+1}N} + \|(x_{k+ck}, \dots, x_n)\|_M \right) \\
& \leq \mathbb{E} \operatorname*{k-max}_{1 \leq i \leq n} |x_i \xi_i| \\
& \leq c_4 \left(\max_{0 \leq \ell \leq ck-1} \| (1/x_i)_{i=1}^{k+\ell} \|^{-1}_{\frac{2e}{\ell+1}N} + \|(x_{k+ck}, \dots, x_n)\|_M \right),
\end{aligned}
$$

where N, M are *Orlicz functions* (see the definitions below) and $\| \cdot \|_N$, $\| \cdot \|_M$ are the corresponding Orlicz norms. The Orlicz functions N, M are computed in terms of the distribution function of the random variables under consideration. The constants c, c_1, c_2, c_3, c_4 depend – mildly – on the distribution function of the random variables and on k (of the order of $\ln k$ or $1/\ln k$), but – and this is the important point – they do not depend in any way on the number n and on the scalars $x_1, \dots, x_n$. The precise statements are given in Section 3. We would like to note that Orlicz functions appear naturally in the connection with log-concave distributions. For example in the important work of Gluskin and Kwapień [14] Orlicz functions were used to obtain tail and moment estimates for sums of independent random variables. Recently, Latala [29] proved tail comparison theorem for log-concave vectors.

In problems where only a small number of random variables is involved, numerical computations will give sufficient estimates for order statistics. However, in the case when a large number of random variables is involved, numerical computations may not be feasible. Our formulae allow easy computations also in that situation.

Finally let us mention that throughout this paper we use the following notation. For a random variable ξ on a probability space $(\Omega, \mathfrak{A}, \mathbb{P})$ we denote its distribution function by G_ξ and $1 - G_\xi$ by F_ξ

$$
G_\xi(t) = \mathbb{P}(\{\xi \leq t\}) \qquad \text{and} \qquad F_\xi(t) = \mathbb{P}(\{\xi > t\}).
$$

Acknowledgment. We would like to thank Hermann König, Kiel, for discussions.

2 Preliminaries. Orlicz functions and norms.

In this section we recall some facts about Orlicz functions and norms. For more details and other properties of Orlicz spaces we refer to [26, 30, 35].

A left continuous convex function $M : [0, \infty) \to [0, \infty]$ is called Orlicz function or Young function, if $M(0) = 0$ and if M is neither the function that is constant 0 nor the function that takes the value 0 at 0 and is ∞ elsewhere. The corresponding Orlicz norm on $\mathbb{R}^n$ is defined by

$$
\|x\|_M = \inf \left\{ \rho > 0 \,\middle|\, \sum_{i=1}^n M(|x_i|/\rho) \leq 1 \right\}. \tag{3}
$$

Note that the expression for $\|\cdot\|_M$ makes also sense if the function M is merely positive and increasing. Although in that case the expression need not be a norm, we keep the same notation $\|\cdot\|_M$. We often use formula (3) in a slightly different form, namely

$$1/\|x\|_M = \sup\left\{\rho > 0 \,\middle|\, \sum_{i=1}^{n} M\left(\rho\,|x_i|\right) \le 1\right\}.$$

Clearly, $M \le \bar{M}$ implies $\|\cdot\|_M \le \|\cdot\|_{\bar{M}}$. Moreover, if M is an Orlicz function and $s \ge 1$, then

$$sM(t) \le M(st) \tag{4}$$

for every $t \ge 0$. In particular, this implies

$$\|\cdot\|_{sM} \le s\|\cdot\|_M. \tag{5}$$

The dual function M^* to an Orlicz function M is defined by

$$M^*(s) = \sup_{0\le t<\infty} (t\cdot s - M(t)).$$

For instance, for $M(t) = \frac{1}{q}\, t^q$, $q \ge 1$, the dual function is $M^*(t) = \frac{1}{q^*}\, t^{q^*}$ with $\frac{1}{q} + \frac{1}{q^*} = 1$.

Let the function $p = p_M : [0,\infty) \to [0,\infty]$ be given by

$$p(t) = \begin{cases} 0 & t = 0 \\ M'(t) & M(t) < \infty \\ \infty & M(t) = \infty, \end{cases}$$

where M' is the left hand side derivative of M. Then p is increasing and the left hand side inverse q of the increasing function p is

$$q(s) = \inf\{t \in [0,\infty) \mid p(t) > s\}.$$

Then

$$M^*(s) = \int_0^s q(t)dt.$$

To a given random variable ξ we associate an Orlicz function $M = M_\xi$ in the following way:

$$M(s) = \int_0^s \int_{\frac{1}{t}\le|\xi|} |\xi| d\mathbb{P}dt = \int_{\frac{1}{s}\le|\xi|} \left(s|\xi| - 1\right) d\mathbb{P}. \tag{6}$$

The equality here follows by changing the order of integration and the convexity of M follows by the definition of convexity. We prefer to keep in mind both formulae for M. Note that equivalently one can write

$$M(s) = \mathbb{E}\left(s|\xi| - 1\right)_+,$$

where, as usual, $h_+(x)$ denotes $h(x)$ if $h(x) \geq 0$ and 0 otherwise.

We claim that the dual function $M^* = M^*_\xi$ is given on $[0, \int |\xi| d\mathbb{P}]$ by

$$M^*\left(\int_{t \leq |\xi|} |\xi| d\mathbb{P}\right) = \mathbb{P}(|\xi| \geq t) \tag{7}$$

and $M^*(s) = \infty$ for $s > \int |\xi| d\mathbb{P}$.

Indeed, by definition

$$M^*(s) = \sup_{0 \leq w}(w \cdot s - M(w)) = \sup_{0 \leq w}\left(w \cdot s - \int_0^w \int_{\frac{1}{u} \leq |\xi|} |\xi| d\mathbb{P} du\right)$$

$$= \sup_{0 \leq w} \int_0^w \left(s - \int_{\frac{1}{u} \leq |\xi|} |\xi| d\mathbb{P}\right) du.$$

If $s > \int |\xi| d\mathbb{P}$ then the supremum is equal to ∞. Now fix $t \geq 0$, set

$$s = \int_{t \leq |\xi|} |\xi| d\mathbb{P}$$

and consider the function

$$\phi(w) := \int_0^w \left(s - \int_{\frac{1}{u} \leq |\xi|} |\xi| d\mathbb{P}\right) du.$$

It is easy to see that ϕ is increasing on $[0, 1/t]$ and decreasing on $[1/t, \infty)$. Therefore,

$$M^*(s) = \sup_{0 \leq w} \phi(w) = \phi(1/t) = \int_0^{1/t} \int_{t \leq |\xi| < 1/u} |\xi| \; d\mathbb{P} \; du.$$

Changing the order of integration we obtain

$$M^*(s) = \int_{t \leq |\xi|} \int_0^{1/|\xi|} |\xi| \; du \; d\mathbb{P} = \int_{t \leq |\xi|} d\mathbb{P} = \mathbb{P}(|\xi| \geq t),$$

which proves (7).

In the Gaussian case we have

$$F(t) = \mathbb{P}\left(\{|\xi| > t\}\right) = \sqrt{\tfrac{2}{\pi}} \int_t^\infty e^{-\frac{s^2}{2}} ds$$

and thus

$$M(s) = \sqrt{\tfrac{2}{\pi}} \int_0^s \int_{\frac{1}{t}}^\infty u e^{-\frac{u^2}{2}} du dt = \sqrt{\tfrac{2}{\pi}} \int_0^s e^{-\frac{1}{2t^2}} dt. \tag{8}$$

This implies that on the interval $[0, \sqrt{2/\pi}]$ M^* is given by

$$M^*(s) = \int_0^s \frac{1}{\sqrt{2\ln\left(\sqrt{\frac{2}{\pi}}\frac{1}{u}\right)}} du.$$

For $s > \sqrt{2/\pi}$, $M^*(s) = \infty$.

3 The main results

Now we consider certain functions associated with a random variable $\xi : \Omega \to \mathbb{R}$.

The function $F : [0, \infty) \to [0, \infty)$ is given by

$$F(t) = \mathbb{P}(|\xi| > t). \tag{9}$$

We assume that F is strictly decreasing on $[0, \infty)$ and $F(0) = 1$. In particular, F is invertible.

The function $N : [0, \infty) \to [0, \infty)$ is defined by

$$N(t) = \ln \frac{1}{F(t)} \tag{10}$$

and is assumed to be convex. In particular, N is an Orlicz function. For such a function N and $k \in \mathbb{N}$ we put

$$N_j = \frac{2e}{k-j+1} N, \qquad j = 1, \ldots, k. \tag{11}$$

Furthermore, let us observe that under assumptions above for all $t \geq 0$ and all $s \geq 1$ we have

$$F(st) \leq F(t)^s. \tag{12}$$

Indeed, by (4) we have $sN(t) \leq N(st)$, i.e. $-s \ln F(t) \leq -\ln F(st)$, which is equivalent to (12).

The following theorem generalizes results from [20, 21], where similar estimates were obtained for Gaussian distributions. Of course, the Gaussian case is simpler and the corresponding formulae are less involved. We discuss the details in Remark 1 after the theorem.

Theorem 1 *Let $1 \leq k \leq \frac{n}{2}$ and let $\xi_1, \ldots, \xi_n$ be i.i.d. copies of a random variable ξ. Let F, N and N_j, $j = 1, \ldots, k$, be as specified in (9), (10) and (11). Then for all $0 < x_1 \leq x_2 \leq \cdots \leq x_n$*

$$c_1 \max_{1 \leq j \leq k} \| (1/x_i)_{i=j}^n \|_{N_j}^{-1} \leq \mathbb{E}\, k\text{-}\min_{1 \leq i \leq n} |x_i \xi_i| \leq 16e^2\, C_N\, \ln(k+1) \max_{1 \leq j \leq k} \| (1/x_i)_{i=j}^n \|_{N_j}^{-1},$$

where $c_1 = 1 - \frac{1}{\sqrt{2\pi}}$ and $C_N = \max\{N(1), 1/N(1)\}$.

Moreover, the lower estimate does not require the condition "N is an Orlicz function".

Remark (the Gaussian case). In [20, 21] it was shown that for $N(0,1)$ random variables g_i, $i=1,\ldots,n$ and for all $0<x_1\le x_2\le\cdots\le x_n$

$$(13)\quad c_0 \max_{1\le j\le k} \frac{k+1-j}{\sum_{i=j}^n 1/x_i} \le \mathbb{E}\ k\text{-}\min_{1\le i\le n} |x_i g_i| \le 2\sqrt{2\pi}\ \ln(k+1) \max_{1\le j\le k} \frac{k+1-j}{\sum_{i=j}^n 1/x_i}.$$

where $c_0=\left(1-\frac{1}{\sqrt{2\pi}}\right)\frac{1}{2e}\sqrt{\frac{\pi}{2}}$. In Section 6 we show that the Gaussian distribution satisfies the conditions of Theorem 1. Thus the estimate (13) can be obtained from Theorem 1 (with different absolute constants).

Our second theorem provides bounds for expectations of k-max. As in Theorem 1 we assume that F is strictly decreasing, $F(0)=1$, and that $N=-\ln F$ is a convex function, where F is given by (9). Note that such a function F satisfies

$$(14)\qquad \int_{t\le|\xi_1|} |\xi_1| d\mathbb{P} \le \left(1+\frac{1}{N(t)}\right) t\cdot F(t)$$

for all positive t. We verify this. Since $F=e^{-N}$ and N is convex

$$\int_{t\le|\xi_1|} |\xi_1| d\mathbb{P} = t\cdot F(t) + \int_t^\infty F(s)ds = t\cdot F(t) + \int_t^\infty e^{-N(s)}ds.$$

Using (4), we have $N(s)\ge \frac{s}{t}N(t)$ for $s\ge t$. Therefore

$$\begin{aligned}\int_{t\le|\xi_1|} |\xi_1| d\mathbb{P} &\le t\cdot F(t) + \int_t^\infty e^{-\frac{s}{t}N(t)}ds \\ &\le t\cdot F(t) + \frac{t}{N(t)}e^{-N(t)} = t\cdot F(t) + \frac{t}{N(t)}F(t),\end{aligned}$$

which implies (14).

Theorem 2 *Let $\xi_1,\ldots,\xi_n$ be i.i.d. copies of a random variable ξ. Let F, M, and N be as specified in (9), (6), and (10). Let $1<k\le n$ and $k_0=\left[\frac{4(k-1)}{F(1)}\right]$. Assume that $k+k_0\le n$. Then for all $x_1\ge x_2\ge\cdots\ge x_n>0$*

$$\frac{1}{4}\left(\max_{0\le\ell\le k_0-1} \|\,(1/x_i)_{i=1}^{k+\ell}\,\|^{-1}_{\frac{2e}{\ell+1}N} + \left(1+\frac{\ln(8(k-1))}{N(1)}\right)^{-1} \|(x_{k+k_0},\ldots,x_n)\|_M\right)$$

$$\le \mathbb{E}\,\text{k-max}_{1\le i\le n}|x_i\xi_i| \le c\left(C_N \ln(k+1) \max_{0\le\ell\le k_0-1} \|\,(1/x_i)_{i=1}^{k+\ell}\,\|^{-1}_{\frac{2e}{\ell+1}N} + \|(x_{k+k_0},\ldots,x_n)\|_M\right),$$

where $C_N=\max\{N(1),1/N(1)\}$, and c is an absolute positive constant.

Remark. The case $k = 1$ was obtained in [18] (see also Lemma 11 below): Let

$$M(s) = \int_0^s \int_{\frac{1}{t} \le |\xi_1|} |\xi_1| d\mathbb{P} dt = \int_{\frac{1}{s} \le |\xi|} (s|\xi| - 1)\, d\mathbb{P}.$$

Then, for all $x \in \mathbb{R}^n$ one has

$$c_1 \|x\|_M \le \int_\Omega \max_{1 \le i \le n} |x_i \xi_i(\omega)| d\mathbb{P}(\omega) \le c_2 \|x\|_M,$$

In particular, in the Gaussian case (8),

$$M(s) = \sqrt{\tfrac{2}{\pi}} \int_0^s e^{-\frac{1}{2t^2}} dt.$$

4 k-min

We need the following two simple lemmas. Similar lemmas were used in [20, 21]. For the sake of completeness we provide the proofs.

Lemma 3 *Let* $0 < x_1 \le x_2 \le ... \le x_n$. *Let* $\xi_1, \dots, \xi_n$ *be i. d. random variables. Let* $F(t) = \mathbb{P}\{|\xi_1| > t\}$ *and* $G(t) = 1 - F(t)$. *Then*

$$\mathbb{P}\left\{ \min_{1 \le i \le n} |x_i \xi_i| \le t \right\} \le \sum_{i=1}^n G\left(t/x_i\right).$$

Moreover, if the ξ_i*'s are independent then for every* $t > 0$

$$\mathbb{P}\left\{ \min_{1 \le i \le n} |x_i \xi_i| > t \right\} = \prod_{i=1}^n F\left(t/x_i\right).$$

Proof. Denote $A_k(t) = \{\omega \mid |x_k \xi_k(\omega)| > t\} = \{\omega \mid |\xi_k(\omega)| > t/x_k\}$ and

$$A(t) = \{\omega \mid \min_{k \le n} |x_k \xi_k(\omega)| > t\} = \bigcap_{k \le n} A_k(t).$$

Then

$$\mathbb{P}\left(A(t)\right) \ge 1 - \sum_{k=1}^n \mathbb{P}\left(A_k(t)^c\right) = 1 - \sum_{k=1}^n G\left(t/x_k\right),$$

which proves the first estimate. The second estimate is trivial. $\square$

For the second lemma we need the following Proposition, proved in [20].

Proposition 4 *Let $1 \le k \le n$. Let a_i, $i = 1, \dots, n$ be nonnegative real numbers. Assume*

$$0 < a := \frac{e}{k} \sum_{i=1}^{n} a_i < 1.$$

Then

$$\sum_{l=k}^{n} \sum_{\substack{A \subset \{1,2,\dots,n\} \\ |A|=l}} \prod_{i \in A} a_i < \frac{1}{\sqrt{2\pi k}} \frac{a^k}{1-a}.$$

Lemma 5 *Let $1 \le k \le n$. Let $0 < x_1 \le x_2 \le \dots \le x_n$ and $\xi_1, \dots, \xi_n$ be i.i.d. random variables. Let $G(t) = \mathbb{P}\{|\xi_1| \le t\}$ and*

$$a = a(t) = \frac{e}{k} \sum_{i=1}^{n} G\left(t/x_i\right).$$

Assume that t is such that $0 < a < 1$. Then

$$\mathbb{P}\left\{k\text{-}\min_{1 \le i \le n} |x_i \xi_i| \le t\right\} \le \frac{1}{\sqrt{2\pi k}} \frac{a^k}{1-a}. \tag{15}$$

Remark. Note that if G is continuous and $G(s) = 0$ if and only if $s = 0$ then the condition on t in Lemma 5 above corresponds to the condition

$$0 < t < \|(1/x_i)_{i=1}^n\|_H^{-1},$$

where $H = \frac{e}{k} G$.

Proof of Lemma 5. We have

$$\begin{aligned}
&\mathbb{P}\left\{\omega \,\middle|\, k\text{-}\min_{1 \le i \le n} |x_i \xi_i(\omega)| \le t\right\} \\
&= \mathbb{P}\left\{\omega \,\middle|\, \exists i_1, \dots, i_k \ge 1 : |\xi_{i_j}(\omega)| \le \frac{t}{x_{i_j}}\right\} \\
&= \mathbb{P}\left(\bigcup_{\ell=k}^{n} \bigcup_{\substack{A \subset \{1,\dots,n\} \\ |A|=\ell}} \left\{\omega \,\middle|\, \forall i \in A : |\xi_i(\omega)| \le \frac{t}{x_i} \text{ and } \forall i \notin A : |\xi_i(\omega)| > \frac{t}{x_i}\right\}\right) \\
&= \sum_{l=k}^{n} \sum_{\substack{A \subset \{1,\dots,n\} \\ |A|=l}} \prod_{i \in A} \mathbb{P}\left\{\omega \,\middle|\, |\xi_i(\omega)| \le \frac{t}{x_i}\right\} \prod_{i \notin A} \mathbb{P}\left\{\omega \,\middle|\, |\xi_i(\omega)| > \frac{t}{x_i}\right\}.
\end{aligned}$$

It follows

$$\begin{aligned}
\mathbb{P}\left\{\omega \,\middle|\, k\text{-}\min_{1 \le i \le n} |x_i \xi_i(\omega)| \le t\right\} &\le \sum_{l=k}^{n} \sum_{\substack{A \subset \{1,\dots,n\} \\ |A|=l}} \prod_{i \in A} \mathbb{P}\left\{\omega \,\middle|\, |\xi_i(\omega)| \le \frac{t}{x_i}\right\} \\
&\le \sum_{l=k}^{n} \sum_{\substack{A \subset \{1,\dots,n\} \\ |A|=l}} \prod_{i \in A} G\left(t/x_i\right).
\end{aligned}$$

Proposition 4 implies the desired result. $\square$

Lemma 6 *Let $H : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. For every k with $1 \leq k \leq n$ and every $0 < x_1 \leq x_2 \leq \cdots \leq x_n$ there is a partition of nonempty sets $A_1, \ldots, A_k$ of the set $\{1, \ldots, n\}$ such that*

$$\min_{1\leq j\leq k} \left\| \left(\frac{1}{x_i}\right)_{i=j}^{n} \right\|_{\frac{H}{k-j+1}} \leq 4\max\{H(1), 1/H(1)\} \min_{1\leq j\leq k} \left\| \left(\frac{1}{x_i}\right)_{i\in A_j} \right\|_{H}. \tag{16}$$

We want to emphasize that it is important that the partition consists of exactly k sets. Our proof shows that the partition can be taken as intervals, that is $A_j = \{n_j + 1, \ldots, n_{j+1}\}$ for an increasing sequence $0 = n_0 < n_1 < \ldots < n_k = n$.

Proof. We may assume that $H(1) = 1$. Indeed, as H is convex and as $H(0) = 0$, $H(s) \leq \frac{s}{t}H(t)$ for all $0 < s < t$. Thus if $H(1) \leq 1$, then

$$H(1)\|y\|_{\frac{H}{H(1)}} \leq \|y\|_H \leq \|y\|_{\frac{H}{H(1)}}$$

for every $y \in \mathbb{R}^n$. Similarly, if $H(1) > 1$

$$\|y\|_{\frac{H}{H(1)}} \leq \|y\|_H \leq H(1)\|y\|_{\frac{H}{H(1)}}.$$

We consider three cases.

Case 1:

$$\frac{1}{x_1} \leq \frac{1}{4} \left\| \left(\frac{1}{x_i}\right)_{i=1}^{n} \right\|_{\frac{H}{k}}. \tag{17}$$

Note that $H(1) = 1$ implies $t = \|(t, 0, \ldots, 0)\|_H$ for every $t > 0$, in particular $1/x_1 = \|(1/x_1, 0, \ldots, 0)\|_H$. We put $n_0 = 0$ and after having chosen $n_0, \ldots, n_\ell < n$ we define $n_{\ell+1} \leq n$ to be the largest integer such that

$$\left\| \left(\frac{1}{x_i}\right)_{i=n_\ell+1}^{n_{\ell+1}} \right\|_{H} \leq \frac{1}{2} \left\| \left(\frac{1}{x_i}\right)_{i=1}^{n} \right\|_{\frac{H}{k}}. \tag{18}$$

We define

$$B_\ell = \{n_{\ell-1} + 1, \ldots, n_\ell\}, \qquad \ell = 1, \ldots, L.$$

These sets are basically the partition we are looking for, except for a slight change that is necessary in order to get exactly k sets.

We verify first that such a partition exists. For this we have to show that each B_ℓ contains at least one element, i.e. $B_\ell \neq \emptyset$. In other words, we show that $0 = n_0 < n_1 < \ldots < n_L = n$. Indeed, if $n_{l-1} < n$, then $n_{\ell-1} + 1 \in B_\ell$ because

$$\frac{1}{4} \left\| \left(\frac{1}{x_i}\right)_{i=1}^{n} \right\|_{\frac{H}{k}} \geq \frac{1}{x_1} \geq \frac{1}{x_{n_{\ell-1}+1}} = \left\| \left(0, \ldots, 0, \frac{1}{x_{n_{\ell-1}+1}}, 0, \ldots, 0\right) \right\|_{H}.$$

In the last equality we used again that $H(1)=1$. Thus $B_\ell \neq \emptyset$ and $n_L = n$ which means that the partition is well defined.

We show now that $L > k$. By (18) for every $\varepsilon \in (0,1)$ and for $\ell = 0, \dots, L-1$ we have

$$\sum_{i=n_\ell+1}^{n_{\ell+1}} H\left((2-\varepsilon)\left\|\left(\frac{1}{x_i}\right)_{i=1}^n\right\|_{\frac{H}{k}}^{-1}\frac{1}{x_i}\right) \le 1,$$

which implies

$$\sum_{i=1}^{n} H\left((2-\varepsilon)\left\|\left(\frac{1}{x_i}\right)_{i=1}^n\right\|_{\frac{H}{k}}^{-1}\frac{1}{x_i}\right) \le L.$$

Therefore

$$\left\|\left(\frac{1}{x_i}\right)_{i=1}^n\right\|_{\frac{H}{L}} \le \frac{1}{2}\left\|\left(\frac{1}{x_i}\right)_{i=1}^n\right\|_{\frac{H}{k}}.$$

This implies $L > k$ and below we use that the inequality is strict.

We claim that for all $\ell = 1, \dots, k$ one has

$$\left\|\left(\frac{1}{x_i}\right)_{i\in B_\ell}\right\|_{H} \ge \frac{1}{4}\left\|\left(\frac{1}{x_i}\right)_{i=1}^n\right\|_{\frac{H}{k}}. \tag{19}$$

Suppose that there is ℓ with $1 \le \ell \le k$ such that

$$\left\|\left(\frac{1}{x_i}\right)_{i\in B_\ell}\right\|_{H} < \frac{1}{4}\left\|\left(\frac{1}{x_i}\right)_{i=1}^n\right\|_{\frac{H}{k}}. \tag{20}$$

Since $L > k \ge \ell$ we have $n_\ell + 1 \le n$. $\|\cdot\|_H$ is a norm. Therefore, by the triangle inequality and since $H(1)=1$,

$$\left\|\left(\frac{1}{x_i}\right)_{i=n_{\ell-1}+1}^{n_\ell+1}\right\|_{H} \le \left\|\left(\frac{1}{x_i}\right)_{i\in B_\ell}\right\|_{H} + \left\|\left(0,\dots,0,\frac{1}{x_{n_\ell+1}}\right)\right\|_{H} = \left\|\left(\frac{1}{x_i}\right)_{i\in B_\ell}\right\|_{H} + \frac{1}{x_{n_\ell+1}}.$$

By (17) and (20)

$$\left\|\left(\frac{1}{x_i}\right)_{i=n_{\ell-1}+1}^{n_\ell+1}\right\|_{H} < \frac{1}{2}\left\|\left(\frac{1}{x_i}\right)_{i=1}^n\right\|_{\frac{H}{k}}.$$

This contradicts the definition of n_ℓ.

Now we define the partition $A_1, \dots, A_k$. We put $A_\ell = B_\ell$ for $1 \le \ell \le k-1$ and

$$A_k = \bigcup_{\ell=k}^{L} B_\ell.$$

Then, by (19),

$$\min_{1\le j\le k}\left\|\left(\frac{1}{x_i}\right)_{i=j}^n\right\|_{\frac{H}{k-j+1}} \le \left\|\left(\frac{1}{x_i}\right)_{i=1}^n\right\|_{\frac{H}{k}}$$

$$\le 4\min_{1\le \ell\le k}\left\|\left(\frac{1}{x_i}\right)_{i\in B_\ell}\right\|_H \le 4\min_{1\le \ell\le k}\left\|\left(\frac{1}{x_i}\right)_{i\in A_\ell}\right\|_H, \tag{21}$$

which proves (16).

Case 2:

$$\frac{1}{x_1} > \frac{1}{4}\left\|\left(\frac{1}{x_i}\right)_{i=1}^n\right\|_{\frac{H}{k}} \qquad \text{and for all } \ j\le k \ \text{ one has } \ \frac{1}{x_j} > \frac{1}{4}\left\|\left(\frac{1}{x_i}\right)_{i=j}^n\right\|_{\frac{H}{k+1-j}}.$$

We choose $A_j=\{j\}$ for $j=1,\dots,k-1$ and $A_k=\{k,\dots,n\}$. Then for every $j\le k$

$$\left\|\left(\frac{1}{x_i}\right)_{i\in A_j}\right\|_H \ge \frac{1}{x_j} > \frac{1}{4}\left\|\left(\frac{1}{x_i}\right)_{i=j}^n\right\|_{\frac{H}{k+1-j}},$$

which proves (16).

Case 3:

$$\frac{1}{x_1} > \frac{1}{4}\left\|\left(\frac{1}{x_i}\right)_{i=1}^n\right\|_{\frac{H}{k}} \qquad \text{and there exists } \ j\le k \ \text{ such that } \ \frac{1}{x_j} \le \frac{1}{4}\left\|\left(\frac{1}{x_i}\right)_{i=j}^n\right\|_{\frac{H}{k+1-j}}.$$

Let m be the smallest integer such that $m>1$ and

$$\frac{1}{x_m} \le \frac{1}{4}\left\|\left(\frac{1}{x_i}\right)_{i=m}^n\right\|_{\frac{H}{k+1-m}}. \tag{22}$$

For $1\le \ell < m$ we choose $A_\ell=\{\ell\}$. Then

$$\left\|\left(\frac{1}{x_i}\right)_{i\in A_\ell}\right\|_H = \frac{1}{x_\ell} > \frac{1}{4}\left\|\left(\frac{1}{x_i}\right)_{i=\ell}^n\right\|_{\frac{H}{k+1-\ell}}$$

and therefore

$$\min_{1\le j<m}\left\|\left(\frac{1}{x_i}\right)_{i=j}^n\right\|_{\frac{H}{k-j+1}} \le 4\min_{1\le j<m}\left\|\left(\frac{1}{x_i}\right)_{i\in A_j}\right\|_H.$$

Now we consider the sequence $0<x_m\le x_{m+1}\le\cdots\le x_n$ and proceed as in *Case 1*. The assumption of *Case 1* is fulfilled by (22). The procedure of *Case 1* gives a partition $A_m,\dots,A_k$ of $\{m,\dots,n\}$ satisfying (21)

$$4\min_{m\le \ell\le k}\left\|\left(\frac{1}{x_i}\right)_{i\in A_\ell}\right\|_H \ge \left\|\left(\frac{1}{x_i}\right)_{i=m}^n\right\|_{\frac{H}{k+1-m}}.$$

This completes the proof. $\Box$

Lemma 7 *Let $p > 0$, $1 \le k \le n$, and $0 < x_1 \le x_2 \le ... \le x_n$. Let $\xi_1, \dots, \xi_n$ be i.i.d. random variables, $F(t) = \mathbb{P}(\{|\xi_1| > t\})$, $N(t) = \ln \frac{1}{F(t)}$, and $N_j = \frac{2e}{k-j+1} N$, $j = 1, \dots, k$. Then*

$$\left(1 - \tfrac{1}{\sqrt{2\pi}}\right) \max_{1 \le j \le k} \| (1/x_i)_{i=j}^n \|_{N_j}^{-p} \le \mathbb{E}\, \text{k-min}_{1\le i\le n} |x_i \xi_i|^p.$$

Proof. Let $c = \left(1 - \frac{1}{\sqrt{2\pi}}\right)^{1/p}$. It is enough to show that for every $k \le n$

$$c \| (1/x_i)_{i=1}^n \|_{N_1}^{-1} \le \left(\mathbb{E}\, k\text{-} \min_{1 \le i \le n} |x_i \xi_i|^p \right)^{1/p}. \tag{23}$$

Indeed, assume that (23) is true. Fix $j \le k$. Since

$$\mathbb{E}\ k\text{-}\min_{1\le i\le n} |x_i\xi_i|^p \ge \mathbb{E}\ (k-j+1)\text{-} \min_{j\le i\le n} |x_i\xi_i|^p,$$

(23) implies

$$\left(\mathbb{E}\ k\text{-}\min_{1\le i\le n} |x_i\xi_i|^p\right)^{1/p} \ge c \| (1/x_i)_{i=j}^n \|_{N_j}^{-1},$$

for all $1 \le j \le k$.

Now we show estimate (23). Fix $\varepsilon > 0$ small enough and put

$$A = \| (1/x_i)_{i=1}^n \|_{N_1}^{-1} - \varepsilon.$$

We use that $1 - t \le -\ln t$ for $t > 0$ and that $N_1 = \frac{2e}{k} N = \frac{2e}{k} \ln \frac{1}{F}$ and we obtain

$$\begin{aligned} a &:= \tfrac{e}{k} \sum_{i=1}^n G(A/x_i) = \tfrac{e}{k} \sum_{i=1}^n (1 - F(A/x_i)) \\ &\le \tfrac{e}{k} \sum_{i=1}^n \ln \frac{1}{F(A/x_i)} = \frac{1}{2} \sum_{i=1}^n N_1(A/x_i) \le 1/2. \end{aligned}$$

Applying Lemma 5, we get

$$\mathbb{P}\left\{k\text{-}\min_{1\le i\le n} |x_i\xi_i|^p \ge A^p\right\} \ge 1 - \frac{1}{\sqrt{2\pi k}}\ \frac{a^k}{1-a} \ge 1 - \frac{1}{\sqrt{2\pi}},$$

as $a \le 1/2$. This implies

$$\mathbb{E}\ k\text{-}\min_{1\le i\le n} |x_i\xi_i|^p \ge A^p\ \mathbb{P}\left\{k\text{-}\min_{1\le i\le n} |x_i\xi_i|^p \ge A^p\right\} \ge \left(1 - \tfrac{1}{\sqrt{2\pi}}\right)\ A^p.$$

Sending ε to 0 we obtain the desired result. $\Box$

Lemma 8 *Let $p > 0$ and $0 < x_1 \le x_2 \le ... \le x_n$ be real numbers. Let $\xi_1, \dots, \xi_n$ be i.i.d. random variables. Let $F(t) = \mathbb{P}(|\xi_1| > t)$ be strictly decreasing and $N = -\ln F$ be an Orlicz function. Then*

$$\left(1 - \tfrac{1}{\sqrt{2\pi}}\right) \left\| \left(\frac{1}{x_i}\right)_{i=1}^{n} \right\|_{2eN}^{-p} \le \mathbb{E} \min_{1\le i\le n} |x_i \xi_i|^p \le (1 + \Gamma(1+p)) \left\| \left(\frac{1}{x_i}\right)_{i=1}^{n} \right\|_{N}^{-p}.$$

Remark 1. If N is an Orlicz function then by (5)

$$(2e)^{-p} \| \cdot \|_N^{-p} \le \| \cdot \|_{2eN}^{-p}.$$

Remark 2. The left hand side inequality does not require the condition "N is an Orlicz function."

Proof. The left hand inequality follows from Lemma 7.

To prove the right hand side inequality we choose

$$t_0 = \left\| \left(\frac{1}{x_i}\right)_{i=1}^{n} \right\|_{N}^{-p}.$$

Then for all $t \ge t_0$

$$\sum_{i=1}^{n} \ln\left(1/F\left(t^{1/p}/x_i\right)\right) \ge 1.$$

By (12) for all $t \ge t_0$ and all x_i

$$(t/t_0)^{\frac{1}{p}} \ln \frac{1}{F(t_0^{\frac{1}{p}}/x_i)} \le \ln \frac{1}{F(t^{\frac{1}{p}}/x_i)}.$$

By Lemma 3,

$$\mathbb{P}\left\{ \min_{1\le i\le n} |x_i\xi_i|^p > t \right\} = \prod_{i=1}^{n} F\left(t^{\frac{1}{p}}/x_i\right) = \exp\left(-\sum_{i=1}^{n} \ln \frac{1}{F\left(t^{\frac{1}{p}}/x_i\right)} \right),$$

and thus for all $t \ge t_0$

$$\mathbb{P}\left\{ \min_{1\le i\le n} |x_i\xi_i|^p > t \right\} \le \exp\left(-(t/t_0)^{\frac{1}{p}} \sum_{i=1}^{n} \ln \frac{1}{F\left(t_0^{\frac{1}{p}}/x_i\right)} \right) \le \exp\left(-\left(\frac{t}{t_0}\right)^{\frac{1}{p}} \right).$$

Therefore

$$\begin{aligned}
\mathbb{E} \min_{1\le i\le n} |x_i\xi_i|^p &= \int_0^\infty \mathbb{P}\left\{ \min_{1\le i\le n} |x_i\xi_i| > t^{\frac{1}{p}} \right\} dt \\
&= \int_0^{t_0} \mathbb{P}\left\{ \min_{1\le i\le n} |x_i\xi_i| > t^{\frac{1}{p}} \right\} dt + \int_{t_0}^\infty \mathbb{P}\left\{ \min_{1\le i\le n} |x_i\xi_i| > t^{\frac{1}{p}} \right\} dt \\
&\le t_0 + \int_{t_0}^\infty \exp\left(-\left(\frac{t}{t_0}\right)^{\frac{1}{p}} \right) dt.
\end{aligned}$$

We substitute $t = t_0 s^p$, then

$$\mathbb{E} \min_{1\le i\le n} |x_i\xi_i|^p \le t_0 + t_0 p \int_1^\infty s^{p-1} e^{-s} ds \le t_0 \left(1 + p\Gamma(p)\right),$$

which completes the proof. $\square$

Lemma 9 *Let $1 \le k \le n$ and $0 < x_1 \le x_2 \le \ldots \le x_n$. Let $\xi_1, \ldots, \xi_n$ be i.i.d. random variables. Let $F(t) = \mathbb{P}(|\xi_1| > t)$ be strictly decreasing, and let $N = -\ln F$ be an Orlicz function. Let $M_j = N/(k-j+1)$, $j = 1, \ldots, k$. Then*

$$\mathbb{E}\ k\text{-}\min_{1\le i\le n} |x_i\xi_i| \le 8e\ln(k+1)\ C_N \max_{1\le j\le k} \|\,(1/x_i)_{i=j}^n\,\|_{M_j}^{-1}$$

where $C_N = \max\{N(1), 1/N(1)\}$.

Proof. The case $k = 1$ follows form Lemma 8. We assume $k \ge 2$.

Let $A_1, \ldots A_k$ be the partition of $\{1\ldots n\}$ given by Lemma 6. Then for all $q \ge 1$

$$\begin{aligned}
\mathbb{E}\ k\text{-}\min_{1\le i\le n} |x_i\xi_i| &\le \mathbb{E} \max_{1\le j\le k} \min_{i\in A_j} |x_i\xi_i| \le \mathbb{E}\left(\sum_{j=1}^k \left|\min_{i\in A_j} |x_i\xi_i|\right|^q\right)^{\frac{1}{q}} \\
&\le \left(\mathbb{E}\sum_{j=1}^k \left|\min_{i\in A_j} |x_i\xi_i|\right|^q\right)^{\frac{1}{q}} = \left(\sum_{j=1}^k \mathbb{E}\left|\min_{i\in A_j} |x_i\xi_i|\right|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

By Lemma 8 the latter expression is less than

$$(1+\Gamma(1+q))^{\frac{1}{q}} \left(\sum_{j=1}^k \left\|\left(\frac{1}{x_i}\right)_{i\in A_j}\right\|_N^{-q}\right)^{\frac{1}{q}} \le 2qk^{1/q} \max_{1\le j\le k} \left\|\left(\frac{1}{x_i}\right)_{i\in A_j}\right\|_N^{-1}.$$

The choice $q = \ln(k+1)$ gives

$$\mathbb{E}\ k\text{-}\min_{1\le i\le n} |x_i\xi_i| \le 2e\ln(k+1) \max_{1\le j\le k} \left\|\left(\frac{1}{x_i}\right)_{i\in A_j}\right\|_N^{-1}.$$

By Lemma 6 this expression is smaller than

$$8e\ C_N \max_{1\le j\le k} \left\|\left(\frac{1}{x_i}\right)_{i=j}^n\right\|_{M_j}^{-1}.$$

$\square$

Proof of Theorem 1. The lower estimate follows from Lemma 7. Since, by (5),

$$\|\cdot\|_{N_j} \le 2e\|\cdot\|_{M_j},$$

the upper estimate follows by Lemma 9. $\square$

5 k-max

In this section we prove Theorem 2. We require a result from [18]. Let f be a random variable with continuous distribution and such that $\mathbf{E}|f| < \infty$. Let $t_n = t_n(f) = 0$, $t_0 = t_0(f) = \infty$, and for $j = 1, \ldots, n-1$

$$t_j = t_j(f) = \sup \left\{ t \mid \mathbb{P}\{\omega \mid |f(\omega)| > t\} \geq \tfrac{j}{n} \right\}. \tag{24}$$

Since f has the continuous distribution, we have for every $j \geq 1$

$$\mathbb{P}\{\omega \mid |f(\omega)| \geq t_j\} = \tfrac{j}{n}.$$

For $j = 1, \ldots, n$ define the sets

$$\Omega_j = \Omega_j(f) = \{\omega \mid t_j \leq |f(\omega)| < t_{j-1}\}. \tag{25}$$

For all $j = 1, \ldots, n$ we have

$$\Omega_j = \{\omega \mid t_j \leq |f(\omega)| < t_{j-1}\} = \{\omega \mid t_j \leq |f(\omega)|\} \setminus \{\omega \mid t_{j-1} \leq |f(\omega)|\}.$$

Therefore

$$\mathbb{P}(\Omega_j) = \tfrac{j}{n} - \tfrac{j-1}{n} = \tfrac{1}{n}.$$

For $j = 1, \ldots, n$ let

$$y_j = y_j(f) := \int_{\Omega_j} |f(\omega)| d\mathbb{P}(\omega). \tag{26}$$

Then

$$\sum_{j=1}^{n} y_j = \mathbf{E}|f| \qquad \text{and} \qquad t_j \leq n y_j < t_{j-1} \qquad \text{for all} \quad j = 1, \ldots, n.$$

In [18], Corollary 2 we proved the following statement.

Lemma 10 *Let $f_1, \ldots, f_n$ be i.i.d. random variables such that $\int |f_i(\omega)| d\mathbb{P}(\omega) = 1$. Let M be an Orlicz function such that for all $k = 1, \ldots, n$*

$$M^*\left(\sum_{j=1}^{k} y_j \right) = \tfrac{k}{n}.$$

Then, for all $x \in \mathbb{R}^n$

$$c_1 \|x\|_M \leq \int_\Omega \max_{1 \leq i \leq n} |x_i f_i(\omega)| d\mathbb{P}(\omega) \leq c_2 \|x\|_M,$$

where c_1 and c_2 are absolute positive constants.

This can be reformulated in the following way.

Lemma 11 *Let $\xi_1, \ldots, \xi_n$ be i.i.d. random variables such that $\int |\xi_i(\omega)| d\mathbb{P}(\omega) = 1$. Let M be the Orlicz function such that for all $s \geq 0$*

$$M(s) = \int_0^s \int_{\frac{1}{t} \leq |\xi_1|} |\xi_1| d\mathbb{P} dt = \int_{\frac{1}{s} \leq |\xi|} (s|\xi| - 1)\, d\mathbb{P}.$$

Then, for all $x \in \mathbb{R}^n$

$$c_1 \|x\|_M \leq \int_\Omega \max_{1 \leq i \leq n} |x_i \xi_i(\omega)| d\mathbb{P}(\omega) \leq c_2 \|x\|_M,$$

where c_1 and c_2 are absolute positive constants.

Proof. By definition

$$\sum_{i=1}^k y_i = \int_{t_k \leq \xi_1} |\xi_1(\omega)|\ d\mathbb{P}(\omega)$$

and

$$\mathbb{P}\left(\{\omega \ \mid \ t_k \leq \xi_1(\omega)\}\right) = \frac{k}{n}.$$

Therefore the Orlicz function M^* defined by

$$M^* \left(\int_{t \leq |\xi_1|} |\xi_1(\omega)| d\mathbb{P}(\omega) \right) = \mathbb{P}\{\omega | t \leq |\xi_1(\omega)|\}$$

satisfies the condition of Lemma 10. It is left to observe that the dual function to M^* is

$$M(s) = \int_0^s \int_{\frac{1}{t} \leq |\xi_1|} |\xi_1| d\mathbb{P} dt.$$

This has been verified in Section 2 (see formulae (6) and (7)). $\square$

For the next lemma we need the following simple claim.

Claim 12 *Let $(x_i)_{i=1}^n$ be a sequence. Then for every $j \leq n - k$ one has*

$$\operatorname*{k\text{-}max}_{1 \leq i \leq n} |x_i| \leq \operatorname*{j\text{-}min}_{1 \leq i \leq k+j-1} |x_i| + \max_{k+j \leq i \leq n} |x_i|.$$

Proof. If the numbers $|x_1|, \ldots, |x_{k+j-1}|$ contain the k biggest of the numbers $|x_1|, \ldots, |x_n|$, then

$$\operatorname*{j\text{-}min}_{1 \leq i \leq k+j-1} |x_i| = \operatorname*{k\text{-}max}_{1 \leq i \leq k+j-1} |x_i| = \operatorname*{k\text{-}max}_{1 \leq i \leq n} |x_i|.$$

On the other hand, if the numbers $|x_1|,\dots,|x_{k+j-1}|$ do not contain the k biggest of the numbers $|x_1|,\dots,|x_n|$, then at least one of those is contained in the numbers $|x_{k+j}|,\dots,|x_n|$ and therefore

$$\max_{k+j\le i\le n}|x_i| \ge \text{k-}\max_{1\le i\le n}|x_i|.$$

$\square$

Lemma 13 *Let $x_1\ge x_2\ge\cdots\ge x_n>0$. Let $\xi_1,\dots,\xi_n$ be i.i.d. random variables and $F(t)=\mathbb{P}(\{|\xi_1|>t\})$. Suppose that F is strictly decreasing and $N=-\ln F$ is an Orlicz function. Assume that M is the Orlicz function such that for all $s\ge 0$*

$$M(s)=\int_0^s\int_{\frac{1}{t}\le|\xi|}|\xi|d\mathbb{P}dt.$$

Then we have

$$\mathbb{E}\,\text{k-}\max_{1\le i\le n}|x_i\xi_i| \le c\min_{1\le j\le n-k}\left\{C_N\ln(k+1)\max_{0\le\ell\le j-1}\|\,(1/x_i)_{i=1}^{k+\ell}\,\|^{-1}_{\frac{2e}{\ell+1}N}+\|(x_{k+j},\dots,x_n)\|_M\right\},$$

where c is an absolute constant and $C_N=\max\{N(1),1/N(1)\}$.

Proof. Claim 12 implies

$$\mathbb{E}\,\text{k-}\max_{1\le i\le n}|x_i\xi_i| \le \min_{1\le j\le n-k}\left(\mathbb{E}\,\text{j-}\min_{1\le i\le k+j-1}|x_i\xi_i|+\mathbb{E}\max_{k+j\le i\le n}|x_i\xi_i|\right).$$

Applying Theorem 1 to the sequence $x_{k+j-1}\le x_{k+j-2}\le\cdots\le x_1$, we observe that

$$\mathbb{E}\,\text{j-}\min_{1\le i\le k+j-1}|x_i\xi_i| \le 16e^2C_N\ln(k+1)\max_{1\le\ell\le j}\|\,(1/x_i)_{i=1}^{k+j-\ell}\,\|^{-1}_{\frac{2e}{j-\ell+1}N}.$$

This is the same as

$$\mathbb{E}\,\text{j-}\min_{1\le i\le k+j-1}|x_i\xi_i| \le 16e^2C_N\ln(k+1)\max_{0\le\ell\le j-1}\|\,(1/x_i)_{i=1}^{k+\ell}\,\|^{-1}_{\frac{2e}{\ell+1}N}.$$

On the other hand, Lemma 11 implies

$$\mathbb{E}\max_{k+j\le i\le n}|x_i\xi_i| \le c\,\|(x_{k+j},\dots,x_n)\|_M,$$

where c is an absolute constant. This completes the proof. $\square$

Lemma 14 *Let $x_1\ge x_2\ge\cdots\ge x_n>0$. Let $\xi_1,\dots,\xi_n$ be i.i.d. random variables and $F(t)=\mathbb{P}(|\xi_1|>t)$. For $k>1$ let*

$$N_{F,k}(t)=\frac{F(1/t)}{4(k-1)}.$$

Then

$$\mathbb{E}\,\underset{1\le i\le n}{\text{k-max}}\,|x_i\xi_i| \ge \max\left\{\frac{1}{2}\ \|(x_k,\dots,x_n)\|_{N_{F,k}},\ \max_{1\le \ell\le n-k}\mathbb{E}\underset{1\le i\le k+\ell-1}{\ell\text{-min}}|x_i\xi_i|\right\}.$$

In particular, if $N(t)=\ln\frac{1}{F(t)}$*, then*

$$\mathbb{E}\,\underset{1\le i\le n}{\text{k-max}}\,|x_i\xi_i| \ge \max\left\{\frac{1}{2}\ \|(x_k,\dots,x_n)\|_{N_{F,k}},\ (1-\tfrac{1}{\sqrt{2\pi}})\max_{1\le \ell\le n-k}\ \max_{1\le j\le \ell}\ \|\,(1/x_i)_{i=1}^{k+\ell-j}\,\|^{-1}_{\frac{2e}{\ell-j+1}N}\right\}.$$

Proof. First we show

$$\mathbb{E}\,\underset{1\le i\le n}{\text{k-max}}\,|x_i\xi_i| \ge \frac{1}{2}\ \|(x_k,\dots,x_n)\|_{N_{F,k}}.$$

We have

$$\mathbb{P}\left\{\omega\,\middle|\,\underset{1\le i\le n}{\text{k-max}}\,|x_i\xi_i(\omega)|\le t\right\}\le\sum_{j=0}^{k-1}\sum_{\substack{A\subset\{1,\dots n\}\\|A|=j}}\prod_{i\in A}F\left(\frac{t}{x_i}\right)\prod_{i\notin A}\left(1-F\left(\frac{t}{x_i}\right)\right).$$

Since $x_1\ge x_2\ge\cdots\ge x_n>0$ and $|A^c|\ge n-k+1$, we observe

$$\mathbb{P}\left\{\omega\,\middle|\,\underset{1\le i\le n}{\text{k-max}}\,|x_i\xi_i(\omega)|\le t\right\}\le\sum_{j=0}^{k-1}\sum_{|A|=j}\prod_{i\in A}F\left(\frac{t}{x_i}\right)\prod_{i=k}^{n}\left(1-F\left(\frac{t}{x_i}\right)\right).$$

Now we apply the Hardy-Littlewood-Polya inequality ([23]), which states that for non-negative numbers $a_1,\dots,a_m$ one has

$$\sum_{\substack{A\subset\{1,\dots m\}\\|A|=j}}\prod_{i\in A}a_i\le\binom{m}{j}\left(\frac{1}{m}\ \sum_{i=1}^{m}a_i\right)^j\le\frac{1}{j!}\left(\sum_{i=1}^{m}a_i\right)^j.$$

This implies

$$\mathbb{P}\left\{\omega\,\middle|\,\underset{1\le i\le n}{\text{k-max}}\,|x_i\xi_i(\omega)|\le t\right\}\le\sum_{j=0}^{k-1}\frac{1}{j!}\left(\sum_{i=1}^{n}F\left(\frac{t}{x_i}\right)\right)^j\prod_{i=k}^{n}\left(1-F\left(\frac{t}{x_i}\right)\right).$$

Since $F\left(\frac{t}{x_i}\right)\le 1$ and $1-x\le e^{-x}$ for $x\ge 0$, one has

$$\mathbb{P}\left\{\omega\,\middle|\,\underset{1\le i\le n}{\text{k-max}}\,|x_i\xi_i(\omega)|\le t\right\}\le\sum_{j=0}^{k-1}\frac{1}{j!}\left(k-1+\sum_{i=k}^{n}F\left(\frac{t}{x_i}\right)\right)^j\exp\left(-\sum_{i=k}^{n}F\left(\frac{t}{x_i}\right)\right).$$

Let

$$\alpha=\alpha(t)=\frac{1}{k-1}\ \sum_{i=k}^{n}F\left(\frac{t}{x_i}\right).$$

Then

$$\begin{aligned}\mathbb{P}\left\{\omega \left| \underset{1\le i\le n}{\text{k-max}} |x_i\xi_i(\omega)| \le t\right.\right\} \qquad &\le e^{-\alpha(k-1)}\sum_{j=0}^{k-1}\frac{1}{j!}\left((\alpha+1)(k-1)\right)^j \\ &\le e^{-\alpha(k-1)}(1+\alpha)^{k-1}\sum_{j=0}^{k-1}\frac{1}{j!}(k-1)^j \\ &\le e^{-\alpha(k-1)}(1+\alpha)^{k-1}e^{k-1} \\ &= \exp\left((k-1)\left(-\alpha+1+\ln\left(1+\alpha\right)\right)\right).\end{aligned}$$

Now put

$$t_0 := \|(x_k,\dots,x_n)\|_{N_{F,k}} \ge 0.$$

If $t_0 = 0$ we are done. If $t_0 > 0$ then for every $0 < \varepsilon < t_0$

$$\alpha(t_0-\varepsilon) = \frac{1}{k-1}\ \sum_{i=k}^{n} F\left(\frac{t_0-\varepsilon}{x_i}\right) > 4.$$

Since $k > 1$, this implies

$$\mathbb{P}\left\{\omega \left| \underset{1\le i\le n}{\text{k-max}} |x_i\xi_i(\omega)| \le t_0-\varepsilon\right.\right\} \le \exp((k-1)(-3+\ln 5)) \le 1/2.$$

Thus

$$\mathbb{E}\,\underset{1\le i\le n}{\text{k-max}} |x_i\xi_i| \ge (t_0-\varepsilon)\ \mathbb{P}\left\{\omega \left| \underset{1\le i\le n}{\text{k-max}} |x_i\xi_i(\omega)| \ge t_0-\varepsilon\right.\right\} \ge \frac{t_0-\varepsilon}{2}.$$

Letting ε tend to 0 we obtain the first part of the desired estimate.

Now we show the second part of the estimate. We observe that for all $l \le n-k+1$

$$\underset{1\le i\le k+\ell-1}{\ell\text{-min}} |x_i\xi_i(\omega)| = \underset{1\le i\le k+\ell-1}{\text{k-max}} |x_i\xi_i(\omega)|.$$

This implies

$$\mathbb{E}\,\underset{1\le i\le n}{\text{k-max}} |x_i\xi_i| \ge \max_{1\le\ell\le n-k+1} \mathbb{E}\ \underset{1\le i\le k+\ell-1}{\ell\text{-min}} |x_i\xi_i|.$$

Finally, the "In particular" part of Lemma 14 follows from Lemma 7. Note that in Lemma 7 the sequence x is in increasing order while in Lemma 14 it is in decreasing order. □

In the next lemma we provide a lower estimate on $\|\cdot\|_{N_{F,k}}$, appearing in Lemma 14.

Lemma 15 *Let $1 < k \le n$. Let $\xi_1,\dots,\xi_n$ be symmetric i.i.d. random variables with $\int |\xi_i(\omega)|d\mathbb{P}(\omega) = 1$. Let $F(t) = \mathbb{P}(|\xi_1| > t)$ be a strictly decreasing function such that $N = -\ln F$ is an Orlicz function. Let M be the Orlicz function defined by*

$$M(s) = \int_0^s \int_{\frac{1}{t}\le|\xi_1|} |\xi_1| d\mathbb{P} dt.$$

Let

$$N_{F,k}(t) = \frac{F\,(1/t)}{4(k-1)}.$$

Let $k_0 = \left[\frac{4(k-1)}{F(1)}\right]$ *and assume that* $k + k_0 \le n$. *Then for all* $x_1 \ge \cdots \ge x_n > 0$

$$\|(x_{k+k_0}, \ldots, x_n)\|_M \le \left(1 + \frac{\ln(8(k-1))}{N(1)}\right)\ \|(x_k, \ldots, x_n)\|_{N_{F,k}}.$$

Remark. The condition $N = -\ln F$ *is an Orlicz function* in this Lemma can be substituted with the condition *there is a constant* $c_2 \ge 1$ *such that for all* $s \in (0, 1/c_2]$ *and* $t \in (0, 1/(4c_2^2)]$

$$F^{-1}(s)F^{-1}(t) \ge F^{-1}(c_2 st) \tag{27}$$

and such that for all $t \ge F^{-1}(\frac{1}{c_2})$ *we have*

$$\int_{t \le |\xi_1|} |\xi_1| d\mathbb{P} \le c_2 t F(t). \tag{28}$$

Then the conclusion will be

$$\|(x_{k+k_0}, \ldots, x_n)\|_M \le F^{-1}\,(\alpha)\ \ \|(x_k, \ldots, x_n)\|_{N_{F,k}},$$

where $\alpha = 1/(4c_2^2(k-1))$.

Proof. Since both functions $\|\cdot\|_M$ and $\|\cdot\|_{N_{F,k}}$ are homogeneous, we may assume that $\|(x_{k+k_0}, \ldots, x_n)\|_M = 1$. Thus, without loss of generality, we can assume that $\sum_{i=k+k_0}^{n} M(x_i) = 1$ (otherwise we pass from the sequence $\{x_i\}_i$ to $\{x_i/(1+\varepsilon)\}_i$ for an suitably small $\varepsilon > 0$).

We put

$$A := F^{-1}(\alpha) = 1 + \frac{\ln(8(k-1))}{N(1)}.$$

Note that by (4), $N(A) \ge AN(1) \ge \ln 8 > 2$.

Case 1: $x_{k+k_0} \ge 1/A$. Then $x_k \ge x_{k+1} \ge \ldots \ge x_{k+k_0} \ge 1/A$.

Since F is a decreasing function, $1/F^{-1}$ is increasing and

$$\sum_{i=k}^{n} F\left((x_i A)^{-1}\right) \ge \sum_{i=k}^{k+k_0} F\left((x_i A)^{-1}\right) \ge (k_0+1)F(1) > 4(k-1).$$

This means that

$$\|(x_k, \ldots, x_n)\|_{N_{F,k}} \ge 1/A.$$

Case 2: $x_{k+k_0} < 1/A$. Then $1/A > x_{k+k_0} \ge \ldots \ge x_n$.

Since $\int_{\frac{1}{t}\leq|\xi_1|}|\xi_1|d\mathbb{P}$ is an increasing function of t, we observe

$$M(s)=\int_0^s\int_{\frac{1}{t}\leq|\xi_1|}|\xi_1|d\mathbb{P}dt\leq s\int_{\frac{1}{s}\leq|\xi_1|}|\xi_1|d\mathbb{P}.$$

By (14), applied with $t=1/s$, we obtain that for all positive s

$$\int_{1/s\leq|\xi_1|}|\xi_1|d\mathbb{P}\leq\left(1+\frac{1}{N(1/s)}\right)\frac{1}{s}\,F(1/s).$$

Recall that N is increasing and $N(A)>2$. Thus for all $s\leq 1/A$ we have

$$M(s)\leq s\int_{\frac{1}{s}\leq|\xi_1|}|\xi_1|d\mathbb{P}\leq 2\,F(1/s).$$

By the condition of Case 2, $x_i\leq 1/A$ for $i\geq k+k_0$. This implies

$$1=\sum_{i=k+k_0}^{n}M(x_i)\leq 2\sum_{i=k+k_0}^{n}F\left(\frac{1}{x_i}\right). \tag{29}$$

Now, by (4), we have $N(y)\geq\beta N(y/\beta)$ for every $y\geq 0$ and $\beta\geq 1$. Since $N=-\ln F$, we observe

$$F(y)\leq F(y/\beta)^{\beta}$$

for every $y\geq 0$ and $\beta\geq 1$. Since F is decreasing, it implies

$$F(y)\leq F(y/\beta)\;F(1)^{\beta-1}$$

for every $y\geq\beta\geq 1$. Applying the last inequality with $y=1/x_i$ and $\beta=A$, we obtain for every $i\geq k+k_0$

$$F(1/x_i)\leq F(1/(Ax_i))\;F(1)^{A-1}.$$

By (29),

$$\sum_{i=k}^{n}F\left(1/(Ax_i)\right)\geq\frac{1}{F(1)^{A-1}}\sum_{i=k+k_0}^{n}F\left(1/x_i\right)\geq\frac{1}{2F(1)^{A-1}}.$$

Now, by the choice of A,

$$A-1=\frac{\ln(8(k-1))}{\ln(1/F(1))},$$

and hence

$$2F(1)^{A-1}=\frac{1}{4(k-1)}.$$

Thus,

$$\sum_{i=k}^{n}F\left(1/(Ax_i)\right)\geq\frac{1}{4(k-1)},$$

which implies

$$\|(x_k,\dots,x_n)\|_{N_{F,k}} \geq 1/A.$$

It completes the proof. $\square$

Now we prove Theorem 2.

Proof of Theorem 2. Let $k_0 = \left[\frac{4(k-1)}{F(1)}\right]$. First we show the right hand side inequality. With $j = k_0$ in Lemma 13

$$\mathbb{E}\,\underset{1\le i\le n}{\text{k-max}}\,|x_i\xi_i| \le c \min_{1\le j\le n-k}\left\{C_N \ln(k+1) \max_{0\le \ell\le j-1} \|\,(1/x_i)_{i=1}^{k+\ell}\,\|^{-1}_{\frac{2e}{\ell+1}N} + \|(x_{k+j},\dots,x_n)\|_M\right\}.$$

To show the left hand side inequality we apply Lemma 14 with $l = k_0$:

$$\mathbb{E}\,\underset{1\le i\le n}{\text{k-max}}\,|x_i\xi_i| \ge \max\left\{\frac{1}{2}\,\|(x_k,\dots,x_n)\|_{N_{F,k}}, \left(1-\tfrac{1}{\sqrt{2\pi}}\right) \max_{1\le j\le k_0} \|\,(1/x_i)_{i=1}^{k+k_0-j}\,\|^{-1}_{\frac{2e}{k_0-j+1}N}\right\}.$$

By Lemma 15 we have for all x with $x_1 \ge \dots \ge x_n > 0$

$$\|(x_{k+k_0},\dots,x_n)\|_M \le A\,\|(x_k,\dots,x_n)\|_{N_{F,k}},$$

where $A = 1 + \frac{\ln(8(k-1))}{N(1)}$. Thus

$$\begin{aligned}
\mathbb{E}\,\underset{1\le i\le n}{\text{k-max}}\,|x_i\xi_i| &\ge \frac{1}{4A}\,\|(x_{k+k_0},\dots,x_n)\|_M + \frac{1}{2}\left(1-\tfrac{1}{\sqrt{2\pi}}\right) \max_{1\le j\le k_0} \|\,(1/x_i)_{i=1}^{k+k_0-j}\,\|^{-1}_{\frac{2e}{k_0-j+1}N}\\
&= \frac{1}{4A}\,\|(x_{k+k_0},\dots,x_n)\|_M + \frac{1}{2}\left(1-\tfrac{1}{\sqrt{2\pi}}\right) \max_{0\le \ell\le k_0-1} \|\,(1/x_i)_{i=1}^{k+\ell}\,\|^{-1}_{\frac{2e}{\ell+1}N}.
\end{aligned}$$

$\square$

6 The Gaussian case

In this section we verify that Gaussian $N(0,1)$-variables satisfy the hypotheses of Theorems 1 and 2. Hence in this section we consider only standard Gaussian variables and we denote them by ξ. Accordingly,

$$F(t) = \mathbb{P}\left(\{|\xi| > t\}\right) = \sqrt{\tfrac{2}{\pi}}\int_t^\infty e^{-\frac{s^2}{2}}\,ds$$

for $t \ge 0$ and $N = \ln\frac{1}{F} = -\ln F$. We check the properties of the functions F and N in several claims.

Claim 16 *For every $t > 0$ and $A > 0$*

$$F(t) = \sqrt{\tfrac{2}{\pi}} \int_t^\infty e^{-\frac{s^2}{2}} ds \leq \sqrt{\tfrac{2}{\pi}}\ \frac{1}{t}\ \exp\left(-\frac{t^2}{2}\right) \tag{30}$$

and

$$\sqrt{\tfrac{2}{\pi}}\ A\ \exp\left(-\left(t+A\right)^2/2\right) \leq F(t). \tag{31}$$

In particular, for $A = 1/t$

$$\sqrt{\tfrac{2}{\pi}}\ \frac{1}{et}\ \exp\left(-\left(t^2 + 1/t^2\right)/2\right) \leq F(t). \tag{32}$$

Proof. Both estimates follow immediately by integration. Indeed, the upper estimate follows from

$$t \int_t^\infty e^{-\frac{s^2}{2}} ds \leq \int_t^\infty s e^{-\frac{s^2}{2}} ds = e^{-\frac{t^2}{2}},$$

while the lower estimate follows from

$$\int_t^\infty e^{-\frac{s^2}{2}} ds \geq \int_t^{t+A} e^{-\frac{s^2}{2}} ds \geq A \exp\left(-\left(t+A\right)^2/2\right).$$

□

Claim 17 *N is an Orlicz function.*

Proof. N is an increasing function on $[0, \infty)$ such that $N(t) = 0$ if and only if $t = 0$. We have to show that N is convex. To do so, we show that $N''(t) \geq 0$ for $t \geq 0$.

$$N''(t) = \left(\frac{e^{-\frac{t^2}{2}}}{\int_t^\infty e^{-\frac{s^2}{2}} ds}\right)' = \frac{-te^{-\frac{t^2}{2}} \int_t^\infty e^{-\frac{s^2}{2}} ds + e^{-t^2}}{\left(\int_t^\infty e^{-\frac{s^2}{2}} ds\right)^2} = \frac{e^{-\frac{t^2}{2}} \left(e^{-\frac{t^2}{2}} - t \int_t^\infty e^{-\frac{s^2}{2}} ds\right)}{\left(\int_t^\infty e^{-\frac{s^2}{2}} ds\right)^2},$$

which is non-negative by (30). □

Claim 17 shows that F and N satisfy the hypothesis of Theorem 1 and 2. Next we estimate the Orlicz norm N_j.

Claim 18 *Let $N = -\ln F$,*

$$H(t) = \begin{cases} t & \text{for } 0 \leq t < 1 \\ t^2 & \text{for } t \geq 1. \end{cases}$$

Then H is an Orlicz function and for every $t \geq 0$

$$(2\pi e)^{-1/2} \; H(t) \leq N(t) \leq 4.5 \; H(t).$$

In particular, if $k \leq n$ and N_j, $j \leq k$, as in (11) then for every $t \geq 0$

$$\sqrt{\frac{2e}{\pi}} \, \frac{1}{k-j+1} \; H(t) \leq N_j(t) \leq \frac{9e}{k-j+1} \; H(t).$$

Proof. Clearly, H is an Orlicz function.

For every $0 \leq t \leq \sqrt{\pi/8}$ we have

$$\frac{1}{2} \leq 1 - \sqrt{\tfrac{2}{\pi}} \; t \leq F(t) = 1 - \sqrt{\tfrac{2}{\pi}} \int_0^t e^{-\frac{s^2}{2}} ds \leq 1 - \sqrt{\tfrac{2}{e\pi}} \; t.$$

Since $(x-1)/2 \leq \ln x \leq x - 1$ on $[1,2]$, we observe for $0 \leq t \leq \sqrt{\pi/8}$

$$N(t) = \ln \frac{1}{F(t)} \leq \frac{1}{F(t)} - 1 \leq \frac{1}{1 - \sqrt{\frac{2}{\pi}} \; t} - 1 \leq \frac{\sqrt{\frac{2}{\pi}} \; t}{1 - \sqrt{\frac{2}{\pi}} \; t} \leq \sqrt{\tfrac{8}{\pi}} \; t$$

and

$$N(t) = \ln \frac{1}{F(t)} \geq \frac{1}{2} \left(\frac{1}{F(t)} - 1 \right) \geq \frac{1}{2} \left(\frac{1}{1 - \sqrt{\frac{2}{e\pi}} \; t} - 1 \right) \geq \frac{t}{\sqrt{2e\pi}}.$$

This shows the desired result for $0 \leq t \leq \sqrt{\pi/8}$.

Consider now the function $f(t) = N(t) - t^2/2$ and observe that $f(0) = 0$. By (30) we have $f'(t) \geq 0$ for $t \geq 0$. Thus, for every $t \geq 0$ one has $N(t) \geq t^2/2$.

Finally, applying (31) with $A = \sqrt{\pi/2}$, we have for $t \geq \sqrt{\pi/8}$ (then $t + A \leq 3t$) that

$$F(t) \geq \exp\left(-9t^2/2\right).$$

This implies $t^2/2 \leq N(t) \leq 9t^2/2$ for $t \geq \sqrt{\pi/8}$. In particular, $H/\sqrt{2\pi e} \leq N \leq 9H/2$.
□

References

[1] B. C. Arnold, N. Narayanaswamy, *Relations, Bounds and Approximations for Order Statistics*, Lecture Notes in Statistics, 53, Berlin etc.: Springer-Verlag. viii (1989).

[2] N. Balakrishnan, W.W.S. Chen, *Handbook of Tables for order Statistics from lognormal Distributions with Applications*, Amsterdam, Netherlands, Kluwer Academic Publishers (1999).

[3] N. Balakrishnan, A.C. Cohen, *Order Statistics and Inference*, New York, NY: Academic Press (1991).

[4] R Baraniuk, M Davenport, R DeVore, M Wakin, *The Johnson-Lindenstrauss lemma meets compressed sensing*, preprint.

[5] N. Balakrishnan, E. Castillo and J.M. Sarabia, editors, *Advances in Distribution Theory, Order Statistics, and Inference Series: Statistics for Industry and Technology*, Birkhäuser (2006).

[6] Y. Benjamini, M. Leshno, *Statistical Methods for Data Mining*, Data Mining and Knowledge Discovery Handbook, Springer US (2005).

[7] N. Balakrishnan and C. R. Rao, editors, *Handbook of Statistics 16: Order Statistics: Theory and Methods Elsevier*, Amsterdam (1999a).

[8] N. Balakrishnan and C. R. Rao, editors, *Handbook of Statistics 16: Order Statistics: Applications*, Elsevier, Amsterdam (1999b).

[9] E. J. Candes, J. Romberg, T. Tao, *Robust Uncertaintity Principles: Exact Signal Reconstruction from Highly Incomplete Frequency Information*, IEEE Trans. Inf. Theory, to appear.

[10] A. Cohen, W. Dahmen and R. DeVore, *Compressed Sensing and Best k-term Approximation*, preprint.

[11] H. A. David, H. N. Nagaraja, *Order statistics*, 3rd ed., Wiley Series in Probability and Statistics. Chichester: John Wiley & Sons (2003).

[12] A. Dimitriyuk, Y. Gordon, *Generalizing the Johnson-Lindenstrauss lemma to k-dimensional affine subspaces*, preprint.

[13] D. Donoho, *Compressed Sensing*, IEEE Trans. Information Theory, 52 (2006), 1289–1306.

[14] E.D. Gluskin and S. Kwapień *Tail and moment estimates for sums of independent random variables*, Stud. Math. 114 (1995), 303-309.

[15] Y. Gordon, *Majorization of Gaussian processes and Geometric Applications*, Prob. Th. Rel. Fields, 91, No. 2 (1992), 251–267.

[16] Y. Gordon, O. Guédon, M. Meyer and A. Pajor, *On the Euclidean sections of some Banach spaces and operator spaces*, Math. Scandinavica 91 (2002), 247–268.

[17] Y. Gordon, A. E. Litvak, S. Mendelson, A. Pajor, *Gaussian averages of interpolated bodies and applications to approximate reconstruction*, J. Approx. Theory, 149 (2007), 59–73.

[18] Y. Gordon, A. E. Litvak, C. Schütt, E. Werner, *Orlicz Norms of Sequences of Random Variables*, Ann. of Prob., 30 (2002), 1833–1853.

[19] Y. Gordon, A. E. Litvak, C. Schütt, E. Werner, *Geometry of spaces between zonoids and polytopes*, Bull. Sci. Math., 126 (2002), 733–762.

[20] Y. Gordon, A. E. Litvak, C. Schütt, E. Werner, *On the minimum of several random variables*, Proc. Am. Math. Soc. 134, No. 12, 3665–3675 (2006).

[21] Y. Gordon, A. E. Litvak, C. Schütt, E. Werner, *Minima of sequences of Gaussian random variables*, C.R. Acad. Sci. Paris, Ser 1, Math., 340 (2005), 445–448.

[22] O. Guèdon, *Gaussian Version of a Theorem of Milman and Schechtman*, Positivity 1, No.1 (1997), 1–5.

[23] G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities*, 2nd ed., Cambridge, The University Press. XII (1952).

[24] W. B. Johnson, J. Lindenstrauss, *Extensions of Lipschitz Mappings into a Hilbert Space*, Contemp. Math. 26 (1984), 189–206.

[25] M. Junge, *The optimal order for the p-th moment of sums of independent random variables with respect to symmetric norms and related combinatorial estimates*, Positivity 10 (2006), 201-230.

[26] M.A. Krasnosel'skii and Ya. B. Rutickii, *Convex Functions and Orlicz Functions*, P. Noordhoff, Groningen (1961).

[27] S. Kwapien, C. Schütt, *Some combinatorial and probabilistic inequalities and their application to Banach space theory*, Studia Math. 82 (1985), 91–106.

[28] S. Kwapien, C. Schütt, *Some combinatorial and probabilistic inequalities and their application to Banach space theory. II*, Studia Math. 95 (1989), 141–154.

[29] R. Latała, preprint.

[30] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I*, Springer-Verlag (1977).

[31] S. Mallat and O. Zeitouni, *Optimality of the Karhunen-Loeve basis in nonlinear reconstruction*, preprint.

[32] V. A. Marchenko and L. A. Pastur, *Distribution of eigenvalues in certain sets of random matrices*, Mat. Sb. (N.S.), 72 (1967), 407–535 (Russian).

[33] S. Mendelson, A. Pajor, N. Tomczak-Jaegermann, *Reconstruction and Subgaussian Operators in Asymptotic Geometric Analysis*, GAFA, Geom. and Funct. Anal., 17 (2007), 1248–1282.

[34] S. Montgomery-Smith, *Rearrangement invariant norms of symmetric sequence norms of independent sequences of random variables*, Isr. J. Math. 131 (2002), 51–60.

[35] M.M. Rao und Z.D. Ren, *Theory of Orlicz Spaces*, Marcel Dekker (1991).

[36] M. Rudelson *Lower estimates for the singular values of random matrices*, C. R. Math. Acad. Sci. Paris 342 (2006), no. 4, 247–252.

[37] S. J. Szarek, *Spaces with large distance to ℓ_∞^n and random matrices*, Amer. J. Math. 112 (1990), no. 6, 899–942.

[38] S. J. Szarek, *Condition numbers of random matrices*, J. Complexity 7 (1991), no. 2, 131–149.

[39] L. Wasserman, *All of Statistics: A Concise Course in Statistical Inference*, Springer Texts in Stat. (2004).

Y. Gordon, Dept. of Math., Technion, Haifa 32000, Israel, gordon@techunix.technion.ac.il

A. E. Litvak, Dept. of Math. and Stat. Sciences, University of Alberta, Edmonton, AB, Canada T6G 2G1, alexandr@math.ualberta.ca

C. Schütt, Mathematisches Seminar, Christian Albrechts Universität, 24098 Kiel, Germany, schuett@math.uni-kiel.de

E. Werner, Dept. of Math., Case Western Reserve University, Cleveland, Ohio 44106, U.S.A. and Université de Lille 1, UFR de Mathématique, 59655 Villeneuve d'Ascq, France, emw2@po.cwru.edu